

involve

a journal of mathematics

An elementary inequality about the Mahler measure

Konstantin Stulov and Rongwei Yang



An elementary inequality about the Mahler measure

Konstantin Stulov and Rongwei Yang

(Communicated by Andrew Granville)

Let $p(z)$ be a degree n polynomial with zeros z_j , $j = 1, 2, \dots, n$. The total distance from the zeros of p to the unit circle is defined as $\text{td}(p) = \sum_{j=1}^n ||z_j| - 1|$. We show that up to scalar multiples, $\text{td}(p)$ sits between $M(p) - 1$ and $m(p)$. This leads to an equivalent statement of Lehmer's problem in terms of $\text{td}(p)$. The proof is elementary.

1. Introduction

Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial with complex coefficients of degree n . The Mahler measure $M(p)$ [Everest and Ward 1999] is defined as

$$M(p) = \exp\left(\int_0^{2\pi} \log |p(e^{i\theta})| \frac{d\theta}{2\pi}\right).$$

We denote $\log M(p)$ by $m(p)$. Jensen's formula implies that

$$M(p) = |a_n| \prod_{|z_j| > 1} |z_j|,$$

where throughout this paper the z_j , $j = 1, 2, \dots, n$, are the zeros of $p(z)$, counting multiplicity. We also assume that $|a_n| = 1$. It is then clear that $M(p) \geq 1$, and

$$0 \leq m(p) = \log((M(p) - 1) + 1) \leq M(p) - 1,$$

and when $M(p)$ is close to 1, $m(p)$ is close to $M(p) - 1$. Lehmer's problem is to verify that for integer-coefficient monic polynomials, $m(p)$ is either 0 (for products of cyclotomic polynomials and possibly a factor of z^k) or is bounded away from 0 by a fixed positive constant. This is a deep and unsolved problem.

For a polynomial p of degree n , the associated polynomial $p^*(z)$ is defined as $z^n \overline{p(1/\bar{z})}$. We say p is reciprocal if $p = cp^*$ for some complex number c of modulus 1. One sees that the zeros of a reciprocal p off the unit circle appear in conjugate reciprocal pairs. Interestingly, Lehmer's problem was unsolved only for reciprocal polynomials. A key ingredient of this paper is the total distance from the

MSC2010: 11CXX.

Keywords: Mahler measure, total distance.

zeros of p to the unit circle T defined to be

$$\text{td}(p) = \sum_{j=1}^n \left| |z_j| - 1 \right|.$$

Theorem. *For every complex polynomial $p(z) = \sum_{j=0}^n a_j z^j$, with $|a_n| = |a_0| = 1$, we have*

$$m(p) \leq \text{td}(p) \leq 2(M(p) - 1).$$

If p is reciprocal, then $2m(p) \leq \text{td}(p)$. Further, the equalities hold only if $\text{td}(p) = 0$.

Therefore, Lehmer’s problem can be stated equivalently as follows: There is an $\epsilon > 0$ such that if p has integer coefficients with $|a_n| = |a_0| = 1$ and $\text{td}(p) \neq 0$, then $\text{td}(p) \geq \epsilon$.

2. Proof

Lemma 1. *If $t_j, j = 1, 2, \dots, k$ are numbers in the interval $[0, 1]$, then*

$$\sum_{j=1}^k (1 - t_j) \leq \frac{1}{\prod_{j=1}^k t_j} - 1,$$

where equality holds only if $t_j = 1$ for each j .

Proof. The inequality is trivial if one of the t_j is 0. Now, we assume $t_j > 0$ for each j . We prove by induction. It is easy to see that the lemma is true for $k = 1$. Assume the lemma is true for k . For s and t in $(0, 1]$, one checks that

$$\frac{1}{ts} - \left(\frac{1}{t} + 1 - s \right) = \frac{(1-s)(1-ts)}{ts} \geq 0, \tag{2-1}$$

and hence

$$\frac{1}{ts} - 1 \geq \frac{1}{t} - s.$$

Therefore

$$\begin{aligned} \sum_{j=1}^k (1 - t_j) + (1 - t_{k+1}) &\leq \frac{1}{\prod_j^k t_j} - 1 + (1 - t_{k+1}) \\ &= \frac{1}{\prod_j^k t_j} - t_{k+1} \leq \frac{1}{\prod_j^{k+1} t_j} - 1. \end{aligned} \quad \square$$

If $\{\lambda_j : j = 1, 2, \dots\}$ is a subset of the open unit disk \mathbb{D} , the associated Blaschke product is defined as

$$B(z) = \prod_{j=1}^{\infty} \frac{z - \lambda_j}{1 - \overline{\lambda_j} z}, \quad z \in \mathbb{D}.$$

Clearly, the product is convergent for each z if and only if $\sum_{j=0}^{\infty} (1 - |\lambda_j|) < \infty$ [Garnett 2007]. In this case $B(z)$ is a bounded analytic function on \mathbb{D} . It follows immediately from Lemma 1 that

$$\sum_{j=1}^{\infty} (1 - |\lambda_j|) \leq \frac{1}{|B(0)|} - 1.$$

Proof of the Theorem. For a polynomial $p(z)$, since $\prod_{j=1}^n |z_j| = \frac{|a_0|}{|a_n|}$, we have

$$\frac{M(p)}{|a_0|} - 1 = \frac{1}{\prod_{|z_j| \leq 1} |z_j|} - 1 \geq \sum_{|z_j| \leq 1} (1 - |z_j|) \tag{2-2}$$

by Lemma 1. On the other hand, inductively using that $(a - 1) + (b - 1) < ab - 1$ for $a, b > 1$, we have

$$\sum_{|z_j| > 1} (|z_j| - 1) \leq \prod_{|z_j| > 1} |z_j| - 1 = \frac{M(p)}{|a_n|} - 1.$$

Here the equality is allowed only because there may not be a z_j with $|z_j| > 1$. Combining with (2-2), we have $\text{td}(p) \leq M(p)(1/|a_n| + 1/|a_0|) - 2$. In the case $|a_0| = |a_n| = 1$, we have

$$\text{td}(p) \leq 2(M(p) - 1), \tag{2-3}$$

with equality occurring only if $\text{td}(p) = 0$. The dominance of $m(p)$ by $\text{td}(p)$ is an easy consequence of the inequality $\log(1 + t) \leq t$. To be precise,

$$m(p) = \sum_{|z_k| > 1} \log |z_k| \leq \sum_{|z_k| > 1} (|z_k| - 1) \leq \text{td}(p).$$

We establish a stronger inequality for reciprocal polynomials with $|a_0| = |a_n| = 1$. Let z_1, z_2, \dots, z_k be the zeros of such a p that are outside of the unit circle, where $2k \leq n$. Then $m(p) = \log |z_1| + \log |z_2| + \dots + \log |z_k|$ and

$$\text{td}(p) = \sum_{j=1}^k (|z_j| - 1) + \left(1 - \frac{1}{|z_j|}\right).$$

Let $f(t) = t - (1/t) - 2 \log t$, $t \geq 1$. One easily checks that f is strictly increasing and $f(1) = 0$. It follows that $|z_j| - 1/|z_j| > 2 \log |z_j|$ for each $1 \leq j \leq k$, and hence $2m(p) \leq \text{td}(p)$, with equality precisely when $k = 0$, which occurs if and only if $\text{td}(p) = 0$ since $|a_0| = |a_n| = 1$. □

Example. Consider Lehmer’s polynomial

$$G(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

It is well-known that eight of its zeros lie in the unit circle and the other two are real and form a reciprocal pair. Since $M(G) \approx 1.1763$, we have

$$\begin{aligned}\mathrm{td}(G) &\approx (1.1763 - 1) + (1 - 1/1.1763) \approx 0.3262, \\ 2m(G) &\approx 2 \times 0.1624 = 0.3248, \\ 2(M(p) - 1) &\approx 0.3526.\end{aligned}$$

Our Theorem has some interesting implications. We need two more definitions to state them. Define

$$\begin{aligned}\Delta(p) &= \max\{||\alpha| - 1| : p(\alpha) = 0\}, \\ \delta(p) &= \min\{||\alpha| - 1| : p(\alpha) = 0\}.\end{aligned}$$

Then it is clear that

$$\delta(p) \leq \frac{\mathrm{td}(p)}{n} \leq \Delta(p). \quad (2-4)$$

When p is reciprocal and α is a zero of p , $1/\alpha$ is also a zero. Since $t - 1 \geq 1 - 1/t$ for $t \geq 1$, we have

$$\Delta(p) = \max\{|\alpha| - 1 : p(\alpha) = 0\} = \max\{|\alpha| : p(\alpha) = 0\} - 1.$$

Likewise

$$\delta(p) = 1 - \max\{|\alpha| : |\alpha| \leq 1, p(\alpha) = 0\}.$$

For simplicity, we let

$$\lambda(p) = \max\{|\alpha| : p(\alpha) = 0\}$$

and let

$$\lambda'(p) = \max\{|\alpha| : |\alpha| \leq 1, p(\alpha) = 0\}.$$

In [Smyth 2008], $\lambda(p)$ is called the house of the zeros of p . Geometrically, $\lambda(p)$ is the modulus of the zero that is the farthest from the unit circle, while $\lambda'(p)$ is the modulus of the zero that is the nearest to the unit circle. The next proposition then follows easily from (2-4).

Proposition. *For a reciprocal complex polynomial p of degree $n \geq 2$,*

$$\lambda(p) \geq 1 + \frac{\mathrm{td}(p)}{n} \quad \text{and} \quad \lambda'(p) \geq 1 - \frac{\mathrm{td}(p)}{n}.$$

Regarding $\lambda(p)$, there is an unsolved conjecture by Schinzel and Zassenhaus that states that there is an absolute constant C so that if p is a monic irreducible polynomial of degree n with integer coefficients, then $\lambda(p) \geq 1 + C/n$. This inequality will follow easily from a positive answer to Lehmer's problem. Indeed, one has $\lambda(p) \geq 1 + m(p)/n$ [Smyth 2008]. But in view of Theorem, Proposition provides a better inequality for reciprocal polynomials.

Acknowledgment

The authors thank the referee for helpful and stimulating comments.

References

- [Everest and Ward 1999] G. Everest and T. Ward, *Heights of polynomials and entropy in algebraic dynamics*, Springer, London, 1999. MR 2000e:11087
- [Garnett 2007] J. B. Garnett, *Bounded analytic functions*, 1st ed., Graduate Texts in Mathematics **236**, Springer, New York, 2007. MR 2007e:30049
- [Smyth 2008] C. Smyth, “The Mahler measure of algebraic numbers: a survey”, pp. 322–349 in *Number theory and polynomials*, London Math. Soc. Lecture Note Ser. **352**, Cambridge Univ. Press, 2008. MR 2009j:11172

Received: 2012-07-09

Revised: 2013-02-12

Accepted: 2013-02-16

kstulov@stanford.edu

*Institute for Computational and Mathematical Engineering,
Stanford University, Stanford, NY 94305, United States*

ryang@albany.edu

*Department of Mathematics and Statistics,
University of Albany, State University of New York,
Albany, NY 12047, United States*

involve

msp.org/involve

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsgdam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rjplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION


Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2013 is US \$105/year for the electronic version, and \$145/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2013 Mathematical Sciences Publishers

involve

2013

vol. 6

no. 4

Embeddedness for singly periodic Scherk surfaces with higher dihedral symmetry VALMIR BUCAJ, SARAH CANNON, MICHAEL DORFF, JAMAL LAWSON AND RYAN VIERTEL	383
An elementary inequality about the Mahler measure KONSTANTIN STULOV AND RONGWEI YANG	393
Ecological systems, nonlinear boundary conditions, and Σ -shaped bifurcation curves KATHRYN ASHLEY, VICTORIA SINCAVAGE AND JEROME GODDARD II	399
The probability of randomly generating finite abelian groups TYLER CARRICO	431
Free and very free morphisms into a Fermat hypersurface TABES BRIDGES, RANKEYA DATTA, JOSEPH EDDY, MICHAEL NEWMAN AND JOHN YU	437
Irreducible divisor simplicial complexes NICHOLAS R. BAETH AND JOHN J. HOBSON	447
Smallest numbers beginning sequences of 14 and 15 consecutive happy numbers DANIEL E. LYONS	461
An orbit Cartan type decomposition of the inertia space of $SO(2m)$ acting on \mathbb{R}^{2m} CHRISTOPHER SEATON AND JOHN WELLS	467
Optional unrelated-question randomized response models SAT GUPTA, ANNA TUCK, TRACY SPEARS GILL AND MARY CROWE	483
On the difference between an integer and the sum of its proper divisors NICHOLE DAVIS, DOMINIC KLYVE AND NICOLE KRAGHT	493
A Pexider difference associated to a Pexider quartic functional equation in topological vector spaces SAEID OSTADBASHI, ABBAS NAJATI, MAHSA SOLAIMANINIA AND THEMISTOCLES M. RASSIAS	505



1944-4176(2013)6:4;1-7