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A k-ranking of a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that if $f(u) = f(v)$, then every u-v path contains a vertex w such that $f(w) > f(u)$. The rank number of G, denoted $\chi_r(G)$, is the minimum k such that a k-ranking exists for G. It is shown that given a graph G and a positive integer t, the question of whether $\chi_r(G) \le t$ is NP-complete. However, the rank number of numerous families of graphs have been established. We study and establish rank numbers of some more families of graphs that are combinations of paths and cycles.

1. Introduction

Let G be an undirected graph with no loops and no multiple edges. A function $f : V(G) \to \{1, 2, \ldots, k\}$ is a (vertex) *k-ranking* of G if for $u, v \in V(G)$, $f(u) = f(v)$ implies that every u-v path contains a vertex w such that $f(w) > f(u)$. By definition, every ranking is a proper coloring. The *rank number* of G, denoted $\chi_r(G)$, is the minimum value of k such that G has a k-ranking. If the value of k is not important then f will be referred to simply as a ranking of G.

Interest in rankings of graphs was sparked by its many applications to other fields, including designs of very large scale integration (VLSI) layouts, Cholesky factorizations of matrices in parallel, and scheduling problems of assembly steps in manufacturing systems [Duff and Reid 1983; Iyer et al. 1991; Leiserson 1980; Liu 1990; Sen et al. 1992]. The optimal tree node ranking problem is identical to the problem of generating a minimum-height node separator tree for a tree. Node separator trees are extensively used in VLSI layout [Leiserson 1980]. Ranking of graphs is used in communication networks in which information flow between the nodes has to be monitored. An application of graph ranking to scheduling of assembly steps in manufacturing system is discussed in [Iyer et al. 1991].

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Bodlaender et al. [1995] show that for a graph G and a positive integer t, the question of whether $\chi_r(G) \leq t$ is NP-complete. However, the rank number of numerous families of graphs have been established [Alpert 2010; Bruoth and Horňák 1999; Dereniowski and Nadolski 2006; Hsieh 2002; Novotny et al. 2009; Ortiz et al. 2010; Sergel et al. 2011]. Bodlaender et al. [1995] established that $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$, where P_n is a path on n vertices. They showed that a k-ranking for $P_n = v_1 v_2 \cdots v_n$, where $k = \chi_r(P_n)$, can be obtained by labeling v_i with $\gamma + 1$, where 2^γ is the largest power of 2 that divides i. Throughout this paper, this particular scheme of ranking of a path will be referred as a standard ranking.

In this paper, we study and establish rank numbers of some more families of graphs, called flower graphs, lollipop graphs, star-flower graphs, and spider-flower graphs, which are defined in the following sections. These graphs can be considered as combinations of paths and cycles. We restate some known results that are used throughout this paper.

Lemma 1 [Ghoshal et al. 1996]. *Let H be a subgraph of G. Then $\chi_r(H) \leq \chi_r(G)$.*

Lemma 2 [Sergel et al. 2011]. *Let H_1 and H_2 be two vertex-disjoint graphs such that $\chi_r(H_1) = \chi_r(H_2) = k$. Let G be a connected supergraph of $H_1 \cup H_2$. Then $\chi_r(G) \geq k + 1$.*

Theorem 3 [Bodlaender et al. 1995]. $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$, *where P_n is a path on n vertices.*

If z is an integer, any ranking of P_{2^z} must have a label $r > z$. Hence:

Lemma 4. *Let $\chi_r(P_n) = j$, and let f be a χ_r-ranking of P_n such that $f(v_n) = k < j$, where v_n is an end vertex of P_n. Then $n \leq \sum_{i=k}^{j} 2^{i-1}$.*

The rank number of a cycle on n vertices, where $n \geq 3$, is as follows:

Theorem 5 [Bruoth and Horňák 1999]. $\chi_r(C_n) = \lceil \log_2 n \rceil + 1$, *where C_n is a cycle on $n > 2$ vertices.*

2. Flower graphs

A *flower graph* is a graph consisting of c cycles that share a common vertex. Figure 1 gives an example.

Theorem 6. *Let G be a flower graph where C_n is the largest cycle in G. Then $\chi_r(G) = \chi_r(C_n)$.*

Proof. Let G be a flower graph with largest cycle C_n and let $\chi_r(C_n) = k$. Since C_n is a subgraph of G, we have $\chi_r(G) \geq k$ by Lemma 1.

Now, consider a labeling f such that each cycle is given its χ_r-ranking so that the vertex with the largest label would be the center vertex x. Then, let $f(x) = k$. This is a valid k-ranking of G, since for any two vertices u, v on the same cycle in G, if

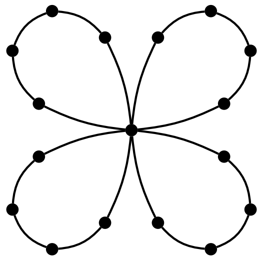

Figure 1. A flower graph where all of the cycles are the same size.

$f(u) = f(v)$, then any u-v path will contain some vertex w such that $f(w) > f(u)$ because f restricted to each cycle is a valid ranking. In addition, if the two vertices are on different cycles, then any u-v path will contain x, and $f(x) = k > f(u)$.

Therefore, $k \le \chi_r(G) \le k$ and $\chi_r(G) = \chi_r(C_n)$. □

3. Lollipop graphs

A lollipop graph $L_{a,b}$ consists of a path of order a and a cycle of order b joined by an edge, as shown in Figure 2.

In this section, we determine the rank number of $L_{a,b}$ for all values of a and b. In determining the rank number of lollipop graphs, we consider three cases: $\chi_r(P_a) < \chi_r(C_b)$, $\chi_r(P_a) = \chi_r(C_b)$, and $\chi_r(P_a) > \chi_r(C_b)$.

Theorem 7. *Let $L_{a,b}$ be a lollipop graph where $\chi_r(P_a) < \chi_r(C_b)$. Then $\chi_r(L_{a,b}) = \chi_r(C_b)$.*

Proof. Let $L_{a,b}$ be a lollipop graph with $\chi_r(P_a) < \chi_r(C_b)$ and let $\chi_r(C_b) = k$. Since C_b is a subgraph of $L_{a,b}$, $\chi_r(L_{a,b}) \ge k$ by Lemma 1.

Now, consider a labeling f of $L_{a,b}$ where P_a is labeled according to the standard ranking of a path and the cycle C_b is given a valid k-ranking such that the vertex adjacent to P_a is labeled k. Note that f restricted to C_b and P_a respectively are valid rankings. Also, for any two vertices u, v where one is on P_a and the other is on C_b, if $f(u) = f(v)$, then any u-v path will contain the vertex on C_b labeled k, and $k > f(u)$. Thus f is a valid k-ranking.

Thus $k \le \chi_r(L_{a,b}) \le k$, and so if $\chi_r(P_a) < \chi_r(C_b)$, then $\chi_r(L_{a,b}) = \chi_r(C_b)$. □

Theorem 8. *If $\chi_r(P_a) = \chi_r(C_b)$, then $\chi_r(L_{a,b}) = \chi_r(P_a) + 1$.*

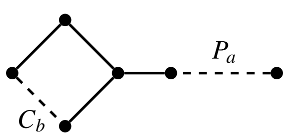

Figure 2. Lollipop graph.

Proof. Let $L_{a,b}$ be a lollipop graph and let $\chi_r(P_a) = \chi_r(C_b) = k$. Since $L_{a,b}$ is the connected supergraph of P_a and C_b, and since $\chi_r(P_a) = \chi_r(C_b) = k$, we have $\chi_r(L_{a,b}) \geq k+1$ by Lemma 2.

Now, consider a labeling f of $L_{a,b}$ as in the proof of Theorem 7, and let $f(x) = k+1$, where x is the vertex on C_b that is adjacent to P_a. Clearly f is a valid $(k+1)$-ranking, since the restrictions of f to C_b and P_a are valid rankings, and for any two vertices u, v, one on P_a and the other on C_b, if $f(u) = f(v)$, then any u-v path will contain x which is labeled $k+1$ and $k+1 > f(u)$.

Therefore $k+1 \leq \chi_r(L_{a,b}) \leq k+1$, and so if $\chi_r(P_a) = \chi_r(C_b)$, then $\chi_r(L_{a,b}) = \chi_r(P_a)+1$. □

Theorem 9. *If $\chi_r(P_a) > \chi_r(C_b)$, then*

$$\chi_r(L_{a,b}) = \begin{cases} \chi_r(P_a) & \text{if } 2^{\chi_r(P_a)-1} \leq a \leq \left(\displaystyle\sum_{i=\chi_r(C_b)}^{\chi_r(P_a)} 2^{i-1}\right) - 1, \\ \chi_r(P_a)+1 & \text{otherwise.} \end{cases}$$

Proof. Let $L_{a,b}$ be a lollipop graph where $\chi_r(P_a) > \chi_r(C_b)$, and let $\chi_r(P_a) = j$ and $\chi_r(C_b) = k$. Since $L_{a,b}$ can be labeled to have a valid $(j+1)$-ranking by giving C_b a valid k-ranking, P_a a valid j-ranking, and by changing the label of the vertex on C_b adjacent to P_a to $j+1$, we have $\chi_r(L_{a,b}) \leq j+1$. Also, since P_a is a subgraph of $L_{a,b}$, we have $\chi_r(L_{a,b}) \geq j$ by Lemma 1. Thus, $j \leq \chi_r(L_{a,b}) \leq j+1$.

Let $L_{a,b}$ be a lollipop graph such that

$$2^{j-1} \leq a \leq \left(\sum_{i=k}^{j} 2^{i-1}\right) - 1.$$

Now consider a labeling f of $L_{a,b}$ defined as follows. Label C_b so that it has a valid k-ranking, with the vertex adjacent to P_a labeled k. Beginning with the vertex of P_a that is joined to C_b, label the vertices of P_a starting with the label of the $(2^{k-1}+1)$-st vertex in the standard ranking of $P_{a+2^{k-1}}$. Since

$$2^{j-1} \leq a \leq \left(\sum_{i=k}^{j} 2^{i-1}\right) - 1,$$

we have

$$a + 2^{k-1} \leq \left(\sum_{i=k}^{j} 2^{i-1}\right) - 1 + 2^{k-1} = 2^j - 1.$$

However, $\chi_r(P_{2^j-1}) = j$, which means $\chi_r(P_{a+2^{k-1}}) \leq j$ and thus f uses at most j labels.

Let x be the vertex on C_b that is adjacent to P_a. The restriction of f to P_{a+1}, the induced subgraph induced by $V(P_a) \cup \{x\}$, is part of the standard ranking of $P_{a+2^{k-1}}$ and hence is a valid ranking. Also, f restricted to C_b is a valid ranking, and for any two vertices u, v, one on P_a and the other in $V(C_b) \setminus \{x\}$, if $f(u) = f(v)$, then any u-v path contains the x and $f(x) = k > f(u)$. Thus f is a valid j-ranking.

Thus, if $\chi_r(P_a) > \chi_r(C_b)$ and $2^{j-1} \leq a \leq \left(\sum_{i=k}^{j} 2^{i-1}\right) - 1$, then $j \leq \chi_r(L_{a,b}) \leq j$, and hence $\chi_r(L_{a,b}) = \chi_r(P_a)$ in this case.

Now, let $L_{a,b}$ be a lollipop graph such that $a > \left(\sum_{i=k}^{j} 2^{i-1}\right) - 1$. Consider any χ_r-ranking of $L_{a,b}$. Since $\chi_r(C_b) = k$, a label of at least $k + \delta < j$, where $\delta \geq 0$, must go on C_b. (Note that if C_b has a label j, and since $\chi_r(P_a) = j$, there must be a label $j + 1$.) Assume, without loss of generality, that $k + \delta$ is the largest label on C_b and that the vertex labeled $k + \delta$ is the vertex which is adjacent to P_a. Then by Lemma 4, if there is no vertex with label $j + 1$ on P_a, then

$$a < \sum_{i=k+\delta}^{j} 2^{i-1}.$$

This is a contradiction, and thus a vertex on P_a must have label $j + 1$. This means $\chi_r(L_{a,b}) \geq j + 1$. Thus if $\chi_r(P_a) > \chi_r(C_b)$ and $a > \left(\sum_{i=k}^{j} 2^{i-1}\right) - 1$, then $\chi_r(L_{a,b}) = \chi_r(P_a) + 1$. □

4. Star-flower graphs

A star-flower graph is a graph that consists of c vertex-disjoint cycles each appended to a center vertex x by an edge. The largest cycle in a star-flower graph will be referred to as C_n. An example of a star-flower graph is shown in Figure 3.

Theorem 10. *Let G be a star-flower graph such that no two cycles in G have the same rank number. Then $\chi_r(G) = \chi_r(C_n)$.*

Proof. Let G be a star-flower graph where no two cycles in G have the same rank number and let $\chi_r(C_n) = k$. Since C_n is a subgraph of G, $\chi_r(G) \geq k$ by Lemma 1.

Consider a labeling f of G in which each cycle is labeled using its χ_r-ranking such that the vertices which are adjacent to x are labeled with the highest label

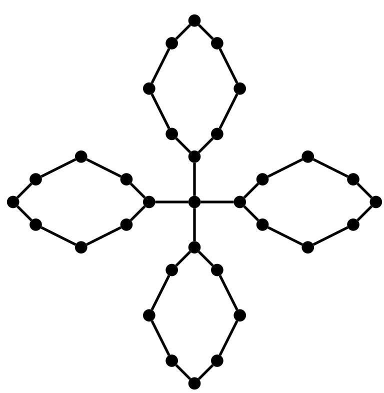

Figure 3. Star-flower graph.

needed in each cycle. Now let $f(x) = 1$. This is a valid k-ranking of G, since f restricted to each cycle is a valid ranking and because for any two vertices u, v with $f(u) = f(v)$, if u and v are on different cycles or one of them is x, then any u-v path contains the vertex on the larger cycle adjacent to x which is greater than $f(u)$. Thus $k \leq \chi_r(G) \leq k$, and hence if no cycles in G have the same rank number, then $\chi_r(G) = \chi_r(C_n)$. □

Theorem 11. *Let G be a star-flower graph such that G has two or more cycles with the same rank number, and let w be the largest repeated rank number among the cycles. Then, $\chi_r(G) = \chi_r(C_n)$ if there exists q such that $w < q < \chi_r(C_n)$ and such that there is no cycle C in G with $\chi_r(C) = q$. In the opposite case, $\chi_r(G) = \chi_r(C_n) + 1$.*

Proof. Let G be a star-flower graph with two or more cycles with the same rank number, and let w be the largest repeated rank number among the cycles. Also, let $\chi_r(C_n) = k$. Note that $k \leq \chi_r(G) \leq k+1$. The lower bound follows from Lemma 1, since C_n is a subgraph of G. The upper bound follows from the fact that G can be given a valid $(k+1)$-ranking by giving the cycles valid k-rankings and labeling x with $k+1$.

Suppose G is a star-flower graph such that there exists some q, where $w < q < k$, for which there is no cycle with a rank number q. Consider a labeling f of G as follows. Label each cycle using its χ_r-ranking so that the vertices adjacent to x are given the highest label needed in each cycle. Now let $f(x) = q$.

The restriction of f to each cycle is a valid ranking. For any two vertices u and v, where both are on different cycles with rank number less than q, any u-v path contains x and $f(x) = q > f(u)$ if $f(u) = f(v)$. Also, if both u and v are on different cycles where one or both of the cycles have a rank number greater than q, and if $f(u) = f(v)$, then any u-v path contains the vertex on the larger cycle which is greater than $f(u)$. Thus f is a valid k-ranking, and therefore $\chi_r(G) \leq k$. Hence in this case $\chi_r(G) = \chi_r(C_n)$.

Now, let G be a star-flower graph such that for all integers q in $w \leq q \leq k$ there is at least one cycle with rank number q. If there are two or more cycles in G with rank number k (that is, $w = k$), then by Lemma 2 we have $\chi_r(G) \geq k+1$. Otherwise, since there are at least two cycles with a rank number of w, the connected supergraph of these cycles must have a rank number of at least $w+1$ by Lemma 2. Since there is a cycle with a rank number of $w+1$, the subgraph of G formed by taking the connected supergraph of this cycle and the subgraph with rank number $w+1$ must have a rank number of at least $w+2$, also by Lemma 2. Continuing this argument, since there are cycles with rank numbers of $w+2$ through k, we see that $\chi_r(G) \geq k+1$. Thus, if G is a star-flower graph with two or more cycles with the same rank number and if there is at least one cycle with rank number q for all integers q in $w \leq q \leq k$, then $\chi_r(G) = \chi_r(C_n) + 1$. □

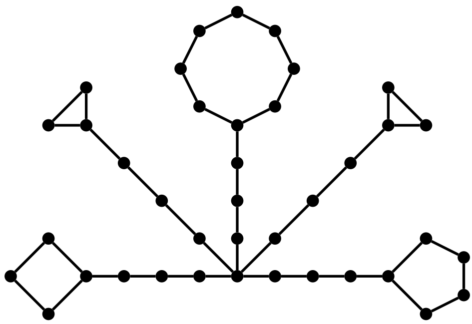

Figure 4. Spider-flower graph.

5. Spider-flower graphs

A spider-flower graph consists of three or more lollipop graphs

$$L_{a,b_1}, L_{a,b_2}, L_{a,b_3}, \ldots, L_{a,b_n}$$

that are appended to a center vertex x (the pendant vertex in each lollipop graph is adjacent to x in the spider-flower graph). Figure 4 illustrates a spider-flower graph which consists of five lollipop graphs. The paths of the lollipop graphs comprising the spider-flower graph are of the same length by definition. The largest cycle in a spider-flower graph will be referred to as C_n, and the paths of the lollipop graphs which comprise the spider-flower graph will be referred to as P_a and are the arms of the spider-flower graph. Note that any spider-flower graph has at least three arms by definition.

To determine the rank number of a spider-flower graph, as with lollipop graphs, we will consider three main cases of spider-flower graphs: $\chi_r(C_n) < \chi_r(P_a)$, $\chi_r(C_n) = \chi_r(P_a)$, and $\chi_r(C_n) > \chi_r(P_a)$.

Theorem 12. *Let G be a spider-flower graph such that $\chi_r(C_n) < \chi_r(P_a)$. Then*

$$\chi_r(G) = \begin{cases} \chi_r(P_a)+1 & \text{if } 2^{\chi_r(P_a)-1} \le a < \sum\limits_{i=\chi_r(C_n)}^{\chi_r(P_a)} 2^{i-1}, \\ \chi_r(P_a)+2 & \text{otherwise.} \end{cases}$$

Proof. Suppose G is a spider-flower graph with $\chi_r(C_n) < \chi_r(P_a)$. Let $\chi_r(P_a) = j$ and let $\chi_r(C_n) = k$. Then $j+1 \le \chi_r(G) \le j+2$. The lower bound follows from Lemma 2 since there are at least two vertex-disjoint copies of P_a in G. The upper bound is true because G can be given a valid $(j+2)$-ranking as follows. Label each of the cycles in G using its χ_r-ranking, label the vertex on each arm that is adjacent to the cycles with $j+1$, label the remaining vertices of each arm using the standard ranking of a path, and label x with $j+2$.

Now, suppose G has arms of order a such that

$$2^{\chi_r(P_a)-1} \leq a < \sum_{i=\chi_r(C_n)}^{\chi_r(P_a)} 2^{i-1}.$$

Consider a labeling f of G as follows. Label the cycles in G using a k-ranking where the vertices on each cycle that are adjacent to the arms are labeled with k. Label each arm using the labeling scheme described in the proof of Theorem 9. Finally, let $f(x) = j+1$.

The restriction of f to each lollipop graph is a valid ranking by similar arguments as in the proof of Theorem 9. Also, for any two vertices u, v with $f(u) = f(v)$, if u and v occur on different cycles or on different arms, or if one is on a cycle and the other is on a nonadjacent arm, then any u-v path contains x and $f(x) = j+1 > f(u)$. This means f is a valid $j+1$-ranking. Thus $\chi_r(G) \leq j+1$, and hence if

$$\chi_r(C_n) < \chi_r(P_a) \quad \text{and} \quad 2^{j-1} \leq a < \sum_{i=w}^{j} 2^{i-1},$$

then $\chi_r(G) = \chi_r(P_a) + 1$.

Suppose $a \geq \sum_{i=k}^{j} 2^{i-1}$. Since $\chi_r(P_a) = j$, we have $2^{j-1} \leq a < 2^j$, so

$$\sum_{i=k}^{j} 2^{i-1} \leq a < 2^j.$$

There is at least one lollipop graph $L_{a,b}$ which is a subgraph of G that has rank number $j+1$ by Theorem 9. There are also at least two other arms in G which are vertex-disjoint from $L_{a,b}$, and the rank number of the connected supergraph H of these two arms must also be at least $j+1$ by Lemma 2. So, by applying Lemma 2 again, the rank number of the connected supergraph of $L_{a,b}$ and H must be at least $j+2$, and thus $\chi_r(G) \geq j+2$. Therefore, if

$$\chi_r(C_n) < \chi_r(P_a) \quad \text{and} \quad \sum_{i=w}^{j} 2^{i-1} \leq a < 2^j,$$

then $\chi_r(G) = \chi_r(P_a) + 2$. □

Theorem 13. *Let G be a spider-flower graph such that $\chi_r(C_n) = \chi_r(P_a)$. Then $\chi_r(G) = \chi_r(C_n) + 2$.*

Proof. Let $\chi_r(P_a) = \chi_r(C_n) = k$. Using the same arguments as in the proof of the second case in Theorem 12, we have $\chi_r(G) \geq k+2$.

Now consider a labeling f of G as follows. Label the largest cycle(s) using a valid k-ranking, where the vertex which is adjacent to the arm of G is labeled k.

Label all of the other cycles using their χ_r-ranking and placing the largest label on the vertex adjacent to the arm. Now, label the vertex on each of the arms which is adjacent to the cycles $k+1$. Label the remainder of each of the arms according to the standard labeling of a path. Finally, let $f(x) = k+2$.

Note that f restricted to a cycle or an arm is a valid ranking. Also, for any two vertices u, v where both are on different cycles or both are on different arms, if $f(u) = f(v)$ then any u-v path will contain x and $f(x) = k+2 > f(u)$. Finally, for any two vertices u, v where one is on a cycle and the other is on an adjacent arm, if $f(u) = f(v)$, then any u-v path will contain the vertex on the arm of G which is adjacent to the cycle and is labeled $k+1 > f(u)$. Hence, f is a valid $(k+2)$-ranking and thus $\chi_r(G) \leq k+2$.

Therefore if $\chi_r(C_n) = \chi_r(P_a)$, then $\chi_r(G) = \chi_r(C_n)+2$. □

Now we consider the case where $\chi_r(C_n) > \chi_r(P_n)$ in a spider-flower graph.

Lemma 14. *Let G be a spider-flower graph such that $\chi_r(C_n) > \chi_r(P_a)$. Then $\chi_r(C_n) \leq \chi_r(G) \leq \chi_r(C_n)+1$.*

Proof. C_n is a subgraph of G, and thus by Lemma 1, $\chi_r(C_n) \leq \chi_r(G)$.

Consider a labeling f of G as follows. Label all cycles using their χ_r-ranking by placing the highest label on the vertices adjacent to the arms, and then relabel these vertices with $\chi_r(C_n)$. Label the arms of G according to the standard labeling of a path, and then label x with $\chi_r(C_n)+1$. Note that f restricted to a cycle or an arm is a valid ranking. If two vertices u, v are on different cycles, different arms, or one on a cycle and the other on a nonadjacent arm, then any u-v path contains the vertex x, and $f(x) = \chi_r(C_n)+1 > f(u)$. Also, since $\chi_r(C_n) > \chi_r(P_a)$, if u is on a cycle and v is on an adjacent arm, then any u-v path contains vertex adjacent to the arm labeled $\chi_r(C_n) > f(v)$.

Therefore f is a valid $(\chi_r(C_n)+1)$-ranking of G, and so $\chi_r(G) \leq \chi_r(C_n)+1$. □

For the rest of the paper, we consider d the largest repeated rank number among the cycles. If no two cycles have the same rank number, then we assume $d = 0$.

Theorem 15. *Let G be a spider-flower graph such that $\chi_r(C_n) > \chi_r(P_a)$. Suppose G has two or more cycles with the same rank number, the greatest of these being d. Let $\chi_r(P_a) < d \leq \chi_r(C_n)$. Then $\chi_r(G) = \chi_r(C_n)$ if there exists t such that $d < t < \chi_r(C_n)$ and such that there is no cycle C in G with $\chi_r(C) = t$. In the opposite case, $\chi_r(G) = \chi_r(C_n)+1$.*

Proof. Let G be a spider-flower graph where two or more cycles have the same rank number, the greatest of these being d, and $\chi_r(P_a) < d \leq \chi_r(C_n)$. Also, let $\chi_r(C_n) = k$. Suppose there exists some t such that $d < t < k$ and there is no cycle with rank number t in G. Consider the labeling f of G as follows. The cycles in G are labeled using their χ_r-ranking, where the vertices adjacent to the arms are given the largest label for each cycle. For those cycles with rank numbers less than d, replace the largest label with d. Label the arms using the standard labeling of a path. Finally, since there is some t, where $d < t < k$, for which there is no cycle with a rank number t, label x with the greatest such t.

Note that f restricted to a cycle or an arm is a valid ranking. For any two vertices u, v where both are on different arms of G, if $f(u) = f(v)$, then any u-v path will contain x, and $f(x) = t > \chi_r(P_a) \geq f(u)$. Also, for any two vertices u, v where both are on different cycles of G or where one is on a cycle and the other is on a nonadjacent arm, if at least one of the cycles has a rank number greater than $t > d$, then if $f(u) = f(v)$, any u-v path will contain the vertex on the larger cycle adjacent to the arm which has a label greater than $f(u)$. And, for any two vertices u, v where both are on different cycles of G or where one is on a cycle and the other is on a nonadjacent arm, if none of the cycles have a rank number greater than t, then if $f(u) = f(v)$, any u-v path will contain the center vertex x, and $f(x) = t > d \geq f(u)$. Finally, for any two vertices u, v where one is on a cycle and the other is on the arm adjacent to that cycle, if $f(u) = f(v)$, then any u-v path will contain the vertex on the cycle which is adjacent to the arm with label $q \geq d > \chi_r(P_a) \geq f(u)$. Thus f is a valid k-ranking, and hence $\chi_r(G) \leq \chi_r(C_n)$. Therefore, by Lemma 14, we have $\chi_r(G) = \chi_r(C_n)$.

Now, suppose for all t in $d < t < k$ there is a cycle with rank number t in G. If there are two or more cycles with a rank number of k, then $\chi_r(G) \geq k+1$ by Lemma 2. Otherwise, since there are at least two cycles with a rank number of d, the rank number of the connected supergraph of these two cycles must be at least $d+1$. Then using similar arguments as in the proof of Theorem 11, $\chi_r(G) \geq k+1$. Thus, by Lemma 14, $\chi_r(G) = \chi_r(C_n) + 1$. □

Theorem 16. *Let G be a spider-flower graph such that $\chi_r(C_n) > \chi_r(P_a)$. Suppose G has two or more cycles with the same rank number, the greatest of these being d, and that $d \leq \chi_r(P_a) < \chi_r(C_n)$. Let b be the largest number for which*

$$2^{\chi_r(P_a)-1} \leq a < \sum_{i=b}^{\chi_r(P_a)} 2^{i-1}.$$

Suppose there are no cycles with rank number r where $b+1 \leq r \leq \chi_r(P_a)$. Then $\chi_r(G) = \chi_r(C_n)$ if there exists t such that $\chi_r(P_a) < t < \chi_r(C_n)$ and such that there is no cycle C in G with $\chi_r(C) = t$. In the opposite case, $\chi_r(G) = \chi_r(C_n) + 1$.

Proof. Assume that there exists t as in the statement. Let $\chi_r(P_a) = j$ and let $\chi_r(C_n) = k$. Consider a labeling f of G as follows. Label the cycles in G using their χ_r-ranking so that the vertex adjacent to the arm has the largest label on the cycle. Relabel the highest-labeled vertices in each of the cycles that have rank numbers less than b with b. Now, label the arms adjacent to cycles with rank numbers greater than j using the standard labeling of a path. The rest of the arms are now adjacent to vertices labeled b. As there are a maximum of $\sum_{i=b}^{j} 2^{i-1} - 1$ vertices on each arm which remain to be labeled, these arms can be labeled with j labels using the standard labeling of a path beginning with the vertex adjacent to the cycle and treating that vertex as if it were the $(2^{b-1}+1)$-st vertex in the path as in the proof of Theorem 9. Finally, let $f(x) = s$, where s is the greatest t in $j < t < k$ for which there is no cycle with a rank number of t.

Note that f restricted to a cycle or an arm is a valid ranking. For any two vertices u, v where both are on different arms, both are on different cycles whose rank numbers are less than s, or one is on a cycle with rank number less than s and the other is on a nonadjacent arm, if $f(u) = f(v)$, then any u-v path will contain the center vertex x, where $f(x) = s > j \geq f(u)$. For any two vertices u, v where both are on cycles where one or both of the rank numbers is greater than s or one is on a cycle with rank number greater than s and the other is on a nonadjacent arm, if $f(u) = f(v)$, then any u-v path will contain the vertex that is adjacent to the arm on the larger cycle and has a label $q > f(u)$. Also, for any two vertices u, v where one is on a cycle and the other is on an adjacent arm, if the rank number of the cycle is greater than j and $f(u) = f(v)$, then any u-v path will contain the vertex on the cycle adjacent to the arm which has a label $z > j \geq f(u)$. If the rank number of the cycle is less than j and $f(u) = f(v)$, then any u-v path will either contain the vertex on the cycle labeled b where $b > f(u)$, or will contain the vertex with a higher label as in the proof of Theorem 9. Therefore f is a valid k-ranking, and hence $\chi_r(G) \leq \chi_r(C_n)$. Thus $\chi_r(G) = \chi_r(C_n)$ by Lemma 14.

Now, let G be a spider-flower graph where for every t such that $j < t < k$, there exists some cycle with rank number equal to t. Since there are at least two disjoint copies of P_a in G, a label of $y \geq j+1$ must be used to separate these arms. However, since for every t in $j < t < k$ there is some cycle with rank number equal to t, $\chi_r(G) \geq k+1$ by the same argument used in the proof of Theorem 11. Thus, by Lemma 14, $\chi_r(G) = \chi_r(C_n) + 1$. □

Theorem 17. *Let G be a spider-flower graph such that $\chi_r(C_n) > \chi_r(P_a)$. Suppose G has two or more cycles with the same rank number, the greatest of these being d, and that $d \leq \chi_r(P_a) < \chi_r(C_n)$. Let b be the largest number for which*

$$2^{\chi_r(P_a)-1} \leq a < \sum_{i=b}^{\chi_r(P_a)} 2^{i-1}.$$

Suppose there is at least one cycle with rank number r for $b+1 \leq r \leq \chi_r(P_a)$. Then $\chi_r(G) = \chi_r(C_n)$ if there exists t such that $\chi_r(P_a)+1 < t < \chi_r(C_n)$ and such that there is no cycle C with $\chi_r(C) = t$. In the opposite case, $\chi_r(G) = \chi_r(C_n)+1$.

Proof. Assume there exists t as in the statement. Let $\chi_r(P_a) = j$ and let $\chi_r(C_n) = k$. Now, consider a labeling f of G as follows. Label G as in the proof of Theorem 16 except that for those cycles with rank number r, where $b+1 \le r \le j$, the vertices on the arms adjacent to the cycles are labeled $j+1$ and the remainder of the arms are labeled according to the standard labeling of a path. Finally, let $f(x) = s$, where s is the largest t in $j+1 < t < k$ for which there is no cycle with rank number t.

For any two vertices u, v where one vertex is on a cycle with rank number r, where $b+1 \le r \le j$, any u-v path will contain the vertex on the arm adjacent to the cycle labeled $j+1$, and $j+1 > f(u)$. For two vertices u, v in any other position of the graph, we can use the same arguments as in the proof of Theorem 16 and conclude that f is a valid k-ranking. Therefore $\chi_r(G) \le \chi_r(C_n)$, and hence $\chi_r(G) = \chi_r(C_n)$ by Lemma 14.

Now, let G be a spider-flower graph where for every t in $j+1 < t < k$ there is some cycle with rank number t. Since there is a cycle with rank number r for each r such that $b+1 \le r \le j$, G contains a cycle with rank number j. This cycle, together with its arm P_a, forms a lollipop graph $L_{a,z}$ with rank number $j+1$ by Theorem 8. Also, since G has at least three lollipop graphs as subgraphs, there are at least two disjoint copies of P_a, namely P and Q, which are also disjoint from $L_{a,z}$. G also has cycles of rank number t, where $j+1 < t < k$. These cycles, P, Q, and $L_{a,z}$ are mutually vertex-disjoint. Then, using a similar argument as in Theorem 11, we get $\chi_r(G) \ge k+1$, and hence by Lemma 14 we get $\chi_r(G) = \chi_r(C_n) + 1$. □

6. Conclusion and future directions

We determined the rank numbers of flower graphs, lollipop graphs, star-flower graphs, and spider-flower graphs. We determined the rank number of each of these graphs for any size of cycles. The spider-flower graphs consists of at least three lollipop graphs that are appended to a vertex. However, the graph where exactly two lollipop graphs are appended to a vertex requires significantly different analysis than the spider-flower graphs. Some cases of this graph were looked into in [McClive 2010].

By definition, the arms of the spider-flower graphs are of the same length. One related graph to consider would be the graph that consists of lollipop graphs of different arm lengths appended to a center vertex. Finding the rank number of this graph requires finding the rank number of a special type of graphs called extended star graphs. We can define an extended star graph to be a graph that consists of paths of any length appended to a single vertex. It is clear that the rank number of an extended star graph is either $\chi_r(P_n)$ or $\chi_r(P_n)+1$, where P_n is the longest path in the extended star graph. However, characterizing extended star graphs with respect to their rank numbers turned out to be extremely difficult.

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