An elementary inequality about the Mahler measure

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Let \( p(z) \) be a degree \( n \) polynomial with zeros \( z_j, j = 1, 2, \ldots, n \). The total distance from the zeros of \( p \) to the unit circle is defined as \( \text{td}(p) = \sum_{j=1}^{n} |z_j| - 1. \) We show that up to scalar multiples, \( \text{td}(p) \) sits between \( M(p) - 1 \) and \( m(p) \). This leads to an equivalent statement of Lehmer’s problem in terms of \( \text{td}(p) \). The proof is elementary.

1. Introduction

Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial with complex coefficients of degree \( n \). The Mahler measure \( M(p) \) [Everest and Ward 1999] is defined as

\[
M(p) = \exp \left( \int_0^{2\pi} \log |p(e^{i\theta})| \frac{d\theta}{2\pi} \right).
\]

We denote \( \log M(p) \) by \( m(p) \). Jensen’s formula implies that

\[
M(p) = |a_n| \prod_{|z_j| > 1} |z_j|,
\]

where throughout this paper the \( z_j, j = 1, 2, \ldots, n \), are the zeros of \( p(z) \), counting multiplicity. We also assume that \( |a_n| = 1 \). It is then clear that \( M(p) \geq 1 \), and

\[
0 \leq m(p) = \log((M(p) - 1) + 1) \leq M(p) - 1,
\]

and when \( M(p) \) is close to 1, \( m(p) \) is close to \( M(p) - 1 \). Lehmer’s problem is to verify that for integer-coefficient monic polynomials, \( m(p) \) is either 0 (for products of cyclotomic polynomials and possibly a factor of \( z^k \)) or is bounded away from 0 by a fixed positive constant. This is a deep and unsolved problem.

For a polynomial \( p \) of degree \( n \), the associated polynomial \( p^*(z) \) is defined as \( z^n p(1/z) \). We say \( p \) is reciprocal if \( p = cp^* \) for some complex number \( c \) of modulus 1. One sees that the zeros of a reciprocal \( p \) off the unit circle appear in conjugate reciprocal pairs. Interestingly, Lehmer’s problem was unsolved only for reciprocal polynomials. A key ingredient of this paper is the total distance from the

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zeros of $p$ to the unit circle $T$ defined to be

$$\text{td}(p) = \sum_{j=1}^{n} |z_j| - 1.$$  

**Theorem.** For every complex polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$, with $|a_n| = |a_0| = 1$, we have

$$m(p) \leq \text{td}(p) \leq 2(M(p) - 1).$$

If $p$ is reciprocal, then $2m(p) \leq \text{td}(p)$. Further, the equalities hold only if $\text{td}(p) = 0$.

Therefore, Lehmer’s problem can be stated equivalently as follows: There is an $\epsilon > 0$ such that if $p$ has integer coefficients with $|a_n| = |a_0| = 1$ and $\text{td}(p) \neq 0$, then $\text{td}(p) \geq \epsilon$.

### 2. Proof

**Lemma 1.** If $t_j$, $j = 1, 2, \ldots, k$ are numbers in the interval $[0, 1]$, then

$$\sum_{j=1}^{k} (1 - t_j) \leq \frac{1}{\prod_{j=1}^{k} t_j} - 1,$$

where equality holds only if $t_j = 1$ for each $j$.

**Proof.** The inequality is trivial if one of the $t_j$ is 0. Now, we assume $t_j > 0$ for each $j$. We prove by induction. It is easy to see that the lemma is true for $k = 1$. Assume the lemma is true for $k$. For $s$ and $t$ in $(0, 1]$, one checks that

$$\frac{1}{ts} - \left(\frac{1}{t} + 1 - s\right) = \frac{(1-s)(1-ts)}{ts} \geq 0,$$

and hence

$$\frac{1}{ts} - 1 \geq \frac{1}{t} - s.$$  

Therefore

$$\sum_{j=1}^{k} (1 - t_j) + (1 - t_{k+1}) \leq \frac{1}{\prod_{j=1}^{k} t_j} - 1 + (1 - t_{k+1})$$

$$= \frac{1}{\prod_{j=1}^{k} t_j} - t_{k+1} \leq \frac{1}{\prod_{j=1}^{k+1} t_j} - 1. \quad \square$$

If $\{\lambda_j : j = 1, 2, \ldots\}$ is a subset of the open unit disk $\mathbb{D}$, the associated Blaschke product is defined as

$$B(z) = \prod_{j=1}^{\infty} \frac{z - \lambda_j}{1 - \lambda_j \bar{z}}, \quad z \in \mathbb{D}.$$
Clearly, the product is convergent for each \( z \) if and only if \( \sum_{j=0}^{\infty} (1 - |\lambda_j|) < \infty \) [Garnett 2007]. In this case \( B(z) \) is a bounded analytic function on \( \mathbb{D} \). It follows immediately from Lemma 1 that
\[
\sum_{j=1}^{\infty} (1 - |\lambda_j|) \leq \frac{1}{|B(0)|} - 1.
\]

Proof of the Theorem. For a polynomial \( p(z) \), since \( \prod_{j=1}^{n} |z_j| = \frac{|a_0|}{|a_n|} \), we have
\[
\frac{M(p)}{|a_0|} - 1 = \frac{1}{\prod_{|z_j| \leq 1} |z_j|} - 1 \geq \sum_{|z_j| \leq 1} (1 - |z_j|)
\]
by Lemma 1. On the other hand, inductively using that \( (a - 1) + (b - 1) < ab - 1 \) for \( a, b > 1 \), we have
\[
\sum_{|z_j| > 1} (|z_j| - 1) \leq \prod_{|z_j| > 1} |z_j| - 1 = \frac{M(p)}{|a_n|} - 1.
\]
Here the equality is allowed only because there may not be a \( z_j \) with \( |z_j| > 1 \). Combining with (2-2), we have \( t_d(p) \leq M(p)(1/|a_n| + 1/|a_0|) - 2 \). In the case \( |a_0| = |a_n| = 1 \), we have
\[
t_d(p) \leq 2(M(p) - 1), \tag{2-3}
\]
with equality occurring only if \( t_d(p) = 0 \). The dominance of \( m(p) \) by \( t_d(p) \) is an easy consequence of the inequality \( \log(1 + t) \leq t \). To be precise,
\[
m(p) = \sum_{|z_k| > 1} \log |z_k| \leq \sum_{|z_k| > 1} (|z_k| - 1) \leq t_d(p).
\]

We establish a stronger inequality for reciprocal polynomials with \( |a_0| = |a_n| = 1 \). Let \( z_1, z_2, \ldots, z_k \) be the zeros of such a \( p \) that are outside of the unit circle, where \( 2k \leq n \). Then \( m(p) = \log |z_1| + \log |z_2| + \cdots + \log |z_k| \) and
\[
t_d(p) = \sum_{j=1}^{k} (|z_j| - 1) + \left( 1 - \frac{1}{|z_j|} \right).
\]
Let \( f(t) = t - (1/t) - 2 \log t, \ t \geq 1 \). One easily checks that \( f \) is strictly increasing and \( f(1) = 0 \). It follows that \( |z_j| - 1/|z_j| > 2 \log |z_j| \) for each \( 1 \leq j \leq k \), and hence \( 2m(p) \leq t_d(p) \), with equality precisely when \( k = 0 \), which occurs if and only if \( t_d(p) = 0 \) since \( |a_0| = |a_n| = 1 \).

Example. Consider Lehmer’s polynomial
\[
G(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.
\]
It is well-known that eight of its zeros lie in the unit circle and the other two are real and form a reciprocal pair. Since 
\[ M(G) \approx 1.1763, \]
we have
\[ M(G) \approx (1.1763 - 1) + (1 - 1/1.1763) \approx 0.3262, \]
\[ 2m(G) \approx 2 \times 0.1624 = 0.3248, \]
\[ 2(M(p) - 1) \approx 0.3526. \]

Our Theorem has some interesting implications. We need two more definitions to state them. Define
\[ \Delta(p) = \max \{|\alpha| - 1 : p(\alpha) = 0\}, \]
\[ \delta(p) = \min \{|\alpha| - 1 : p(\alpha) = 0\}. \]
Then it is clear that
\[ \delta(p) \leq \frac{\text{td}(p)}{n} \leq \Delta(p). \]  \hspace{1cm} (2-4) \]
When \( p \) is reciprocal and \( \alpha \) is a zero of \( p \), \( 1/\alpha \) is also a zero. Since \( t - 1 \geq \frac{1 - 1}{t} \) for \( t \geq 1 \), we have
\[ \Delta(p) = \max \{|\alpha| - 1 : p(\alpha) = 0\} = \max \{|\alpha| : p(\alpha) = 0\} - 1. \]
Likewise
\[ \delta(p) = 1 - \max \{|\alpha| : |\alpha| \leq 1, \; p(\alpha) = 0\}. \]
For simplicity, we let
\[ \lambda(p) = \max \{|\alpha| : p(\alpha) = 0\} \]
and let
\[ \lambda'(p) = \max \{|\alpha| : |\alpha| \leq 1, \; p(\alpha) = 0\}. \]
In [Smyth 2008], \( \lambda(p) \) is called the house of the zeros of \( p \). Geometrically, \( \lambda(p) \) is the modulus of the zero that is the farthest from the unit circle, while \( \lambda'(p) \) is the modulus of the zero that is the nearest to the unit circle. The next proposition then follows easily from (2-4).

**Proposition.** *For a reciprocal complex polynomial \( p \) of degree \( n \geq 2 \),
\[ \lambda(p) \geq 1 + \frac{\text{td}(p)}{n} \quad \text{and} \quad \lambda'(p) \geq 1 - \frac{\text{td}(p)}{n}. \]

Regarding \( \lambda(p) \), there is an unsolved conjecture by Schinzel and Zassenhaus that states that there is an absolute constant \( C \) so that if \( p \) is a monic irreducible polynomial of degree \( n \) with integer coefficients, then \( \lambda(p) \geq 1 + C/n \). This inequality will follow easily from a positive answer to Lehmer's problem. Indeed, one has \( \lambda(p) \geq 1 + m(p)/n \) [Smyth 2008]. But in view of Theorem, Proposition provides a better inequality for reciprocal polynomials.
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References


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