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Extensions of the Euler–Satake characteristic for nonorientable 3-orbifolds and indistinguishable examples

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We compute the $\mathbb{F}_l$-Euler–Satake characteristics of an arbitrary closed, effective 3-dimensional orbifold Q where $\mathbb{F}_l$ is a free group with l generators. We focus on the case of nonorientable orbifolds, extending previous results for the case of orientable orbifolds. Using these computations, we determine examples of distinct 3-orbifolds Q and Q' such that $\chi_\Gamma^{\mathrm{ES}}(Q) = \chi_\Gamma^{\mathrm{ES}}(Q')$ for every finitely generated discrete group Γ.

1. Introduction

This paper completes a program to determine what information about the singular set of an effective, low-dimensional orbifold is determined by the collection of Γ-extensions of the Euler–Satake characteristic. For a finitely generated discrete group Γ and an orbifold Q, the *orbifold of* Γ*-sectors* of Q is a collection of orbifolds of different dimensions containing Q as well as finite singular covers of the singular strata of Q. The Γ*-extension* of an orbifold invariant is defined by applying the invariant to the orbifold of Γ-sectors of Q.

The *Euler–Satake characteristic* $\chi_{\mathrm{ES}}(Q)$ of a closed orbifold Q is a rational number that corresponds to $\chi_{\mathrm{top}}(M)/|G|$ in the case that Q is a *global quotient orbifold*, i.e., is presented by the quotient of a closed manifold M by a finite group G, where χ_{top} denotes the usual Euler characteristic. It was defined in [Satake 1957] where it is referred to as the *Euler characteristic as a V-manifold* and [Thurston 1997] where it is called the *orbifold Euler characteristic*. The Γ-extensions of the Euler–Satake characteristic of Q, denoted $\chi_\Gamma^{\mathrm{ES}}(Q)$, include many interesting orbifold invariants. When $\Gamma = \mathbb{Z}$, $\chi_\Gamma^{\mathrm{ES}}(Q)$ coincides with the Euler characteristic of

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the underlying topological space of Q. When $\Gamma = \mathbb{Z}^2$, $\chi_\Gamma^{\mathrm{ES}}(Q)$ is the stringy orbifold Euler characteristic defined in [Dixon et al. 1985] for global quotients and [Roan 1996] for general orbifolds; see also [Adem and Ruan 2003]. Further extensions corresponding to $\Gamma = \mathbb{Z}^l$ were suggested in [Atiyah and Segal 1989] and defined for global quotients in [Bryan and Fulman 1998]. In [2001; 2003], Tamanoi introduced and studied extensions of orbifold invariants for global quotients, including the Euler–Satake characteristic, corresponding to arbitrary Γ, and this definition was extended to arbitrary orbifolds in [Farsi and Seaton 2010b; 2011].

In [Duval et al. 2010], it was demonstrated that the collection of $\mathbb{Z}^l$-extensions of the Euler–Satake characteristic determine the diffeomorphism type of a closed, effective, orientable 2-dimensional orbifold and that infinitely many were required to do so. In addition, it was demonstrated that the $\chi_\Gamma^{\mathrm{ES}}$ corresponding to any collection of finitely generated discrete groups do not distinguish between certain effective, nonorientable 2-orbifolds.

However, in [Schulte et al. 2011], it was shown that any infinite collection $\chi_{\mathbb{Z}^l}^{\mathrm{ES}}$ along with any infinite collection of $\chi_{\mathbb{F}_l}^{\mathrm{ES}}$ determines the number and type of point singularities of a closed, effective, nonorientable 2-orbifold and that an infinite collection of both is required. In dimension 3, it is shown in [Carroll and Seaton 2013] that any infinite collection of the $\chi_{\mathbb{F}_l}^{\mathrm{ES}}$ determines the number and type of point singularities of a closed, effective, orientable 3-orbifold and that infinitely many are required to do so.

Here, we study the Γ-Euler–Satake characteristics of effective, nonorientable 3-dimensional orbifolds and demonstrate that the above results do not extend to this case. For a closed, effective 3-orbifold Q, we show that the $\chi_\Gamma^{\mathrm{ES}}(Q)$ depend only on the number and type of point singularities of Q and the Euler characteristic of the (topological manifold) boundary of the underlying space of Q. In particular, these invariants can be computed without determining the structure of the 2-dimensional sectors of Q, which can be complicated and difficult to describe in general. We detail a general computation of these invariants for $\Gamma = \mathbb{F}_l$. This computation is used to determine examples of closed, effective 3-orbifolds whose Γ-Euler–Satake characteristics coincide for every finitely generated discrete group Γ, though the point singularities and the topology of the underlying space are different.

This paper is organized as follows. In Section 2, we review the relevant background on orbifolds and Γ-sectors as well as the structure of the singular set of a closed 3-orbifold. In Section 3, we compute the $\mathbb{F}_l$-Euler–Satake characteristics of closed, effective 3-orbifolds. In particular, we demonstrate Proposition 3.6, which reduces the computation of the $\chi_\Gamma^{\mathrm{ES}}(Q)$ for any finitely generated discrete Γ to an expression that does not involve the 2-dimensional sectors of Q. In Section 4, we give an example of distinct orbifolds whose Γ-Euler–Satake characteristics coincide for every Γ.

2. Background and definitions

In this section, we review the relevant background material and fix notation. In Section 2A, we recall the definition of an orbifold Q as well as the Γ-sectors $\widetilde{Q}_\Gamma$ of Q. Note that there are many definitions of orbifolds in the literature that are more or less equivalent and suited to different purposes. Here, we consider an orbifold to be the Morita equivalence class of a proper, étale, Lie groupoid, and the definition of the Γ-sectors is most natural from this perspective. However, we expect this work to be accessible to readers with no knowledge of groupoids, and hence adapt these definitions to the framework in which an orbifold structure is designated by an atlas of orbifold charts. Though many of the required properties of Γ-sectors are developed elsewhere using groupoid language, we explain the ingredients we will need in the language of charts below. In Section 2B, we review the classification of finite subgroups of $\mathrm{O}(3)$, and in Section 2C, we use this classification to describe the topology of a closed, effective 3-orbifold and its singular set.

The reader is referred to [Adem et al. 2007; Moerdijk and Mrčun 2003; Moerdijk 2002] for background on orbifolds from the perspective of Lie groupoids. See [Thurston 1997; Chen and Ruan 2002; Satake 1957] for background from the perspective of orbifold charts, [Boileau et al. 2003; Scott 1983] for more discussion of 3-dimensional orbifolds and [Lerman 2010; Iglesias et al. 2010] for alternate approaches to orbifolds. Note that some of the above references restrict their attention to effective orbifolds. The Γ-sectors of an orbifold are defined for global quotient orbifolds in [Tamanoi 2001; 2003], and are defined for general orbifolds in [Farsi and Seaton 2010b]; see also [Farsi and Seaton 2010a; 2011]. Note that the Γ-sectors extend the definition of the inertia orbifold (see [Kawasaki 1978]) and multi-sectors defined in [Adem et al. 2007; Chen and Ruan 2004].

2A. ***Orbifolds and Γ-sectors.*** By an *orbifold* Q, we will mean a paracompact Hausdorff space $\mathbb{X}_Q$ that is homeomorphic to the orbit space $|\mathcal{G}|$ of a proper, étale Lie groupoid $\mathcal{G}$. We refer to a choice of $\mathcal{G}$ and homeomorphism between $|\mathcal{G}|$ and $\mathbb{X}_Q$ as a *presentation* of Q. For $\mathcal{G}$, we may take an *orbifold atlas* for Q, consisting of charts of the form $\{V, G, \pi\}$ where V is an open neighborhood of the origin in $\mathbb{R}^n$ equipped with the action of the finite group G, which may be taken to be a subgroup of $\mathrm{O}(n)$ with respect to an inner product on $\mathbb{R}^n$, and $\pi : V \to \mathbb{X}_Q$ is a continuous function that induces a homeomorphism of $G\backslash V$ onto an open subset of $\mathbb{X}_Q$. When a chart is labeled $\{V_p, G_p, \pi_p\}$ for a point $p \in \mathbb{X}_Q$, we assume that $\pi_p(0) = p$ and refer to $\{V_p, G_p, \pi_p\}$ as an *orbifold chart at* p. An *injection* of orbifold charts $\{V, G, \pi\} \to \{V', G', \pi'\}$ is a pair (f, λ) where $\lambda : G \to G'$ is an injective homomorphism and $f : V \to V'$ is a λ-equivariant open embedding such that $\pi \circ f = \pi'$. Two charts $\{V, G, \pi\}$ and $\{V', G', \pi'\}$ are said to be *compatible* if for each $p \in \pi(V) \cap \pi'(V')$, there is an orbifold chart $\{V_p, G_p, \pi_p\}$ at p and a pair of injections of $\{V_p, G_p, \pi_p\}$ into $\{V, G, \pi\}$ and $\{V', G', \pi'\}$, respectively. Then an orbifold atlas is a collection of compatible charts whose images in $\mathbb{X}_Q$ cover $\mathbb{X}_Q$, and an equivalence of atlases can be defined which corresponds to Morita equivalence for groupoid presentations. *Diffeomorphic* orbifolds are those presented by Morita equivalent groupoids, e.g, equivalent orbifold atlases.

Note that any two injections (f_1, λ_1) and (f_2, λ_2) of orbifold charts $\{V, G, \pi\} \to \{V', G', \pi'\}$ are related by an element g of G'; that is, $f_2(x) = gf_1(x)$ for each $x \in V$, and $\lambda_2(h) = g\lambda_1(h)g^{-1}$ for each $h \in G$; see [Moerdijk and Pronk 1997, Proposition A.1]. Applying this result to injections between a chart and itself, it follows that the (isomorphism class of the) *isotropy group* G_p of a point $p \in \mathbb{X}_Q$ does not depend on the choice of chart at p, and in fact can be defined as the isotropy group of an arbitrary lift of p into an arbitrary chart. Moreover, though the elements of G_p depend on the choice of chart, their G_p-conjugacy classes in a chart at p are well-defined.

An orbifold Q is *effective* if it is presented by an effective groupoid $\mathcal{G}$, or equivalently if it is equipped with an atlas such that the group action in each chart is effective. It is *closed* if $\mathbb{X}_Q$ is compact and Q does not have boundary as an orbifold; note that we have only considered orbifolds without boundary in the definitions above. When Q is connected, the *dimension* of Q is the dimension of the object space of an étale presentation $\mathcal{G}$ of Q, or equivalently the dimension of the domain of each orbifold chart.

The *Euler–Satake characteristic* $\chi_{\mathrm{ES}}(Q)$ of a closed orbifold Q is defined in terms of a triangulation of Q such that the isomorphism class of the isotropy group is constant on the interior of each simplex. If $\mathcal{T}$ is such a triangulation and for each $\sigma \in \mathcal{T}$, G_σ denotes the isotropy group of a point on the interior of σ, then

$$\chi_{\mathrm{ES}}(Q) = \sum_{\sigma \in \mathcal{T}} \frac{(-1)^{\dim \sigma}}{|G_\sigma|}.$$

Let Γ be a finitely generated discrete group. The simplest description of the Γ-sectors of Q is in terms of a proper étale Lie groupoid $\mathcal{G}$ presenting Q. In this case the collection $\mathrm{HOM}(\Gamma, \mathcal{G})$ of groupoid homomorphisms from Γ to $\mathcal{G}$ inherits the structure of a disjoint union of smooth manifolds, potentially of different dimensions, as well as a smooth action of $\mathcal{G}$. Then $\mathcal{G} \ltimes \mathrm{HOM}(\Gamma, \mathcal{G})$ is itself a proper étale Lie groupoid presenting the orbifold of Γ-sectors $\widetilde{Q}_\Gamma$. Note that $\widetilde{Q}_\Gamma$ always includes a connected component diffeomorphic to Q, called the *nontwisted* Γ*-sector* corresponding to the trivial homomorphisms; all other sectors are called *twisted* Γ*-sectors* If Q is closed, then $\widetilde{Q}_\Gamma$ is a finite union of connected, closed orbifolds.

Alternatively, the Γ-sectors of Q can be defined as follows. Let

$$\mathbb{X}_{\widetilde{Q}_\Gamma} = \{(p, (\varphi_p)_{G_p}) : p \in \mathbb{X}_Q, \varphi_p : \Gamma \to G_p\},$$

where $(\varphi_p)_{G_p}$ denotes the G_p-conjugacy class of φ_p. Given an orbifold chart $\{V_p, G_p, \pi_p\}$ for Q at a point p, define the orbifold chart $\{V_p^{\langle\varphi_p\rangle}, C_{G_p}(\varphi_p), \pi_p^{\varphi_p}\}$ for $\widetilde{Q}_\Gamma$ at $(p, (\varphi_p)_{G_p})$, where $V_p^{\langle\varphi_p\rangle}$ denotes the points in V_p fixed by the image of φ_p, $C_{G_p}(\varphi_p)$ denotes the centralizer of φ_p in G_p, and $\pi_p^{\varphi_p} : V_p^{\langle\varphi_p\rangle} \to \mathbb{X}_{\widetilde{Q}_\Gamma}$ is defined as follows. Given $y \in V_p^{\langle\varphi_p\rangle}$, identify the isotropy group of $\pi_p(y)$ with the isotropy group $(G_p)_y \le G_p$, and then let $\varphi_{\pi_p(y)} : \Gamma \to (G_p)_y$ denote the homomorphism given by restricting the codomain of φ_p to this subgroup. Note that the image $\mathrm{Im}(\varphi_p)$ of φ_p is contained in $(G_p)_y$ precisely when $y \in V_p^{\langle\varphi_p\rangle}$. Note further that $\varphi_{\pi_p(y)}$ is well-defined only up to its $G_{\pi_p(x)}$-conjugacy class. Then we define $\pi_p^{\varphi_p}(y) = (\pi_p(y), (\varphi_{\pi_p(y)})_{(G_p)_y})$. The proof that the $\{V_p^{\langle\varphi_p\rangle}, C_{G_p}(\varphi_p), \pi_p^{\varphi_p}\}$ define an orbifold structure for $\mathbb{X}_{\widetilde{Q}_\Gamma}$ is omitted; it is given by translating the proof in groupoid language given in [Farsi and Seaton 2010b] to this context. It is, as well, a direct generalization of the proof of [Chen and Ruan 2004, Lemma 3.1.1] (see also [Kawasaki 1978]), which is given in atlas language for the case $\Gamma = \mathbb{Z}$. We will, however, require an understanding of the injections between orbifold charts for $\widetilde{Q}_\Gamma$, which we now describe.

Given an injection (f, λ) of orbifold charts $\{V_q, G_q, \pi_q\} \to \{V_p, G_p, \pi_p\}$, we say that a homomorphism $\varphi_q : \Gamma \to G_q$ is *locally covered* by a homomorphism $\varphi_p : \Gamma \to G_p$ (with respect to the choice of charts and injection) if $\lambda \circ \varphi_q = \varphi_p$. Then it is easy to see that $\{f|_{V_q^{\langle\varphi_q\rangle}}, \lambda|_{C_{G_q}(\varphi_q)}\}$ is an injection of the orbifold chart $\{V_q^{\langle\varphi_q\rangle}, C_{G_q}(\varphi_q), \pi_q^{\varphi_q}\}$ into the orbifold chart $\{V_p^{\langle\varphi_p\rangle}, C_{G_p}(\varphi_p), \pi_p^{\varphi_p}\}$. Note that if φ_p locally covers φ_q with respect to the injection (f, λ) as above, then for any other choice of injection $(f', \lambda') : \{V_q, G_q, \pi_q\} \to \{V_p, G_p, \pi_p\}$, there is a $g \in G_p$ such that $g(\lambda \circ \varphi_q)g^{-1} = \varphi_p$; compare [Farsi and Seaton 2010b, Definition 2.6]. In particular, if φ_p and ψ_p are both homomorphisms $\Gamma \to G_p$, then by the characterization of injections of a chart into itself given in [Moerdijk and Pronk 1997, Proposition A.1] and recalled above, φ_p locally covers ψ_p if and only if φ_p and ψ_p are G_p-conjugate. By allowing finite sequences of local coverings (in either direction), we extend the notion of local covering to an equivalence relation on $\bigcup_{p\in\mathbb{X}_Q} \mathrm{HOM}(\Gamma, G_p)$ and let $\approx$ denote this relation. We let $(\varphi_p)_\approx$ denote the $\approx$-class of a homomorphism φ_p and T_Q^Γ the set of equivalence classes. Note that φ_p and φ_q are equivalent if and only if $(p, (\varphi_p)_{G_p})$ and $(q, (\varphi_q)_{G_q})$ are in the same connected component of $\mathbb{X}_{\widetilde{Q}_\Gamma}$.

2B. ***The finite subgroups of* O(3).** In this section, we recall the classification of finite subgroups G of O(3) given in [Benson and Grove 1971, Theorem (2.5.2)] as well as the corresponding orbifold singularities. In each case, we fix a representation of the group G to refer to in the sequel.

First, recall that every element of SO(3) acts as a rotation about a line in $\mathbb{R}^3$. Up to conjugation in SO(3), the finite subgroups of SO(3) consist of the cyclic groups

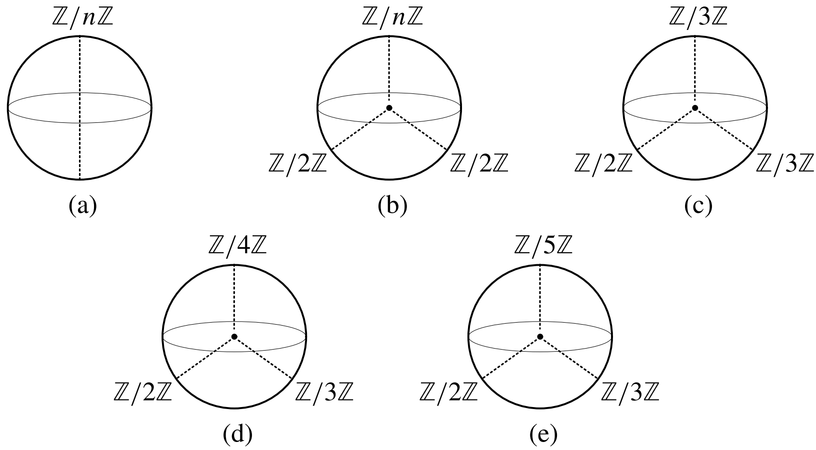

Figure 1. Singular sets in the quotients of $\mathbb{R}^3$ by the finite subgroups of SO(3): (a) a cyclic group $\mathbb{Z}/n\mathbb{Z}$; (b) a dihedral group D_{2n}; (c) the tetrahedral group $\mathbb{T}$; (d) the octahedral group $\mathbb{O}$; (e) the icosahedral group $\mathbb{I}$.

$\mathbb{Z}/n\mathbb{Z}$, the dihedral groups D_{2n} of order $2n$, the tetrahedral group $\mathbb{T}$ of order 12, the octahedral group $\mathbb{O}$ of order 24, and the icosahedral group $\mathbb{I}$ of order 60. The quotient $G\backslash\mathbb{R}^3$ is homeomorphic to $\mathbb{R}^3$ in each case; see [Boileau et al. 2003]. In $\mathbb{Z}_n\backslash\mathbb{R}^3$ the singular set is a line fixed by the entire group $\mathbb{Z}_n$, while for the other groups, the singular set is the origin as well as three rays fixed by cyclic groups; see Figure 1. The Γ-Euler–Satake characteristics of orientable 3-orbifolds, which contain only these singularities, are studied in [Carroll and Seaton 2013].

Let J denote the negative identity element in O(3). A finite subgroup of O(3) generated by J and a finite subgroup G of SO(3) is called a *full group*, denoted G^*. Note that as J is central and $J^2 = I$ is the identity in O(3), G^* is isomorphic to $G \times \mathbb{Z}/2\mathbb{Z}$. There are five classes of full groups corresponding to the five classes of subgroups of SO(3). See Figure 2 for diagrams of the quotient spaces and singular sets in each case, and note that they refer to mixed groups and $\mathcal{P}_{\text{proj}}$, which are defined below. Note that the quotient space $G^*\backslash\mathbb{R}^3$ is homeomorphic to closed half-space in $\mathbb{R}^3$ except in the case of $G = (\mathbb{Z}/n\mathbb{Z})^*$ with n odd, where $(\mathbb{Z}/n\mathbb{Z})^*\backslash\mathbb{R}^3$ is homeomorphic to the cone on $\mathbb{R}P^2$.

- A *full cyclic group* $(\mathbb{Z}/n\mathbb{Z})^*$ has order $2n$ and is generated by A_n and J, where

$$A_n = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) & 0 \\ \sin(2\pi/n) & \cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

satisfies $A_n^n = I$.

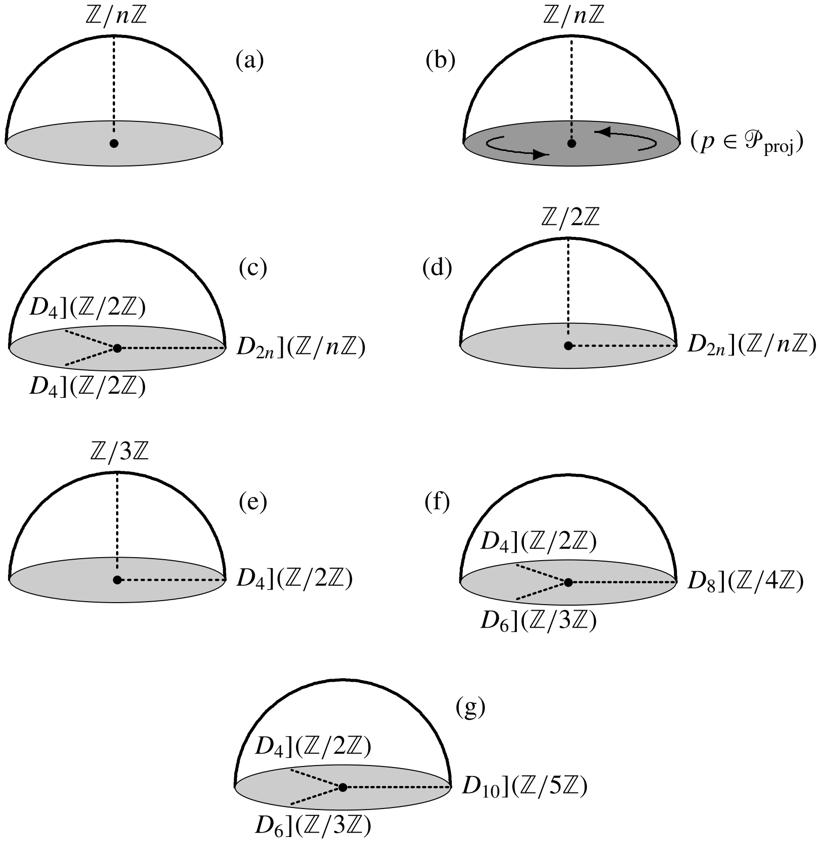

Figure 2. Singular sets corresponding to the finite full subgroups of O(3), where the lightly shaded boundary disks have generic isotropy $\mathbb{Z}/2\mathbb{Z}$ and the darkly shaded boundary disk is identified via the antipodal map as indicated by curved arrows: (a) a full cyclic group $(\mathbb{Z}/n\mathbb{Z})^*$ for n even; (b) a full cyclic group $(\mathbb{Z}/n\mathbb{Z})^*$ for n odd; (c) a full dihedral group D_{2n}^* for n even; (d) a full dihedral group D_{2n}^* for n odd; (e) the full tetrahedral group $\mathbb{T}^*$; (f) the full octahedral group $\mathbb{O}^*$; (g) the full icosahedral group $\mathbb{I}^*$.

- A *full dihedral group* D_{2n}^* has order $4n$ and is generated by A_n, B, and J, where A_n is as above and

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is a rotation about the x-axis through an angle of π. Note that $A_n B = B A_n^{n-1}$.

- The *full tetrahedral group* $\mathbb{T}^*$ has order 24 and is generated by C, D, and J, where
$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$
Note that $C^2 = D^3 = (CD)^3 = I$.

- The *full octahedral group* $\mathbb{O}^*$ has order 48 and is generated by R, S, and J, where
$$R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$
Note that $R^2 = S^4 = (RS)^3 = I$.

- The *full icosahedral group* $\mathbb{I}^*$ is generated by B, E, and J, where B is as above and
$$E = \begin{bmatrix} +\phi/2 & +\overline{\phi}/2 & +1/2 \\ +\overline{\phi}/2 & +1/2 & -\phi/2 \\ -1/2 & +\phi/2 & +\overline{\phi}/2 \end{bmatrix}.$$
Here, $\phi = (1+\sqrt{5})/2$ and $\overline{\phi} = (1-\sqrt{5})/2$. Note that $B^2 = E^5 = (BE)^3 = I$.

A finite subgroup $G < \mathrm{O}(3)$ that is not full or contained in $\mathrm{SO}(3)$ is a *mixed group*. A mixed group G is denoted $H]K$, where H and K are finite subgroups of $\mathrm{SO}(3)$ such that K is a subgroup of H of index 2. Then G is isomorphic to H as a group, but the representation of G on $\mathbb{R}^3$ is given by multiplying those elements in the nontrivial coset of H/K by J. The quotient spaces and singular sets of $G\backslash\mathbb{R}^3$ for mixed groups G are pictured in Figure 3. Again, each quotient space is homeomorphic to closed half-space in $\mathbb{R}^3$ with the exception of $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})\backslash\mathbb{R}^3$ for n even, which is homeomorphic to the cone on $\mathbb{R}P^2$. The four families of mixed subgroups of $\mathrm{O}(3)$ are as follows:

- A *mixed cyclic group* $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ has order $2n$ and is generated by $A_{2n}J$ where A_{2n} is as above.
- A *mixed dihedral group* $D_{4n}]D_{2n}$ of order $4n$ is generated by $A_{2n}J$ and B.
- A *dihedral extending cyclic group* $D_{2n}](\mathbb{Z}/n\mathbb{Z})$ of order $2n$ is generated by A_n and BJ.
- The *octahedral extending tetrahedral group* $\mathbb{O}]\mathbb{T}$ is generated by RJ and SJ.

Every finite subgroup of $\mathrm{O}(3)$ is conjugate to one of the groups listed above. When referring to these groups, we will always mean the group as well as its standard representation on $\mathbb{R}^3$ in coordinates as described above.

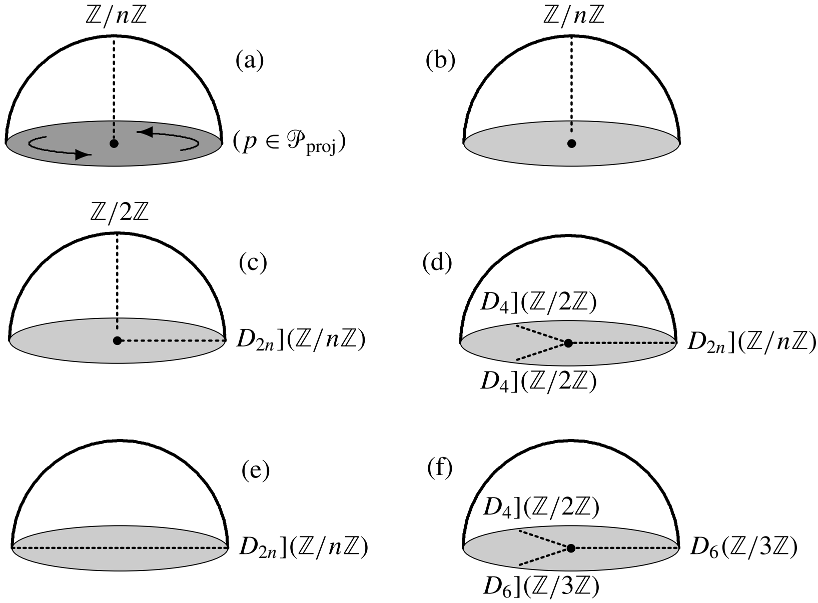

Figure 3. Singular sets corresponding to the finite mixed subgroups of O(3), where the lightly shaded boundary disks have generic isotropy $\mathbb{Z}/2\mathbb{Z}$ and the darkly shaded boundary disk is identified via the antipodal map as indicated by curved arrows: (a) a mixed cyclic group $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ for n even; (b) a mixed cyclic group $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ for n odd; (c) a mixed dihedral group $D_{4n}]D_{2n}$ for n even; (d) a mixed dihedral group $D_{4n}]D_{2n}$ for n odd; (e) a dihedral extending cyclic group $D_{2n}](\mathbb{Z}/n\mathbb{Z})$ for n of either parity; (f) the octahedral extending tetrahedral group $\mathbb{O}]\mathbb{T}$.

2C. ***Closed, effective 3-dimensional orbifolds.*** Let Q be a closed, effective 3-orbifold. Then each point of Q is contained in a neighborhood that is homeomorphic to $G\backslash\mathbb{R}^3$ where G is a finite subgroup of O(3). Inspecting the possible homeomorphism classes of $G\backslash\mathbb{R}^3$, it follows that there is a finite, possibly empty collection $\mathcal{P}_{\mathrm{proj}} = \{p_1, \ldots, p_k\}$ of points in Q such that $\mathbb{X}_Q \smallsetminus \mathcal{P}_{\mathrm{proj}}$ is a topological 3-manifold, potentially with boundary. In particular, $\mathcal{P}_{\mathrm{proj}}$ consists of those points with isotropy group $(\mathbb{Z}/n\mathbb{Z})^*$ for n odd or $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ for n even. Note that Q is orientable if and only if $\mathcal{P}_{\mathrm{proj}} = \varnothing$ and $\mathbb{X}_Q$ is an orientable 3-manifold without boundary. In the case that $\mathbb{X}_Q$ has boundary as a topological manifold, we let $\partial_{\mathrm{top}} Q$ denote the boundary and caution the reader that Q does not have boundary as an orbifold.

The singular set of Q consists of $\partial_{\mathrm{top}} Q \cup \mathcal{P}_{\mathrm{proj}}$ along with a disjoint collection of circles in the interior of $\mathbb{X}_Q \smallsetminus \mathcal{P}_{\mathrm{proj}}$ and a not-necessarily connected graph $\mathfrak{G}$ that is trivalent on the interior of $\mathbb{X}_Q \smallsetminus \mathcal{P}_{\mathrm{proj}}$ and has univalent vertices in $\mathcal{P}_{\mathrm{proj}}$ and $\partial_{\mathrm{top}} Q$. Note that the (isomorphism class of the) isotropy group is constant on each circle as well as the interior of each edge; however, it need not be constant on $\partial_{\mathrm{top}} Q$. Rather, $\partial_{\mathrm{top}} Q$ itself contains a union of circles and a (not necessarily connected) graph with trivalent vertices as well as univalent vertices where $\mathfrak{G}$ intersects $\partial_{\mathrm{top}} Q$. The isotropy type is constant on the circles and the interiors of the edges of this graph and is $\mathbb{Z}/2\mathbb{Z}$ elsewhere in $\partial_{\mathrm{top}} Q$.

By a *point singularity* of Q, we mean a point $p \in \mathbb{X}_Q$ contained in a neighborhood U such that the isotropy group of p is strictly larger than all points in $U \smallsetminus \{p\}$. Equivalently, a point singularity corresponds to a 0-dimensional stratum of $\mathbb{X}_Q$ with respect to the stratification by orbit types. Note that the point singularities of Q correspond to the vertices of the graphs described above.

In the sequel, we will let $\mathcal{P}$ denote the set of point singularities of Q and $\mathcal{P}_\partial$ denote the set of point singularities that occur on $\partial_{\mathrm{top}} Q$. Note that $\mathcal{P}_{\mathrm{proj}}$ is the set of point singularities that occur on nonmanifold points of $\mathbb{X}_Q$. Then $\mathcal{P} \smallsetminus (\mathcal{P}_\partial \cup \mathcal{P}_{\mathrm{proj}})$ is exactly the set of point singularities at which Q is *locally orientable*, i.e., with isotropy group contained in $\mathrm{SO}(3)$.

The Γ-sectors of a closed, effective 3-orbifold Q include the nontwisted Γ-sector of dimension 3 and may include twisted Γ-sectors of dimensions 0, 1, or 2. Each Γ-sector is a closed orbifold, and only the nontwisted Γ-sector is effective. Sectors of dimension 0 are points equipped with the trivial action of a finite group, and it is easy to see that such sectors correspond to homomorphisms $\varphi_p : \Gamma \to G_p$ where p is a point singularity of the orbifold and the image of φ_p fixes a single point. The only closed, effective 1-dimensional orbifolds are circles or mirrored intervals, i.e., intervals with $\mathbb{Z}/2\mathbb{Z}$-isotropy at the endpoints, and so all closed 1-dimensional sectors are given by circles with the trivial action of a finite group or noneffective mirrored intervals.

The 2-dimensional sectors correspond to homomorphisms $\varphi_p : \Gamma \to G_p$ where $p \in \partial_{\mathrm{top}} Q$ and the image of φ_p fixes a plane. However, the 2-dimensional sectors of Q need not correspond to entire connected components of $\partial_{\mathrm{top}} Q$. In fact, the 0- and 1-dimensional singular strata contained in $\partial_{\mathrm{top}} Q$ divide $\partial_{\mathrm{top}} Q$ into regions, and the closures of these regions can be covered by distinct sectors. We illustrate this with the following, considering an open orbifold for simplicity.

Example 2.1. Let $n \geq 4$ be even and let Q denote the orbifold given by the quotient of $\mathbb{R}^3$ by the full dihedral group D_{2n}^*; see Figure 2(c). Then Q is homeomorphic to closed half-space, and the singular strata divide $\partial_{\mathrm{top}} Q$ into three dense regions with isotropy $\mathbb{Z}/2\mathbb{Z}$.

Let $\Gamma = \mathbb{Z}$, and then there are three 2-dimensional sectors. The first, corresponding to homomorphisms that map $1 \in \mathbb{Z}$ to the central element $A_n^{n/2} J \in D_{2n}^*$, is given by the quotient of a plane $\mathbb{R}^2$ by the action of $D_{2n} \times (\mathbb{Z}/2\mathbb{Z})$, where the D_{2n}-factor acts via the standard effective action of a dihedral group on $\mathbb{R}^2$, and the $\mathbb{Z}/2\mathbb{Z}$-factor acts trivially. The resulting orbifold is homeomorphic to a closed quadrant in $\mathbb{R}^2$, where the origin is a *corner reflector* with isotropy $D_{2n} \times (\mathbb{Z}/2\mathbb{Z})$, other points on the (topological) boundary have isotropy $(\mathbb{Z}/2\mathbb{Z})^2$, and points on the interior have isotropy $\mathbb{Z}/2\mathbb{Z}$. The map $(p, (\varphi_p)_{G_p}) \mapsto p$ is a bijection between this sector and the single closed region in $\partial_{\text{top}} Q$ bounded by the two rays with isotropy $D_4](\mathbb{Z}/2\mathbb{Z})$.

The other 2-dimensional sectors cover the respective closures of the other two regions in $\partial_{\text{top}} Q$. They corresponds to the two conjugacy classes of homomorphisms that map $1 \in \mathbb{Z}$ to $A_n^k B J \in D_{2n}^*$ where $k \neq n/2$. Each is given by the quotient of $\mathbb{R}^2$ by $\langle A_n^{n/2}, A_n^k B J, J \rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$, where $A_n^k B J$ acts trivially and the other two factors act via the standard action of $(\mathbb{Z}/2\mathbb{Z})^2 \cong D_4$ on $\mathbb{R}^2$. These sectors are also each homeomorphic to a closed quadrant in $\mathbb{R}^2$, where the origin is a corner reflector with isotropy $D_4 \times (\mathbb{Z}/2\mathbb{Z})$, other points on the (topological) boundary have isotropy $(\mathbb{Z}/n\mathbb{Z})^2$, and points on the interior have isotropy $\mathbb{Z}/2\mathbb{Z}$. The map $(p, (\varphi_p)_{G_p}) \mapsto p$ is a bijection from each of these sectors to the closures of the corresponding regions in $\partial_{\text{top}} Q$.

From this example, it is clear that a description of the 2-dimensional sectors of an arbitrary closed, effective 3-orbifold Q would require a detailed description of the topology of $\partial_{\text{top}} Q$ as well as the configuration of the singular strata it contains. As we will see in Section 3A, however, the sum of the Euler–Satake characteristics of the 2-dimensional sectors of Q depends only on $\chi_{\text{top}}(\partial_{\text{top}} Q)$ and the number and type of point singularities in Q, and hence can be computed using only this information.

3. Computation of $\chi_{\mathbb{F}_l}^{\text{ES}}(Q)$

In this section, we compute the $\mathbb{F}_l$-Euler–Satake characteristics of a closed, effective 3-orbifold Q where $\mathbb{F}_l$ is the free group with l generators. In Section 3A, we simplify this computation by demonstrating Proposition 3.6, which expresses the Γ-Euler–Satake characteristic in terms of quantities involving only the number and type of point singularities of Q as well as $\chi_{\text{top}}(\partial_{\text{top}} Q)$. In Section 3B, we compute these quantities for each of the finite subgroups of O(3) when $\Gamma = \mathbb{F}_l$. The formulas for the $\mathbb{F}_l$-Euler–Satake characteristics are given in Section 3C.

3A. ***General observations.*** Let Γ be a finitely generate discrete group and Q a closed, effective 3-orbifold. The Γ-Euler–Satake characteristic of Q is given by

$$\chi_{\Gamma}^{\text{ES}}(Q) = \chi_{\text{ES}}(\widetilde{Q}_{\Gamma}),$$

the usual Euler–Satake characteristic of the orbifold of Γ-sectors $\widetilde{Q}_\Gamma$ of Q. We let $\widetilde{Q}_{\Gamma,d}$ denote the collection of Γ-sectors of Q of dimension d, and then

$$\widetilde{Q}_\Gamma = \widetilde{Q}_{\Gamma,0} \sqcup \widetilde{Q}_{\Gamma,1} \sqcup \widetilde{Q}_{\Gamma,2} \sqcup \widetilde{Q}_{\Gamma,3},$$

where $\widetilde{Q}_{\Gamma,3} = Q$ consists only of the nontwisted sector. By [Satake 1957, Theorem 4], the Euler–Satake characteristic of an odd-dimensional closed orbifold vanishes; note that Satake assumes that orbifolds do not have singular strata of codimension 1, but his result can be applied to the orientable double-cover of an orbifold and hence extended to arbitrary orbifolds. Therefore, we have that

$$\chi_\Gamma^{\mathrm{ES}}(Q) = \chi_{\mathrm{ES}}(\widetilde{Q}_{\Gamma,0}) + \chi_{\mathrm{ES}}(\widetilde{Q}_{\Gamma,2}). \tag{3-1}$$

As was illustrated in Example 2.1 above, the structure of $\widetilde{Q}_{\Gamma,2}$ is complicated and depends heavily on the graph structure of the singular strata in $\partial_{\mathrm{top}} Q$. However, our first goal of this section is to indicate how $\chi_\Gamma^{\mathrm{ES}}(Q)$ can be computed without determining the structure or number of components of $\widetilde{Q}_{\Gamma,2}$. First, we have the following.

Lemma 3.1. *Let Q be a closed, effective, 3-dimensional orbifold with underlying space $\mathbb{X}_Q$ and let $\mathcal{P}_{\mathrm{proj}}$ denote the finite set of projective points of Q. Then*

$$\chi_{\mathrm{top}}(\mathbb{X}_Q) = \tfrac{1}{2}\chi_{\mathrm{top}}(\partial_{\mathrm{top}} Q) + \tfrac{1}{2}|\mathcal{P}_{\mathrm{proj}}|.$$

Proof. For each $p \in \mathcal{P}_{\mathrm{proj}}$, choose a neighborhood U_p of p homeomorphic to $G_p \backslash \mathbb{R}^3$ and small enough so that $U_p \cap U_q = \varnothing$ for $p \neq q$, each $U_p \cap \partial_{\mathrm{top}} Q = \varnothing$, and $\partial(\overline{U}_p)$ is homeomorphic to $G_p \backslash \mathbb{S}^2$, where by $\partial(\overline{U}_p)$, we mean the boundary of the manifold $\overline{U}_p \smallsetminus \{p\}$. Then the topological space $X = \mathbb{X}_Q \smallsetminus \bigcup_{p\in\mathcal{P}_{\mathrm{proj}}} U_p$ is a topological 3-manifold with boundary given by $\partial X = \partial_{\mathrm{top}} Q \cup \bigcup_{p\in\mathcal{P}_{\mathrm{proj}}} \partial(\overline{U}_p)$. Note that each $\partial(\overline{U}_p)$ is homeomorphic to $\mathbb{R}P^2$ and hence $\chi_{\mathrm{top}}(\partial(\overline{U}_p)) = 1$. Expressing $\mathbb{X}_Q$ as $X \cup \bigcup_{p\in\mathcal{P}_{\mathrm{proj}}} \overline{U}_p$ and noting that the intersection of each $\overline{U}_p$ with X is $\partial(\overline{U}_p)$, we have

$$\chi_{\mathrm{top}}(\mathbb{X}_Q) = \chi_{\mathrm{top}}(X) + \sum_{p\in\mathcal{P}_{\mathrm{proj}}} \chi_{\mathrm{top}}(\overline{U}_p) - \sum_{p\in\mathcal{P}_{\mathrm{proj}}} \chi_{\mathrm{top}}(\partial(\overline{U}_p)).$$

However, as $\overline{U}_p$ is homeomorphic to the cone on $\mathbb{R}P^2$ and hence is contractible, we have $\chi_{\mathrm{top}}(\overline{U}_p) = 1 = \chi_{\mathrm{top}}(\partial(\overline{U}_p))$. It follows that

$$\chi_{\mathrm{top}}(\mathbb{X}_Q) = \chi_{\mathrm{top}}(X).$$

As the Euler characteristic of a 3-manifold is half that of its boundary, we then have

$$\chi_{\mathrm{top}}(\mathbb{X}_Q) = \chi_{\mathrm{top}}(X) = \tfrac{1}{2}\chi_{\mathrm{top}}(\partial X) = \tfrac{1}{2}(\chi_{\mathrm{top}}(\partial_{\mathrm{top}} Q) + |\mathcal{P}_{\mathrm{proj}}|)). \qquad \square$$

Using the fact that $\chi_{\mathrm{top}}(\mathbb{X}_Q) = \chi_{\mathbb{Z}}^{\mathrm{ES}}(Q)$ (see [Farsi and Seaton 2011] or [Seaton 2008]), we have the following.

Corollary 3.2. *Let Q be a closed, effective, 3-dimensional orbifold with underlying space $\mathbb{X}_Q$ and let $\mathcal{P}_{\mathrm{proj}}$ denote the finite set of projective points of Q. Then*

$$\chi_{\mathbb{Z}}^{\mathrm{ES}}(Q) = \tfrac{1}{2}\chi_{\mathrm{top}}(\partial_{\mathrm{top}} Q) + \tfrac{1}{2}|\mathcal{P}_{\mathrm{proj}}|.$$

In particular, as $\chi_{\mathbb{Z}}^{\mathrm{ES}}(Q) = \chi_{\mathrm{ES}}(\widetilde{Q}_{\mathbb{Z},0}) + \chi_{\mathrm{ES}}(\widetilde{Q}_{\mathbb{Z},2})$, we have

$$\chi_{\mathrm{ES}}(\widetilde{Q}_{\mathbb{Z},2}) = \tfrac{1}{2}\chi_{\mathrm{top}}(\partial_{\mathrm{top}} Q) + \tfrac{1}{2}|\mathcal{P}_{\mathrm{proj}}| - \chi_{\mathrm{ES}}(\widetilde{Q}_{\mathbb{Z},0}). \tag{3-2}$$

Hence, using Corollary 3.2, we can compute the Euler–Satake characteristic of the 2-dimensional $\mathbb{Z}$-sectors in terms of the 0-dimensional $\mathbb{Z}$-sectors and $\chi_{\mathrm{top}}(\partial Q)$. We can use this to compute the Euler–Satake characteristic of the 2-dimensional Γ-sectors for arbitrary Γ using the following. For $p \in Q$, let $\mathrm{HOM}(\Gamma, G_p)_d$ denote the collection of homomorphisms $\Gamma \to G_p$ whose image fix a d-dimensional subspace in a chart at p.

Lemma 3.3. *Let Q be a closed, effective, 3-dimensional orbifold and let Γ be a finitely generated discrete group. Then the 2-dimensional Γ-sectors of Q consist of $|\mathrm{HOM}(\Gamma, \mathbb{Z}/2\mathbb{Z})| - 1$ identical copies of the 2-dimensional $\mathbb{Z}$-sectors of Q. In particular, the 2-dimensional $\mathbb{F}_l$-sectors of Q consist of $2^l - 1$ copies of the 2-dimensional $\mathbb{Z}$-sectors of Q.*

Proof. By inspection, the only elements of $\mathrm{O}(3)$ that fix planes are reflections that generate a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence each element in the union $\bigcup_{p\in Q} \mathrm{HOM}(\Gamma, G_p)_2$ has image isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Define the map

$$\Psi : \bigcup_{p\in Q} \mathrm{HOM}(\Gamma, G_p)_2 \longrightarrow \bigcup_{p\in Q} \mathrm{HOM}(\mathbb{Z}, G_p)_2$$

by sending $\varphi_p \in \mathrm{HOM}(\Gamma, G_p)_2$ to the homomorphism in $\mathrm{HOM}(\mathbb{Z}, G_p)_2$ that maps the generator of $\mathbb{Z}$ to the unique generator of the image $\mathrm{Im}(\varphi_p)$ of φ_p. Then as the image of a homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ uniquely characterizes the homomorphism, we have for each $\psi_p \in \mathrm{HOM}(\mathbb{Z}, G_p)_2$ that $\Psi^{-1}(\psi_p)$ consists of every element of $\mathrm{HOM}(\Gamma, \mathrm{Im}(\psi_p))$ except the trivial homomorphism. Therefore, Ψ is a $(|\mathrm{HOM}(\Gamma, \mathbb{Z}/2\mathbb{Z})| - 1)$-to-1 map. It is clear from its construction that Ψ is equivariant with respect to the G_p-actions by conjugation on $\mathrm{HOM}(\mathbb{Z}, G_p)_2$ and $\mathrm{HOM}(\Gamma, G_p)_2$, and moreover that the centralizer of each ψ_p in G_p coincides with the centralizer of $\Psi^{-1}(\psi_p)$ in G_p, so that Ψ induces a $(|\mathrm{HOM}(\Gamma, \mathbb{Z}/2\mathbb{Z})| - 1)$-to-1 map

$$\widetilde{\Psi} : \widetilde{Q}_{\Gamma,2} \to \widetilde{Q}_{\mathbb{Z},2}, \quad (p, (\varphi_p)_{G_p}) \mapsto (p, (\Psi(\varphi_p))_{G_p}).$$

To complete the proof, we demonstrate that the restriction of $\widetilde{\Psi}$ to each 2-dimensional Γ-sector of Q is a diffeomorphism onto a $\mathbb{Z}$-sector of Q. In groupoid language, this can be accomplished by extending Ψ to a map $\mathrm{HOM}(\Gamma, \mathcal{G})_2 \to \mathrm{HOM}(\mathbb{Z}, \mathcal{G})_2$, where $\mathrm{HOM}(\Gamma, \mathcal{G})_2$ denotes the groupoid homomorphisms corresponding to points in 2-dimensional sectors, and then computing directly that the resulting map is in fact equivariant with respect to the $\mathcal{G}$-actions, and hence a Lie groupoid isomorphism when restricted to each sector.

To demonstrate this in terms of an atlas, suppose $(p, (\varphi_p)_{G_p})$ and $(q, (\varphi_q)_{G_q})$ are points in $\mathbb{X}_{\widetilde{Q}_\Gamma}$ contained in orbifold charts for $\widetilde{Q}_\Gamma$ related by an injection. Specifically, suppose $(f, \lambda) : \{V_q, G_q, \pi_q\} \to \{V_p, G_p, \pi_p\}$ is an injection of orbifold charts with respect to which $\varphi_p : \Gamma \to G_p$ locally covers $\varphi_q : \Gamma \to G_q$ for $\varphi_p \in \mathrm{HOM}(\Gamma, G_p)_2$ and $\varphi_q \in \mathrm{HOM}(\Gamma, G_q)_2$. Then

$$\{f|_{V_q^{\langle \varphi_q \rangle}}, \lambda|_{C_{G_q}(\varphi_q)}\}$$

is an injection of the orbifold chart $\{V_q^{\langle \varphi_q \rangle}, C_{G_q}(\varphi_q), \pi_q^{\varphi_q}\}$ for $\widetilde{Q}_\Gamma$ at $(q, (\varphi_q)_{G_q})$ into the chart $\{V_p^{\langle \varphi_p \rangle}, C_{G_p}(\varphi_p), \pi_p^{\varphi_p}\}$ for $\widetilde{Q}_\Gamma$ at $(p, (\varphi_p)_{G_p})$. As Ψ preserves images of homomorphisms, it is easy to see that

$$V_p^{\langle \varphi_p \rangle} = V_p^{\langle \Psi(\varphi_p) \rangle} \quad \text{and} \quad C_{G_p}(\varphi_p) = C_{G_p}(\Psi(\varphi_p)).$$

Moreover, from $\lambda \circ \varphi_q = \varphi_p$, it is easy to see that $\lambda \circ \Psi(\varphi_q) = \Psi(\varphi_p)$. It follows that

$$\{f|_{V_q^{\langle \Psi(\varphi_q) \rangle}}, \lambda|_{C_{G_q}(\Psi(\varphi_q))}\}$$

is an injection of the chart $\{V_q^{\langle \Psi(\varphi_q) \rangle}, C_{G_q}(\Psi(\varphi_q)), \pi_q^{\Psi(\varphi_q)}\}$ for $\widetilde{Q}_\mathbb{Z}$ at $(q, (\Psi(\varphi_q))_{G_q})$ into the chart $\{V_p^{\langle \Psi(\varphi_p) \rangle}, C_{G_p}(\Psi(\varphi_p)), \pi_p^{\Psi(\varphi_p)}\}$ for $\widetilde{Q}_\mathbb{Z}$ at $(p, (\Psi(\varphi_p))_{G_p})$. With this, it follows that there is a bijection between orbifold charts and injections for each connected component of $\widetilde{Q}_{\Gamma,2}$ and its image under $\widetilde{\Psi}$, completing the proof. $\square$

We conclude that for a 3-dimensional closed, effective orbifold Q, the Γ-Euler–Satake characteristics can be determined from the number and type of point singularities of Q as well as the Euler characteristic $\chi_{\mathrm{top}}(\partial_{\mathrm{top}} Q)$. We recall the following, which was also observed in [Carroll and Seaton 2013, Proposition 2.1].

Lemma 3.4. *Let Q be a closed, effective 3-orbifold, let $\mathcal{P}$ denote the collection of point singularities of Q and let Γ be a finitely generated discrete group. Then*

$$\chi_{\mathrm{ES}}(\widetilde{Q}_{\Gamma,0}) = \sum_{p \in \mathcal{P}} \frac{|\mathrm{HOM}(\Gamma, G_p)_0|}{|G_p|}.$$

Proof. First, we have by definition of the Euler–Satake characteristic that

$$\chi_{\mathrm{ES}}(\widetilde{Q}_{\Gamma,0}) = \sum_{p\in\mathcal{P}} \sum_{(\varphi_p)_{G_p}\in \mathrm{HOM}(\Gamma,G_p)_0/G_p} \frac{1}{|C_{G_p}(\varphi_p)|}.$$

Using the fact that $|C_{G_p}(\varphi_p)||(\varphi_p)_{G_p}| = |G_p|$, this is equal to

$$\sum_{p\in\mathcal{P}} \sum_{(\varphi_p)_{G_p}\in \mathrm{HOM}(\Gamma,G_p)_0/G_p} \frac{|(\varphi_p)_{G_p}|}{|G_p|} = \sum_{p\in\mathcal{P}} \frac{|\mathrm{HOM}(\Gamma, G_p)_0|}{|G_p|}. \qquad \square$$

Lemma 3.5. *Let Q be a closed, effective 3-orbifold and let $\mathcal{P}_\partial$ denote the set of point singularities contained in $\partial_{\mathrm{top}} Q$. Then*

$$\chi_{\mathrm{ES}}(\widetilde{Q}_{\mathbb{Z},0}) = \frac{1}{2}|\mathcal{P}_{\mathrm{proj}}| + \sum_{p\in\mathcal{P}_\partial} \frac{|\mathrm{HOM}(\mathbb{Z}, G_p)_0|}{|G_p|}.$$

Proof. For each $p \in \mathcal{P}$ such that $G_p \le \mathrm{SO}(3)$, each element of G_p is a rotation and hence fixes a line. As the image of any element of $\mathrm{HOM}(\mathbb{Z}, G_p)$ must be cyclic, it follows that $\mathrm{HOM}(\mathbb{Z}, G_p)_0 = \varnothing$ for each such p. Recalling that all other point singularities are elements of $\mathcal{P}_{\mathrm{proj}} \cup \mathcal{P}_\partial$ and applying Lemma 3.4, we have

$$\chi_{\mathrm{ES}}(\widetilde{Q}_{\mathbb{Z},0}) = \sum_{p\in\mathcal{P}_{\mathrm{proj}}} \frac{|\mathrm{HOM}(\mathbb{Z}, G_p)_0|}{|G_p|} + \sum_{p\in\mathcal{P}_\partial} \frac{|\mathrm{HOM}(\mathbb{Z}, G_p)_0|}{|G_p|}.$$

If $p \in \mathcal{P}_{\mathrm{proj}}$, then G_p is given either by $(\mathbb{Z}/n\mathbb{Z})^*$ for n odd or $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ for n even. In the former case, it is easy to see that all elements of $(\mathbb{Z}/n\mathbb{Z})^*$ of the form $A_n^k J$ fix a point, while nontrivial elements of the form A_n^k fix a line, so that

$$\frac{|\mathrm{HOM}(\mathbb{Z}, (\mathbb{Z}/n\mathbb{Z})^*)_0|}{|(\mathbb{Z}/n\mathbb{Z})^*|} = \frac{n}{2n} = \frac{1}{2}.$$

In the latter case, $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ is generated by $A_{2n}J$, and odd powers of $A_n J$ fix a point while nontrivial even powers fix a line. We have again

$$\frac{|\mathrm{HOM}(\mathbb{Z}, (\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z}))_0|}{|(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})|} = \frac{n}{2n} = \frac{1}{2}.$$

The claim follows. $\square$

With this, combining Equation (3-2) and Lemma 3.5 yields

$$\chi_{\mathrm{ES}}(\widetilde{Q}_{\mathbb{Z},2}) = \frac{1}{2}\chi_{\mathrm{top}}(\partial_{\mathrm{top}} Q) - \sum_{p\in\mathcal{P}_\partial} \frac{|\mathrm{HOM}(\mathbb{Z}, G_p)_0|}{|G_p|}.$$

Along with Equation (3-1) and Lemmas 3.3 and 3.4, this establishes the following.

Proposition 3.6. *Let Q be a closed, effective 3-orbifold and let Γ be a finitely generated discrete group. Then*

$$\chi_\Gamma^{\mathrm{ES}}(Q) = (|\mathrm{HOM}(\Gamma, \mathbb{Z}/2\mathbb{Z})| - 1)\left(\frac{1}{2}\chi_{\mathrm{top}}(\partial_{\mathrm{top}} Q) - \sum_{p \in \mathcal{P}_\partial} \frac{|\mathrm{HOM}(\mathbb{Z}, G_p)_0|}{|G_p|}\right) + \sum_{p \in \mathcal{P}} \frac{|\mathrm{HOM}(\Gamma, G_p)_0|}{|G_p|}. \quad \text{(3-3)}$$

In particular, $\chi_\Gamma^{\mathrm{ES}}(Q)$ depends only on $\chi_{\mathrm{top}}(\partial_{\mathrm{top}} Q)$ and the number and type of point singularities of Q. For $\Gamma = \mathbb{F}_l$, we have

$$\chi_{\mathbb{F}_l}^{\mathrm{ES}}(Q) = \frac{2^l - 1}{2}\chi_{\mathrm{top}}(\partial_{\mathrm{top}} Q) + \sum_{p \in \mathcal{P} \setminus \mathcal{P}_\partial} \frac{|\mathrm{HOM}(\mathbb{F}_l, G_p)_0|}{|G_p|} + \sum_{p \in \mathcal{P}_\partial} \frac{|\mathrm{HOM}(\mathbb{F}_l, G_p)_0| - (2^l - 1)|\mathrm{HOM}(\mathbb{Z}, G_p)_0|}{|G_p|}.$$

3B. ***Counting point-fixing homomorphisms.*** In view of Proposition 3.6, to complete the computation of the $\mathbb{F}_l$-Euler–Satake characteristic of an arbitrary closed, effective 3-orbifold Q, we need only determine the value of $|\mathrm{HOM}(\mathbb{F}_l, G_p)_0|/|G_p|$ for each G_p corresponding to $p \in \mathcal{P} \setminus \mathcal{P}_\partial$ and that of

$$\big(|\mathrm{HOM}(\mathbb{F}_l, G_p)_0| - (2^l - 1)|\mathrm{HOM}(\mathbb{Z}, G_p)_0|\big)/|G_p|$$

for $p \in \mathcal{P}_\partial$. To organize the computations of these quantities, we make the following observations.

Given an arbitrary finitely generated discrete group Γ, for each finite subgroup $G < \mathrm{O}(3)$ corresponding to a point singularity, we have

$$\mathrm{HOM}(\Gamma, G) = \mathrm{HOM}(\Gamma, G)_0 + \mathrm{HOM}(\Gamma, G)_1 + \mathrm{HOM}(\Gamma, G)_2 + \mathrm{HOM}(\Gamma, G)_3.$$

Clearly, $|\mathrm{HOM}(\Gamma, G)_3| = 1$, as only the trivial homomorphism fixes all of $\mathbb{R}^3$. Similarly, as the only plane-fixing elements of $\mathrm{O}(3)$ generate a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and as a plane in $\mathbb{R}^3$ is fixed by exactly one nontrivial element of $\mathrm{O}(3)$, $\mathrm{HOM}(\Gamma, G)_2$ contains $c(|\mathrm{HOM}(\Gamma, \mathbb{Z}/2\mathbb{Z})| - 1)$ homomorphisms where c is the number of planes in $\mathbb{R}^3$ fixed by an element of G. Finally, by inspection, any 1-dimensional singular stratum of $G\backslash\mathbb{R}^3$ has isotropy group $D_{2n}](\mathbb{Z}/n\mathbb{Z})$ or $\mathbb{Z}/n\mathbb{Z}$. Hence, as each such subgroup fixes a unique line, we have

$$|\mathrm{HOM}(\Gamma, G)_1| = \sum_{n=2}^{\infty} a_n |\mathrm{HOM}(\Gamma, D_{2n}](\mathbb{Z}/n\mathbb{Z}))_1| + b_n |\mathrm{HOM}(\Gamma, \mathbb{Z}/n\mathbb{Z})_1|,$$

where a_n denotes the number of distinct lines in $\mathbb{R}^3$ with isotropy group $D_{2n}](\mathbb{Z}/n\mathbb{Z})$ and b_n denotes the number of distinct lines in $\mathbb{R}^3$ with isotropy group $\mathbb{Z}/n\mathbb{Z}$.

With this, we note that if $\Gamma = \mathbb{F}_l$, then by considering the images of a chosen set of generators for $\mathbb{F}_l$, it is easy to compute that

$$|\mathrm{HOM}(\mathbb{F}_l, \mathbb{Z}/n\mathbb{Z})_1| = n^l - 1.$$

Recall that $D_{2n}](\mathbb{Z}/n\mathbb{Z})$ is generated by A_n and BJ, where A_n acts as a rotation about the z-axis and BJ as a reflection through the yz-plane. Then there are n plane-fixing elements of the form $A_n^k BJ$ for $0 \le k \le n-1$, and hence $n(2^l - 1)$ elements of $\mathrm{HOM}(\mathbb{F}_l, D_{2n}](\mathbb{Z}/n\mathbb{Z}))_2$. Then as each element of $D_{2n}](\mathbb{Z}/n\mathbb{Z})$ fixes the z-axis, there are no point-fixing elements, so that

$$|\mathrm{HOM}(\mathbb{F}_l, D_{2n}](\mathbb{Z}/n\mathbb{Z}))_1| = (2n)^l - n(2^l - 1) - 1.$$

We summarize these observations with the following.

Lemma 3.7. *Let G be a finite subgroup of* $\mathrm{O}(3)$. *For each $n \ge 2$, let a_n denote the number of lines in $\mathbb{R}^3$ with isotropy group $D_{2n}](\mathbb{Z}/n\mathbb{Z})$, let b_n denote the number of lines in $\mathbb{R}^3$ with isotropy group $\mathbb{Z}/n\mathbb{Z}$, and let c denote the number of planes in $\mathbb{R}^3$ fixed by a nontrivial element of G. Then for each $l \ge 1$,*

$$|\mathrm{HOM}(\mathbb{F}_l, G)_0| = |G|^l - c(2^l - 1) - 1 - \sum_{n=2}^{\infty} a_n\big((2n)^l - 2^l n + n - 1\big) + b_n(n^l - 1).$$

We will now apply this result to each of the point-fixing subgroups of $\mathrm{O}(3)$. To simplify the notation, for a finite $G < \mathrm{O}(3)$ such that $G\backslash\mathbb{R}^3$ has nonempty topological boundary (i.e., G is the isotropy group of a point singularity $p \in \mathcal{P}_\partial$), we let

$$\mathbf{S}_\partial(G) := \frac{|\mathrm{HOM}(\mathbb{F}_l, G)_0| - (2^l - 1)|\mathrm{HOM}(\mathbb{Z}, G)_0|}{|G|}$$

denote the corresponding term of $\chi_{\mathbb{F}_l}^{\mathrm{ES}}(Q)$ in Proposition 3.6.

$G = (\mathbb{Z}/n\mathbb{Z})^*$. Recall that $(\mathbb{Z}/n\mathbb{Z})^*$ has order $2n$, and first assume n is even. Then $(\mathbb{Z}/n\mathbb{Z})^*$ contains one plane-fixing element $A_n^{n/2}J$ so that $c = 1$, and a point with isotropy group $(\mathbb{Z}/n\mathbb{Z})^*$ is contained in $\mathcal{P}_\partial$. The z-axis is the only line in $\mathbb{R}^3$ with nontrivial isotropy $\mathbb{Z}/n\mathbb{Z}$, and so $a_k = 0$ for each k, $b_n = 1$, and $b_k = 0$ for $k \ne n$. Applying Lemma 3.7,

$$|\mathrm{HOM}(\mathbb{F}_l, (\mathbb{Z}/n\mathbb{Z})^*)_0| = (n^l - 1)(2^l - 1) \quad (n \text{ even}),$$

and in particular $|\mathrm{HOM}(\mathbb{Z}, (\mathbb{Z}/n\mathbb{Z})^*)_0| = n - 1$. Then

$$\mathbf{S}_\partial\big(((\mathbb{Z}/n\mathbb{Z})^*)_0\big) = \frac{2^l - 1}{2}(n^{l-1} - 1), \qquad (n \text{ even}). \tag{3-4}$$

For n odd, $(\mathbb{Z}/n\mathbb{Z})^*$ contains no plane-fixing elements so that $c = 0$, and a point with isotropy group $(\mathbb{Z}/n\mathbb{Z})^*$ is an element of $\mathcal{P} \smallsetminus \mathcal{P}_\partial$. The z-axis is again the only line in $\mathbb{R}^3$ with nontrivial isotropy $\mathbb{Z}/n\mathbb{Z}$ so the a_k and b_k vanish except for $b_n = 1$. Again applying Lemma 3.7,

$$|\mathrm{HOM}(\mathbb{F}_l, (\mathbb{Z}/n\mathbb{Z})^*)_0| = n^l(2^l - 1) \quad (n \text{ odd}),$$

and so

$$\frac{|\mathrm{HOM}(\mathbb{F}_l, (\mathbb{Z}/n\mathbb{Z})^*)_0|}{|(\mathbb{Z}/n\mathbb{Z})^*|} = \frac{2^l - 1}{2}\, n^{l-1} \quad (n \text{ odd}). \tag{3-5}$$

$G = (\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$. We again have $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ has order $2n$. Assume n is even. Then $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ contains no plane-fixing elements, so $c = 0$, and the corresponding point singularity is in $\mathcal{P} \smallsetminus \mathcal{P}_\partial$. Other than the origin, only the z-axis has nontrivial isotropy $\mathbb{Z}/n\mathbb{Z}$, so the computation is identical to the case of $(\mathbb{Z}/n\mathbb{Z})^*$ for n odd given in Equation (3-5) above. That is,

$$\frac{|\mathrm{HOM}(\mathbb{F}_l, (\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z}))_0|}{|(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})|} = \frac{2^l - 1}{2}\, n^{l-1} \quad (n \text{ even}). \tag{3-6}$$

If n is odd, then $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ contains one plane-fixing element $(A_{2n}J)^n$, and one line in $\mathbb{R}^3$ is fixed by $\mathbb{Z}/n\mathbb{Z}$. The point singularity is contained in $\mathcal{P}_\partial$, and the computation is identical to the case of $(\mathbb{Z}/n\mathbb{Z})^*$ for n even in Equation (3-4). Hence,

$$\mathbf{S}_\partial((\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})) = \frac{2^l - 1}{2}(n^{l-1} - 1) \quad (n \text{ odd}). \tag{3-7}$$

$G = D_{2n}^*$. Recall that D_{2n}^* has order $4n$. Suppose n is even. There are $n+1$ plane-fixing elements given by $A_n^{n/2}J$ and $A_n^k BJ$ for $k = 0, \dots, n-1$ so that $c = n+1$ and the point singularity is an element of $\mathcal{P}_\partial$. The z-axis has isotropy $D_{2n}](\mathbb{Z}/n\mathbb{Z})$, and the n lines spanned by $(\cos(k\pi/n), \sin(k\pi/n), 0)$ for $0 \le k \le n-1$ have isotropy $D_4](\mathbb{Z}/2\mathbb{Z})$. Therefore, $a_2 = n$, $a_n = 1$, and the other a_k and b_k vanish, so that by Lemma 3.7,

$$|\mathrm{HOM}(\mathbb{F}_l, D_{2n}^*)_0| = (2^l - 1)\big((2n)^l - 2^l n + n - 1\big), \quad (n \text{ even}).$$

Therefore,

$$\mathbf{S}_\partial(D_{2n}^*) = 2^{l-2}(2^l - 1)(n^{l-1} - 1) \quad (n \text{ even}). \tag{3-8}$$

If n is odd, there are n plane-fixing elements $A_n^k BJ$ for $k = 0, \dots, n-1$ so $c = n$. A $D_{2n}](\mathbb{Z}/n\mathbb{Z})$ subgroup fixes the z-axis, and n lines in the xy-plane have isotropy $\langle A^k B\rangle \cong \mathbb{Z}/2\mathbb{Z}$, so that $a_n = 1$, $b_2 = n$, and all others vanish. This yields

$$|\mathrm{HOM}(\mathbb{F}_l, D_{2n}^*)_0| = (2^l - 1)\big((2n)^l - n\big), \quad (n \text{ odd}),$$

and so

$$\mathbf{S}_\partial(D_{2n}^*) = \frac{2^l - 1}{2}\big((2n)^{l-1} - 1\big), \quad (n \text{ odd}). \tag{3-9}$$

$G = \mathbb{T}^*$. In this case, the plane fixing elements are the three conjugates of CJ. We have $c = 3$, $a_2 = 3$, $b_3 = 4$, and all a_k and b_k vanish. Then Lemma 3.7 yields

$$|\mathrm{HOM}(\mathbb{F}_l, \mathbb{T}^*)_0| = (24)^l - 3 \cdot 4^l + 3 \cdot 2^l + 3,$$

and

$$\mathbf{S}_\partial(\mathbb{T}^*) = \tfrac{1}{2}(2 \cdot 24^{l-1} - 4^{l-1} - 3^{l-1} - 2^{l-1} + 1). \tag{3-10}$$

$G = \mathbb{O}^*$. Here, the plane fixing elements are the three conjugates of $S^2 J$ and the six conjugates of RJ. We have $c = 9$, $a_2 = 6$, $a_3 = 4$, $a_4 = 3$, and all others vanish. Applying Lemma 3.7,

$$|\mathrm{HOM}(\mathbb{F}_l, \mathbb{O}^*)_0| = 48^l - 3 \cdot 8^l - 4 \cdot 6^l - 6 \cdot 4^l + 27 \cdot 2^l - 15,$$

and

$$\mathbf{S}_\partial(\mathbb{O}^*) = 2^{l-2}(2 \cdot 24^{l-1} - 4^{l-1} - 3^{l-1} - 2^{l-1} + 1). \tag{3-11}$$

$G = \mathbb{I}^*$. The plane fixing elements are the fifteen conjugates of BJ. We have $c = 15$, $a_2 = 15$, $a_3 = 10$, $a_5 = 6$, and the others vanish, so by Lemma 3.7,

$$|\mathrm{HOM}(\mathbb{F}_l, \mathbb{I}^*)_0| = 120^l - 6 \cdot 10^l - 10 \cdot 6^l - 15 \cdot 4^l + 75 \cdot 2^l - 45,$$

and

$$\mathbf{S}_\partial(\mathbb{I}^*) = 2^{l-2}(2 \cdot 60^{l-1} - 5^{l-1} - 3^{l-1} - 2^{l-1} + 1). \tag{3-12}$$

$G = D_{4n}]D_{2n}$. Assume n is even, and then the plane fixing elements are $(A_{2n}J)^k B$ for k odd. Then $c = n$, $a_n = 1$, $b_2 = n$, and the other a_k and b_k vanish. Hence

$$|\mathrm{HOM}(\mathbb{F}_l, D_{4n}]D_{2n})_0| = (2^l - 1)\big((2n)^l - n\big) \quad (n \text{ even}),$$

and

$$\mathbf{S}_\partial(D_{4n}]D_{2n}) = \frac{2^l - 1}{2}\big((2n)^{l-1} - 1\big) \quad (n \text{ even}). \tag{3-13}$$

If n is odd, then the plane fixing elements are $(A_{2n}J)^n$ and $(A_{2n}J)^k B$ for k odd. Hence $c = n + 1$, $a_2 = n$, $a_n = 1$, and the other a_k and b_k vanish so that

$$|\mathrm{HOM}(\mathbb{F}_l, D_{4n}]D_{2n})_0| = (2^l - 1)\big((2n)^l - 2^l n + n - 1\big), \quad (n \text{ odd}),$$

and

$$\mathbf{S}_\partial(D_{4n}]D_{2n}) = 2^{l-2}(2^l - 1)(n^{l-1} - 1), \quad (n \text{ odd}). \tag{3-14}$$

$G = \mathbb{O}]\mathbb{T}$. In this case, the six plane fixing elements are the conjugates of RJ. We have $c = 6$, $a_2 = 3$, $a_3 = 4$, and the other a_k and b_k vanish. Therefore

$$|\mathrm{HOM}(\mathbb{F}_l, \mathbb{O}]\mathbb{T})_0| = 24^l - 4 \cdot 6^l - 3 \cdot 4^l + 3 \cdot 2^{l+2} - 6,$$

and

$$\mathbf{S}_\partial(\mathbb{O}]\mathbb{T}) = 2^{l-2}(2 \cdot 12^{l-1} - 2 \cdot 3^{l-1} - 2^{l-1} + 1). \tag{3-15}$$

3C. *The $\mathbb{F}_l$-Euler–Satake characteristics of a closed, effective 3-orbifold.* Combining Proposition 3.6 with Equations (3-4) through (3-15) as well as [Carroll and Seaton 2013, Theorem 3.1], which computes the terms associated to the finite subgroups of SO(3), we have the following.

Theorem 3.8. *Let Q be a closed, effective 3-orbifold with:*

$\mathfrak{t}$ *point singularities with isotropy* $\mathbb{T}$;
$\mathfrak{o}$ *point singularities with isotropy* $\mathbb{O}$;
$\mathfrak{i}$ *point singularities with isotropy* $\mathbb{I}$;
$\mathfrak{d}$ *point singularities with isotropy* D_{2n} *for each* n;
$\mathfrak{c}_n^{e*}$ *point singularities with isotropy* $(\mathbb{Z}/n\mathbb{Z})^*$ *for each even* n;
$\mathfrak{c}_n^{o*}$ *point singularities with isotropy* $(\mathbb{Z}/n\mathbb{Z})^*$ *for each odd* n;
$\mathfrak{t}^*$ *point singularities with isotropy* $\mathbb{T}^*$;
$\mathfrak{o}^*$ *point singularities with isotropy* $\mathbb{O}^*$;
$\mathfrak{i}^*$ *point singularities with isotropy* $\mathbb{I}^*$;
$\mathfrak{d}_n^{e*}$ *point singularities with isotropy* D_{2n}^* *for each even* n;
$\mathfrak{d}_n^{o*}$ *point singularities with isotropy* D_{2n}^* *for each odd* n;
$\mathfrak{c}_n^{em}$ *point singularities with isotropy* $(\mathbb{Z}/2\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ *for each even* n;
$\mathfrak{c}_n^{om}$ *point singularities with isotropy* $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ *for each odd* n;
$\mathfrak{d}_n^{em}$ *point singularities with isotropy* $D_{4n}]D_{2n}$ *for each even* n;
$\mathfrak{d}_n^{om}$ *point singularities with isotropy* $D_{4n}]D_{2n}$ *for each odd* n;
$\mathfrak{o}^m$ *point singularities with isotropy* $\mathbb{O}]\mathbb{T}$.

Then

$$\begin{aligned}
\chi_{\mathbb{F}_l}^{\mathrm{ES}}(Q) = \frac{2^l-1}{2}\chi_{\mathrm{top}}(\partial_{\mathrm{top}}Q) &+ \frac{\mathfrak{t}}{2}(2\cdot 12^{l-1} - 2\cdot 3^{l-1} - 2^{l-1} + 1) \\
&+ \frac{\mathfrak{o}}{2}(2\cdot 24^{l-1} - 4^{l-1} - 3^{l-1} - 2^{l-1} + 1) \\
&+ \frac{\mathfrak{i}}{2}(2\cdot 60^{l-1} - 5^{l-1} - 3^{l-1} - 2^{l-1} + 1) \\
&+ \frac{\mathfrak{t}^*}{2}(2\cdot 24^{l-1} - 4^{l-1} - 3^{l-1} - 2^{l-1} + 1) \\
&+ 2^{l-2}\mathfrak{o}^*(2\cdot 24^{l-1} - 4^{l-1} - 3^{l-1} - 2^{l-1} + 1) \\
&+ 2^{l-2}\mathfrak{i}^*(2\cdot 60^{l-1} - 5^{l-1} - 3^{l-1} - 2^{l-1} + 1) \\
&+ \mathfrak{o}^m[2^{l-2}(2\cdot 12^{l-1} - 2\cdot 3^{l-1} - 2^{l-1} + 1)] \\
&+ \Big(\frac{2^l-1}{2}\Big)\sum_{n=1}^{\infty}\Big(\mathfrak{d}_n\,(n^{l-1}-1) + \mathfrak{c}_n^{e*}\,(n^{l-1}-1) + \mathfrak{c}_n^{o*}\,n^{l-1} \\
&\qquad + \mathfrak{d}_n^{e*}\,2^{l-1}(n^{l-1}-1) + \mathfrak{d}_n^{o*}\,\big((2n)^{l-1}-1\big) + \mathfrak{c}_n^{em}\,n^{l-1} \\
&\qquad + \mathfrak{c}_n^{om}\,(n^{l-1}-1) + \mathfrak{d}_n^{em}\,\big((2n)^{l-1}-1\big) + \mathfrak{c}_n^{om}\,2^{l-1}(n^{l-1}-1)\Big).
\end{aligned}$$

We note that a closed orbifold Q has a finite number of point singularities, and hence there is a finite number of nonzero terms in the sum over n.

4. Indistinguishable orbifolds

Based on Theorem 3.8, it is easy to see that, in contrast with the orientable case, no collection of the $\chi_{\mathbb{F}_l}^{\mathrm{ES}}(Q)$ determine the point singularities of the closed, effective 3-orbifold Q. In fact, in this section, we describe a pair of closed, effective 3-orbifolds Q_1 and Q_2 such that $\chi_{\Gamma}^{\mathrm{ES}}(Q_1) = \chi_{\Gamma}^{\mathrm{ES}}(Q_2)$ for every finitely generated discrete group Γ.

Let n be an odd integer and let $\overline{\boldsymbol{B}}^3$ denote the closed unit ball in $\mathbb{R}^3$. Let Q_1 be the orbifold formed by gluing together two copies of $(\mathbb{Z}/n\mathbb{Z})^*\backslash\overline{\boldsymbol{B}}^3$ along $(\mathbb{Z}/n\mathbb{Z})^*\backslash\mathbb{S}^2$ so that Q_1 has two point singularities with isotropy $(\mathbb{Z}/n\mathbb{Z})^*$ connected by a segment with isotropy $\mathbb{Z}/n\mathbb{Z}$. See Figure 4, left.

Let Q_2 be the orbifold formed by gluing together two copies of

$$(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})\backslash\overline{\boldsymbol{B}}^3$$

along $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})\backslash\mathbb{S}^2$ so that $\partial_{\mathrm{top}} Q_2$ is homomorphic to $\mathbb{S}^2$ and contains two point singularities with isotropy $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ connected by a segment with isotropy $\mathbb{Z}/n\mathbb{Z}$ contained in the complement of $\partial_{\mathrm{top}} Q_2$. See Figure 4, right.

Let Γ be an arbitrary finitely generated discrete group. Note that $\partial_{\mathrm{top}} Q_1$ is empty so that Proposition 3.6 yields

$$\chi_{\Gamma}^{\mathrm{ES}}(Q_1) = 2\frac{|\mathrm{HOM}(\Gamma, (\mathbb{Z}/n\mathbb{Z})^*)_0|}{|(\mathbb{Z}/n\mathbb{Z})^*|} = \frac{|\mathrm{HOM}(\Gamma, (\mathbb{Z}/n\mathbb{Z})^*)_0|}{n}.$$

A nontrivial homomorphism $\varphi : \Gamma \to (\mathbb{Z}/n\mathbb{Z})^*$ corresponds to a 1-dimensional

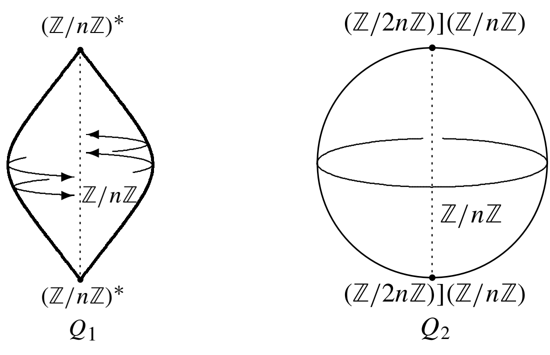

Figure 4. The orbifolds Q_1 and Q_2. Note that the boundary of the region describing Q_1 is identified antipodally in horizontal planes as indicated by the curved arrows, while Q_2 has (topological) boundary homeomorphic to $\mathbb{S}^2$.

sector if its image is contained in $(\mathbb{Z}/n\mathbb{Z})^* \cap \mathrm{SO}(3) = \mathbb{Z}/n\mathbb{Z}$. So using the fact that $(\mathbb{Z}/n\mathbb{Z})^*$ is isomorphic to $\mathbb{Z}/2n\mathbb{Z}$, we have

$$\chi_\Gamma^{\mathrm{ES}}(Q_1) = \frac{|\mathrm{HOM}(\Gamma, \mathbb{Z}/2n\mathbb{Z})| - |\mathrm{HOM}(\Gamma, \mathbb{Z}/n\mathbb{Z})|}{n}.$$

In the case of Q_2, we have $\partial_{\mathrm{top}} Q_2 = \mathbb{S}^2$ so that $\chi_{\mathrm{top}}(\partial_{\mathrm{top}} Q_2) = 2$. Therefore, Proposition 3.6 yields

$$\chi_\Gamma^{\mathrm{ES}}(Q_2) = (|\mathrm{HOM}(\Gamma, \mathbb{Z}/2\mathbb{Z})| - 1)\left(1 - 2\frac{|\mathrm{HOM}(\mathbb{Z}, (\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z}))_0|}{|(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})|}\right) + 2\frac{|\mathrm{HOM}(\Gamma, (\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z}))_0|}{|(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})|}.$$

Then $\mathrm{HOM}(\mathbb{Z}, (\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z}))$ contains $2n$ elements, of which the $n-1$ nontrivial elements of $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z}) \cap \mathrm{SO}(3) = \mathbb{Z}/n\mathbb{Z}$ correspond to 1-dimensional sectors, so that $|\mathrm{HOM}(\mathbb{Z}, (\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z}))_0| = n$. It follows that

$$1 - 2\frac{|\mathrm{HOM}(\mathbb{Z}, (\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z}))_0|}{|(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})|} = 0.$$

Therefore, as $(\mathbb{Z}/2n\mathbb{Z})](\mathbb{Z}/n\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2n\mathbb{Z}$,

$$\chi_\Gamma^{\mathrm{ES}}(Q_2) = \frac{|\mathrm{HOM}(\Gamma, \mathbb{Z}/2n\mathbb{Z})| - |\mathrm{HOM}(\Gamma, \mathbb{Z}/n\mathbb{Z})|}{n}.$$

Hence $\chi_\Gamma^{\mathrm{ES}}(Q_1) = \chi_\Gamma^{\mathrm{ES}}(Q_2)$ for every finitely generated discrete group Γ.

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