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Smallest numbers beginning sequences of 14 and 15 consecutive happy numbers

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It is well known that there exist arbitrarily long sequences of consecutive happy numbers. In this paper we find the smallest numbers beginning sequences of fourteen and fifteen consecutive happy numbers.

1. Introduction

Guy [1994, Problem E34] defines a happy number in the following way: "If you iterate the process of summing the squares of the decimal digits of a number, then it is easy to see that you either reach the cycle $4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$ or arrive at 1. In the latter case you started from a happy number." Written another way, a happy number N is one for which some iteration of the function $S(N) = \sum_{j=0}^{k} a_j^2$ returns a value of 1, where $\sum_{j=0}^{k} a_j 10^j$ is the decimal expansion of N. According to Guy, the problem was first brought to the attention of the Western mathematical world when Reginald Allenby's daughter returned with it from school in Britain. It is thought to have originated in Russia.

The first pair of consecutive happy numbers is 31, 32. The first example of three consecutive happy numbers is 1880, 1881, 1882. The smallest N beginning a sequence of four and five consecutive happy numbers are 7839 and 44488, respectively. El-Sedy and Siksek [2000] were the first to publish a proof that there exist arbitrarily long sequences of happy numbers, although Lenstra is known to have had an unpublished proof before them. Styer [2010] found the smallest examples of sequences of j consecutive happy numbers, for j from 6 to 13.

In this paper, we will use a period (.) to denote the concatenation operator to group sets of digits together within a large number. For convenience and clarity,

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we will also write large strings of 9 by their quantity in parenthesis. For example, $615 \cdot 10^{157} + (10^{155} - 1) \cdot 10^2 + 71$ will be written as 615.(155 nines).71.

Define the function $S\left(\sum_{j=0}^{k} a_j 10^j\right) = \sum_{j=0}^{k} a_j^2$ and

$$N_0 = 7888.(1604938271577 \text{ nines}).1.(345696 \text{ nines}).3.$$

2. Fourteen consecutive happy numbers

Theorem 1. $N_0 = 7888.(1604938271577$ nines$).1.(345696$ nines$).3$ *is the smallest N that begins a sequence of fourteen consecutive happy numbers. Note*: N_0 *has* 1604938617279 *digits.*

Because the S function simply sums the squares of the digits of a number, and because addition is commutative, the ordering of the digits has no effect on the function's output. In other words,

Lemma 1. *For every choice of positive integers A, B, and C,*

$$S(A.B.C) = S(B.A.C) = S(A.C.B) = S(A) + S(B) + S(C).$$

Lemma 2. *N_0 begins a sequence of fourteen consecutive happy numbers.*

Proof. Before the carry:

$$\begin{aligned}
N_0 &= 7888.(1604938271577 \text{ nines}).1.(345696 \text{ nines}).3,\\
S(N_0) &= 130000027999364,\\
N_0 + 1 &= 7888.(1604938271577 \text{ nines}).1.(345696 \text{ nines}).4,\\
S(N_0 + 1) &= 130000027999371,\\
N_0 + 2 &= 7888.(1604938271577 \text{ nines}).1.(345696 \text{ nines}).5,\\
S(N_0 + 2) &= 130000027999380,\\
N_0 + 3 &= 7888.(1604938271577 \text{ nines}).1.(345696 \text{ nines}).6,\\
S(N_0 + 3) &= 130000027999391,\\
N_0 + 4 &= 7888.(1604938271577 \text{ nines}).1.(345696 \text{ nines}).7,\\
S(N_0 + 4) &= 130000027999404,\\
N_0 + 5 &= 7888.(1604938271577 \text{ nines}).1.(345696 \text{ nines}).8,\\
S(N_0 + 5) &= 130000027999419,\\
N_0 + 6 &= 7888.(1604938271577 \text{ nines}).1.(345696 \text{ nines}).9,\\
S(N_0 + 6) &= 130000027999436.
\end{aligned}$$

After the carry:

$$\begin{aligned}
N_0+7 &= 7888.(1604938271577 \text{ nines}).2.(345696 \text{ zeros}).0,\\
S(N_0+7) &= 129999999997982,\\
N_0+8 &= 7888.(1604938271577 \text{ nines}).2.(345696 \text{ zeros}).1,\\
S(N_0+8) &= 129999999997983,\\
N_0+9 &= 7888.(1604938271577 \text{ nines}).2.(345696 \text{ zeros}).2,\\
S(N_0+9) &= 129999999997986,\\
N_0+10 &= 7888.(1604938271577 \text{ nines}).2.(345696 \text{ zeros}).3,\\
S(N_0+10) &= 129999999997991,\\
N_0+11 &= 7888.(1604938271577 \text{ nines}).2.(345696 \text{ zeros}).4,\\
S(N_0+11) &= 129999999997998,\\
N_0+12 &= 7888.(1604938271577 \text{ nines}).2.(345696 \text{ zeros}).5,\\
S(N_0+12) &= 129999999998007,\\
N_0+13 &= 7888.(1604938271577 \text{ nines}).2.(345696 \text{ zeros}).6,\\
S(N_0+13) &= 129999999998018.
\end{aligned}$$

It is not difficult to see that each of these numbers is happy. The iterations of the S function get small rather quickly, and, after at most nine steps, reach 1. □

Lemma 3. *If $N_a < N_0$ is another example of a number beginning a sequence of fourteen consecutive happy numbers, then $S(N_a) < 9^2 \cdot 1604938617279 = 130000027999599$.*

Proof. In order for N_a to be smaller than N_0, it must not contain more digits than N_0. N_0 contains 1604938617279 digits. The largest number containing no more than 1604938617279 digits is $10^{1604938617279} - 1$, or 1604938617279 digits 9, which has an S value of $9^2 \cdot 1604938617279 = 130000027999599$. Therefore, if there were a number $N_a < N_0$ beginning a sequence of fourteen consecutive happy numbers, it would necessarily have $S(N_a) < 130000027999599$. □

We will let N_1 denote any candidate less its final digit. Thus we write $N_a = N_1.x$, where x is the final digit. So, in our case, $N_0 = N_1.3$. Let d be the first (rightmost) non-nine digit of N_1, and let N_2 be the remaining digits of N_1, to the left of d. Thus we have

$$N_1 = N_2.d.(k \text{ nines})$$

for an integer $k \geq 0$.

Lemma 4. $S(N_1+1) \leq S(N_1) + 17$.

Proof.

$$
\begin{aligned}
N_1 &= N_2.d.(k \text{ nines}),\\
N_1+1 &= N_2.(d+1).(k \text{ zeros}),\\
S(N_1) &= S(N_2)+d^2+9^2k,\\
S(N_1+1) &= S(N_2)+(d+1)^2,\\
S(N_1+1)-S(N_1) &= (d+1)^2-d^2-81k \le 9^2-8^2 = 17. \qquad \square
\end{aligned}
$$

Lemma 5. *Let M have four or more digits and let m, f, g, h be integers. Define*

$$M = M_2.f.(m \text{ nines}).g.h,$$

where $m \ge 0$, $0 \le f \le 8$, $0 \le e, g, h \le 9$, and M_2 either is a positive integer or else is possibly vacuous (in which case we define $S(M_2) = 0$). Then

$$S(M+e^2) = S(M_2) + S(f.(m \text{ nines}).g.h + e^2).$$

Proof. Since $e^2 \le 81$, then $g.h + e^2 \le 180$. Now $g.h + e^2 = i.j$ or $g.h + e^2 = 1.i.j$ for some digits i and j. Then we have $M + e^2 = M_2.f.(m \text{ nines}).i.j$ or $M + e^2 = M_2.(f+1).(m \text{ zeros}).i.j$. Now Lemma 1 completes the argument. $\square$

Note that $130000027999599 + 17 = 130000027999616$.

Lemma 6. *If each member of the set $\{M + e^2 \mid e = 2, 3, 4, 5, 6, 7, 8, 9\}$ is happy, then $M > 130000027999616$.*

Proof. Styer [2010], when dealing with fewer than fourteen consecutive happy numbers, did an exhaustive search on all values of M up to the needed bounds for his purposes. In order to reach a bound as high as 130000027999599, we order the digits of M. This makes the search approximately seven million times more efficient.

Write $M = M_2.f.(m \text{ nines}).g.h$ as in Lemma 5. Assume the digits of M_2 are ordered in nondecreasing order. For each m from 0 to 12, we have a separate Maple script that checks every possible M with the digits of M_2 ordered to see if each member of $\{M + e^2 \mid e = 2, 3, 4, 5, 6, 7, 8, 9\}$ is happy. A Maple program shows there are none. (For the relevant Maple worksheets, see [Lyons 2012].) $\square$

Lemma 7. *The final digit x of N_a satisfies $x \ge 3$.*

Proof. We assumed the existence of $N_a < N_0$ that begins a sequence of 14 consecutive happy numbers and we have written $N_a = N_1.x$ where x is a single digit. Suppose $x = 0$, 1, or 2. Then $N_1.e$ is happy with $e = 2, \dots, 9$. Thus $S(N_1) + e^2$ is happy with $e = 2, \dots, 9$. By the previous lemma, we have $S(N_1) > 13000002799916$. But $S(N_a) < 1300002799599$ by Lemma 3. Moreover,

$$S(N_1) = S(N_a) - x^2 \le S(N_a) - 4 < 1300002799595.$$

The upper and lower bounds we have for $S(N_1)$ contradict each other, so $x \ge 3$. $\square$

A set of Maple calculations similar to those in Lemma 6 yields the following lemma:

Lemma 8. *If each member of the set* $\{M+e^2 \mid e = 0, 1, 2, 3, 4, 5, 6, 7\}$ *is happy, then* $M > 130000027999616$.

Lemma 9. *The final digit* x *of* N_a *is* $x = 3$.

Proof. We know that $x \geq 3$ by Lemma 8. Suppose $x \geq 4$. Now the numbers $N_a + u = N_1.x + u$ are happy for $u = 0, 1, \ldots, 14$. If $x \geq 4$ these numbers include $(N_1+1).e$ with $e = 0, 1, \ldots, 7$. Therefore $S(N_1+1) > 13000002777616$. However, by Lemmas 3 and 4,

$$S(N_1+1) < 13000002799599 + 17 = 13000002799616,$$

giving a contradiction. Therefore $x = 3$. □

Lemma 10. *The value* $M_3 = 129999999997982$ *is the only* $M < 130000027999616$ *such that every member of* $\{M+e^2 \mid e = 0, 1, 2, 3, 4, 5, 6\}$ *is a happy number.*

Proof. Maple calculations similar to Lemma 5 give this single example with digits in nondecreasing order. While any other permutation of the leading 11 digits (the M_2 portion of M_3) will also result in every member of $\{M+e^2 \mid e = 0, 1, 2, 3, 4, 5, 6\}$ being a happy number, these permutations will give us an M value which exceeds our bound. □

Lemma 11. *The value of* $S(N_1)$ *must satisfy*

$$129999999997982 - 17 < S(N_1) < 130000027999599.$$

Lemma 12. *The only* M *with* $129999999997982 - 17 < M < 130000027999599$ *such that every member of* $\{M+e^2 \mid e = 3, 4, 5, 6, 7, 8, 9\}$ *is a happy number is* $M = 130000027999355$.

A Maple search over all the numbers within the bounds listed above returned this single result. Call this value M_1.

We now have the following relationships:

$$S(N_1) = S(N_2) + d^2 + 81k = 130000027999355 = M_1,$$
$$S(N_0+7) = S(N_2) + (d+1)^2 = 129999999997982 = M_3,$$
$$M_1 - M_3 = 81k - 2d - 1 = 28001373.$$

We look for integers k and d that satisfy this last relationship and find the sole solution $k = 345696$ and $d = 1$.

Now all that is left is to find the smallest N_2 that will satisfy these three equations. With $d = 1$, it reduces to $S(N_2) = 129999999997978$. Using the methods elaborated by Styer [2010], we easily find that the minimal N_2 with $S(N_2) = 129999999997978$

is $N_2 = 7888.(1604938271577 \text{ nines})$. Putting all this together we see that the smallest N beginning a sequence of fourteen consecutive happy numbers is indeed $N_0 = 7888.(1604938271577 \text{ nines}).1.(345696 \text{ nines}).3$.

3. Fifteen consecutive happy numbers

Using the same methods as outlined above, we have confirmed Styer's previous conjecture that the smallest number beginning a sequence of fifteen consecutive happy numbers is $N = 77.(2222222222222220 \text{ nines}).3.(97388).3$.

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