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We study the inertia space of \mathbb{R}^{2m} with the standard action of the special orthogonal group $\mathrm{SO}(2m)$. In particular, we indicate a decomposition of the inertia space that induces the *orbit Cartan type stratification* of the inertia space recently defined by C. Farsi, M. Pflaum, and the first author for an arbitrary smooth G -manifold where G is a compact Lie group.

1. Introduction

Let G be a compact Lie group, let M be a smooth, left G -manifold, and let $X = G \backslash M$ denote the orbit space of M . The *inertia space* ΛX is a topological space given by a subquotient of $G \times M$ under the diagonal G -action, where G acts by conjugation on the first factor. In [Farsi et al. 2012], an explicit Whitney stratification of the inertia space is presented, called the *orbit Cartan type stratification*, giving the inertia space the structure of a differentiable stratified space. This structure coincides with the notion of a stratified space with smooth structure — see [Pflaum 2001] — and simultaneously a differentiable space in the sense of [Navarro González and Sancho de Salas 2003]. In the case that G acts locally freely, so that $G \backslash M$ is an orbifold, the inertia space has played a major role in the study of the geometry of orbifolds; see [Adem et al. 2007], for instance. In general, the inertia space has appeared in connection with equivariant homology theories in noncommutative geometry [Brylinski 1987].

Recall that a *decomposition* of a topological space X is a locally finite partition of X into locally closed, smooth manifolds, called *pieces*, such that the frontier condition is satisfied: if $R \cap \bar{S} \neq \emptyset$ for pieces R and S , then $R \subseteq \bar{S}$. A *stratification* of X is an equivalence class of essentially identical decompositions, defined by assigning to each point $x \in X$ the germ at x of the piece containing x in a decomposition of a neighborhood of x . A decomposition of X *induces* a stratification

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if the germ assigned to x by the stratification coincides with the germ at x of the piece of the decomposition containing x . See [Pflaum 2001] for background on decomposed and stratified spaces.

In this note, we determine a decomposition of the inertia space for the standard action of the even special orthogonal group $\mathrm{SO}(2m)$ on \mathbb{R}^{2m} that induces the orbit Cartan type stratification. Our goal is to illustrate the computability of the stratification and to develop a large class of examples through which to better understand its properties.

The outline of this paper is as follows. In Section 2, we recall the definition of the inertia space and the orbit Cartan type stratification, and discuss facts about $\mathrm{SO}(2m)$ that we will need. In Section 3, we define the decomposition and prove that it has the required properties, recalling necessary information about the centralizers of elements of the standard maximal torus in $\mathrm{SO}(2m)$. We prove Theorem 3.2 by verifying the decomposition of the inertia space as well as its relationship to the stratification.

2. Background

In this section, we recall the orbit Cartan type stratification of the inertia space and collect the results we will need in the sequel. We use R_θ to indicate the 2×2 matrix

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

that acts (on the left) on \mathbb{R}^2 as a rotation through the angle θ . We say a value of θ is *generic* if θ is not congruent to $0 \pmod{2\pi}$ or $\pi \pmod{2\pi}$. Additionally, we use $\mathrm{diag}(A_1, \dots, A_\ell)$ to indicate the matrix in block form with diagonal blocks A_1, \dots, A_ℓ and 0 elsewhere. We let I_n denote the $n \times n$ identity matrix, or simply I when the dimensions are clear from the context, and we let $\langle x \rangle$ denote the span of an element $x \in \mathbb{R}^{2m}$.

The inertia space of a G -manifold and its stratification. We recall the following from [Farsi et al. 2012]. Note that $\mathrm{SO}(2m)$ is connected, and with respect to the standard action of $\mathrm{SO}(2m)$ on \mathbb{R}^{2m} , the isotropy group of each point $x \in \mathrm{SO}(2m)$ is connected. As this is our intended application, we specialize to this case for simplicity.

Let G be a compact, connected Lie group, let M be a smooth, left G -manifold, and let $X = G \backslash M$ denote the quotient space. The *loop space* ΛM of the G -manifold M is the set

$$\Lambda M := \{(h, x) \in G \times M \mid hx = x\}.$$

The loop space ΛM is clearly invariant under the action of G on $G \times M$ given by

$$g(h, x) = (ghg^{-1}, gx),$$

and the *inertia space* ΛX is defined to be the quotient of the loop space under this action.

Now, assume the isotropy group of each $x \in M$ is connected, and let $(h, x) \in \Lambda M$. Let $H = G_{(h,x)}$ denote the isotropy group of (h, x) in G , which is given by the centralizer $Z_{G_x}(h)$ of h in the isotropy group G_x of x , and choose a linear slice $V_{(h,x)}$ at (h, x) for the G -action on $G \times M$. By a *slice*, we mean a submanifold $V_{(h,x)}$ of $G \times M$ transversal to the orbit of (h, x) and satisfying these properties:

- $V_{(h,x)}$ is closed in its orbit $GV_{(h,x)}$, which is an open neighborhood of (h, x) in $G \times M$.
- $HV_{(h,x)} = V_{(h,x)}$.
- $gV_{(h,x)} \cap V_{(h,x)} \neq \emptyset$ implies $g \in H$.

A *linear slice* is H -equivariantly diffeomorphic to an H -invariant neighborhood of the origin in the normal space $T_{(h,x)}(G \times M)/T_{(h,x)}G(h, x)$ to the orbit at (h, x) , on which H acts linearly. See [Bredon 1972, II, Theorem 4.4] and [Koszul 1953].

As G_x is connected by hypothesis, we have, by [Duistermaat and Kolk 2000, Theorem 3.3.1(i)], that h is contained in the connected component of the identity H° of H . Therefore, we may choose a maximal torus $\mathbb{T}_{(h,x)}$ of H° containing h . We define an equivalence relation \sim on $\mathbb{T}_{(h,x)}$ by declaring that $t_1 \sim t_2$ for $t_1, t_2 \in \mathbb{T}_{(h,x)}$ if there is an open G -invariant neighborhood U of (h, x) such that $U^{t_1} = U^{t_2}$. This is the case if and only if $(GV_{(h,x)})^{t_1} = (GV_{(h,x)})^{t_2}$. We let $\mathbb{T}_{(h,x)}^*$ denote the \sim class of h in $\mathbb{T}_{(h,x)}$.

With this, the stratification of ΛM is given by assigning to (h, x) the germ of the set

$$G \left(V_{(h,x)}^H \cap (\mathbb{T}_{(h,x)}^* \times M) \right), \tag{2-1}$$

and the stratification of ΛX is given by assigning to the orbit $G(h, x)$ the germ of this G -invariant set. It is demonstrated in [Farsi et al. 2012] that ΛM equipped with this stratification has the structure of a differentiable Whitney stratified space, and moreover that ΛX inherits from this G -invariant stratification the structure of a differentiable Whitney stratified space. In particular, the germ at (h, x) of the set defined in (2-1) does not depend on the choice of slice nor on the choice of maximal torus $\mathbb{T}_{(h,x)}$, and the germ at $G(h, x)$ of the corresponding stratification of ΛX does not depend on the choice of representative (h, x) from the orbit $G(h, x)$.

Example 2.1. Consider the case $G = SO(2)$ with its standard action on $M = \mathbb{R}^2$. It is easy to see that

$$\Lambda \mathbb{R}^2 = \{(I, x) : x \in \mathbb{R}^2 \setminus \{0\}\} \cup \{(h, 0) : h \in SO(2)\} \subseteq SO(2) \times \mathbb{R}^2,$$

where $I \in \text{SO}(2)$ denotes the identity matrix. That is, $\Lambda\mathbb{R}^2$ is homeomorphic to \mathbb{R}^2 with a circle attached at the origin. The $\text{SO}(2)$ -isotropy group of points of the form $(h, 0)$ is $\text{SO}(2)$, while all other points have trivial isotropy. In particular, note that the partition of $\Lambda\mathbb{R}^2$ into isotropy types is not a decomposition, as the frontier condition fails at the point $(I, 0)$.

Any invariant neighborhood of a point $(h, 0)$ contains points with nonzero \mathbb{R}^2 -coordinate. Hence, the maximal torus $\mathbb{T}_{(h,0)} = \text{SO}(2)$ consists of two \sim classes: the identity fixing each point in any $\text{SO}(2)$ -invariant neighborhood, and $\text{SO}(2) \setminus \{I\}$, whose elements fix points of the form $(h, 0)$. Clearly, $\mathbb{T}_{(I,x)}$ is trivial for $x \neq 0$. It follows that a decomposition of $\Lambda\mathbb{R}^2$ inducing the orbit Cartan type stratification consists of three pieces:

$$\begin{aligned} \mathcal{P}_1 &= \{(I, x) : x \in \mathbb{R}^2 \setminus \{0\}\}, \\ \mathcal{P}_2 &= \{(h, 0) : h \in \text{SO}(2) \setminus \{I\}\}, \\ \mathcal{P}_3 &= \{(I, 0)\}. \end{aligned}$$

The $\text{SO}(2)$ -action on \mathcal{P}_1 is identified with the standard action on $\mathbb{R}^2 \setminus \{0\}$, while the action is trivial on \mathcal{P}_2 and \mathcal{P}_3 . Hence, the quotient space ΛX is homeomorphic to a ray with a circle attached to its endpoint.

The special orthogonal group $\text{SO}(n)$. The material in this section is well-known, and can be found in [Tapp 2005, Chapter 9]. See also [Bröcker and tom Dieck 1995, IV Section 3; Humphreys 1978, pages 64–5] for a description of the Weil group of $\text{SO}(n)$.

The *special orthogonal group* $\text{SO}(n)$ is the group of $n \times n$ orthogonal matrices with determinant 1. It is a compact, connected Lie group of dimension $n(n - 1)/2$. For an element $k \in \text{SO}(n)$, we let (k) denote the $\text{SO}(n)$ -conjugacy class of k .

If $n = 2m$ is even, then the *standard maximal torus* $\mathbb{T}_{2m}^{\text{st}}$ in $\text{SO}(n)$ is an m -dimensional torus given by the set of matrices of the form

$$\mathbb{T}_{2m}^{\text{st}} := \{\text{diag}(R_{\theta_1}, \dots, R_{\theta_m}) \mid \theta_i \in [0, 2\pi)\}.$$

The center of $\text{SO}(2m)$ is $\{I, -I\}$. The Weil group $N_{\text{SO}(2m)}(\mathbb{T}_{2m}^{\text{st}})/\mathbb{T}_{2m}^{\text{st}}$ is generated by all permutations of the angles $\theta_1, \dots, \theta_m$ as well as all transformations multiplying two angles by $-1 \pmod{2\pi}$.

If $n = 2m + 1$ is odd, then the standard maximal torus $\mathbb{T}_{2m+1}^{\text{st}}$ of $\text{SO}(2m + 1)$ is m -dimensional of the form

$$\mathbb{T}_{2m+1}^{\text{st}} := \{\text{diag}(1, R_{\theta_1}, \dots, R_{\theta_m}) \mid \theta_i \in [0, 2\pi)\},$$

and the center of $\text{SO}(2m + 1)$ is trivial. The Weil group of $\text{SO}(2m + 1)$ is generated by all permutations of $\theta_1, \dots, \theta_m$ and all transformations multiplying any angle by $-1 \pmod{2\pi}$.

3. The decomposition of $\Lambda\mathbb{R}^{2m}$

Statement of the decomposition. Let $k \in \text{SO}(2m)$. As every element of $\text{SO}(2m)$ is conjugate to an element of $\mathbb{T}_{2m}^{\text{st}}$, we may choose an element $h = \text{diag}(R_{\theta_1}, \dots, R_{\theta_m})$ of the $\text{SO}(2m)$ -conjugacy class of k contained in the standard maximal torus. Using the action of the Weil group, we may choose h with the θ_i listed in the following order. We first list all $\theta_i = 0$, followed by all $\theta_i = \pi$. Then, we list the remaining $\theta_i \neq 0$ in such a way that any angles that agree up to a sign $\pmod{2\pi}$ are listed consecutively.

Given such a choice of h , define $(a_0(h), a_\pi(h), \rho(h), s(h))$ as follows. Let $a_0(h)$, with $0 \leq a_0(h) \leq m$, denote the multiplicity of the angle 0; let $a_\pi(h)$, with $0 \leq a_\pi(h) \leq m - a_0(h)$, denote the multiplicity of π ; let ρ denote the (possibly empty) partition of $m - a_0(h) - a_\pi(h)$ indicating the number of generic angles that coincide up to a sign for each angle that occurs. Finally, if it is possible by the action of the Weil group to list all angles that coincide up to a sign with the same sign, we let $s(h) = +$; otherwise, we let $s(h) = -$. As elements of $\mathbb{T}_{2m}^{\text{st}}$ are conjugate in $\text{SO}(2m)$ if and only they are conjugate via an element of $N_{\text{SO}(2m)}(\mathbb{T}_{2m}^{\text{st}})$, it is easy to see that $(a_0(h), a_\pi(h), \rho(h), s(h))$ does not depend on the choice of h , and hence is constant on the conjugacy class of k . Hence, we define

$$(a_0(k), a_\pi(k), \rho(k), s(k)) = (a_0(h), a_\pi(h), \rho(h), s(h)).$$

We refer to $T(k) = (a_0(k), a_\pi(k), \rho(k), s(k))$ as the *type* of k , denoted simply $T = (a_0, a_\pi, \rho, s)$ when k is clear from the context.

Example 3.1. We now illustrate the types of elements of $\mathbb{T}_{2m}^{\text{st}}$.

- (1) The identity element I has type $(m, 0, \emptyset, +)$, while $-I$ has type $(0, m, \emptyset, +)$.
- (2) The element $h = \text{diag}(R_\theta, R_\theta, R_{-\theta}) \in \text{SO}(6)$ with θ generic has type $(0, 0, \{3\}, -)$. Note that any permutation of angles or multiplication of an even number of angles by $-1 \pmod{2\pi}$ will result in angles with different signs.
- (3) The element $h = \text{diag}(R_\theta, R_{-\theta}, R_\phi, R_\phi, R_{-\phi}) \in \text{SO}(10)$ with θ, ϕ generic has type $(0, 0, \{2, 3\}, +)$ because it is conjugate to $\text{diag}(R_\theta, R_\theta, R_\phi, R_\phi, R_\phi)$. On the other hand, $\text{diag}(R_\theta, R_{-\theta}, R_\phi, R_\phi, R_\phi)$ has type $(0, 0, \{2, 3\}, -)$.
- (4) An element $h = \text{diag}(R_0, R_\theta, R_{-\theta}) \in \text{SO}(6)$ with θ generic has type $(1, 0, \{2\}, +)$, because it is conjugate to $\text{diag}(R_0, R_\theta, R_\theta)$ via multiplication of the first and third angles by $-1 \pmod{2\pi}$.

For any k of type (a_0, a_π, ρ, s) , we have $0 \leq a_0 \leq m$ and $0 \leq a_\pi \leq m - a_0$. If $a_0 > 0$ or $a_\pi > 0$, then $s = +$; this follows from the fact that multiplication by $-1 \pmod{2\pi}$ fixes angles 0 and π , as in Example 3.1(4) above. We specify a specific partition $\rho(k)$ by a set with multiplicity, such as $\{1, 1, 2\}$, and adapt ordinary set operations in the obvious way: $\{1\} \cup \{1, 2\} = \{1, 1, 2\}$.

Given $k \in \mathrm{SO}(2m)$ of type (a_0, a_π, ρ, s) with $\rho = \{\rho_1, \dots, \rho_\ell\}$ listed in non-decreasing order, by the above observations, there is an element $h \in (k) \cap \mathbb{T}_{2m}^{\mathrm{st}}$ such that

$$h = \mathrm{diag}(I_{2a_0}, -I_{2a_\pi}, \underbrace{R_{\theta_1}, \dots, R_{\theta_1}}_{\rho_1}, \dots, \underbrace{R_{\theta_\ell}, \dots, R_{\pm\theta_\ell}}_{\rho_\ell}),$$

with each θ_i generic and $\theta_i \neq \pm\theta_j$ for $i \neq j$. In other words, the order and signs of the angles are chosen as above according to the ordering of ρ with at most one sign change, which is required to occur in the last position. We then say that h is in *standard form*. Note that h is unique if and only if $\rho_i \neq \rho_{i+1}$ for each i . We say that $h' \in \mathbb{T}_{2m}^{\mathrm{st}}$ of the same type as h is in the *same standard form as h* if

$$h' = \mathrm{diag}(I_{2a_0}, -I_{2a_\pi}, \underbrace{R_{\phi_1}, \dots, R_{\phi_1}}_{\rho_1}, \dots, \underbrace{R_{\phi_\ell}, \dots, R_{\pm\phi_\ell}}_{\rho_\ell}),$$

with each ϕ_i generic and $\phi_i \neq \pm\phi_j$ for $i \neq j$, so that the repeated angles and the single sign discrepancy, if it occurs, occur in the same positions. Given h and h' in the same standard form, for any $g \in N_{\mathrm{SO}(2m)}(\mathbb{T}_{2m}^{\mathrm{st}})$, we say that ghg^{-1} and $gh'g^{-1}$ are in the *same form*. That is, elements of $\mathbb{T}_{2m}^{\mathrm{st}}$ are in the same form if they are of the same type and can be put into the same standard form by the same element of the Weil group.

With this, we are ready to state our main result, which describes a decomposition of $\Lambda\mathbb{R}^{2m}$ that induces the orbit Cartan type stratification given by (2-1). We state the decomposition for the loop space $\Lambda\mathbb{R}^{2m}$, though a direct consequence is that the quotients of the pieces of the decomposition, which are $\mathrm{SO}(2m)$ -invariant and consist of points of the same isotropy type, define a decomposition of the inertia space $\Lambda(\mathrm{SO}(2m)\backslash\mathbb{R}^{2m})$ that induces the orbit Cartan type stratification.

Theorem 3.2. *For each type $T = (a_0, a_\pi, \rho, s)$, let*

$$P_{T,0} = \{(h, 0) \in \mathrm{SO}(2m) \times \mathbb{R}^{2m} : h \text{ has type } T\},$$

and let

$$P_{T,1} = \{(h, x) \in \mathrm{SO}(2m) \times (\mathbb{R}^{2m} \setminus \{0\}) : hx = x, h \text{ has type } T\}.$$

Then a decomposition of $\Lambda\mathbb{R}^{2m}$ inducing the orbit Cartan type stratification described by (2-1) is given by the following pieces:

- I. $P_{T,0}$ for each type $T = (a_0, a_\pi, \rho, s)$ such that $a_0 > 0$, $a_\pi > 1$, $s = -$, or $a_\pi = 0$ and $1 \notin \rho$,
- II. $P_{(0,1,\rho,+),0} \cup P_{(0,0,\{1\} \cup \rho,+),0}$ for each partition ρ of $m-1$,
- III. $P_{T,1}$ for each type $T = (a_0, a_\pi, \rho, +)$ such that $a_0 > 0$.

Note that an element h fixes a nonzero element of \mathbb{R}^{2m} if and only if $a_0(h) > 0$, and recall that $s(h) = +$ whenever $a_0(h) > 0$ or $a_\pi(h) > 0$.

Centralizers in $SO(2m)$. Let $h \in \mathbb{T}_{2m}^{\text{st}} \leq SO(2m)$ be in standard form. Then the centralizer of h in $SO(2m)$ is determined by the form of h . Specifically, let

$$h = \text{diag}(I_{2a_0}, -I_{2a_\pi}, \underbrace{R_{\theta_1}, \dots, R_{\theta_1}}_{\rho_1}, \dots, \underbrace{R_{\theta_\ell}, \dots, R_{\pm\theta_\ell}}_{\rho_\ell}),$$

with each θ_i generic and $\theta_i \neq \pm\theta_j$ for $i \neq j$, where we may have $a_0 = 0$ or $a_\pi = 0$. The centralizer of h in $SO(2m)$ is a set of matrices in blocks given by

$$\text{diag}(A, B, C_1, \dots, C_\ell),$$

where $A \in O(2a_0)$, $B \in O(2a_\pi)$, and each $C_i \in O(2\rho_i)$. Note that in general, $Z_{SO(2m)}(h)$ contains $\mathbb{T}_{2m}^{\text{st}}$. We first discuss the matrices C_i .

By direct computation, it is easy to see that the only 2×2 matrices that commute with R_θ for θ generic are given by

$$\begin{bmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{bmatrix},$$

i.e., a scalar multiple of a rotation matrix. The only $2\rho_i \times 2\rho_i$ matrices C_i that commute with $\text{diag}(R_\theta, \dots, R_\theta)$, where θ is generic and R_θ occurs ρ_i times, are matrices whose 2×2 blocks are scalar multiples of rotation matrices as above. Similarly, the only $2\rho_i \times 2\rho_i$ matrices that commute with $\text{diag}(R_\theta, \dots, R_\theta, R_{-\theta})$, where θ is generic and R_θ occurs $\rho_i - 1$ times, are matrices whose 2×2 block are as above except for the blocks in the last two rows and columns, excluding the lower-right 2×2 block, which are given by

$$\begin{bmatrix} c_1 & c_2 \\ c_2 & -c_1 \end{bmatrix}.$$

In particular, as these computations require only that θ is generic, all elements of the same form have the same centralizer.

If $a_\pi = 0$, then the set of elements of the same form as h is an open, dense subset of a torus of $SO(2m)$ of dimension ℓ , and $Z_{SO(2m)}(h)$ coincides with the centralizer of this torus. In particular, $Z_{SO(2m)}(h)$ is connected by [Duistermaat and Kolk 2000, Theorem 3.3.1]. Then as the determinant of each block is a continuous function from $Z_{SO(2m)}(h)$ to $\{\pm 1\}$, it must be that each C_i has determinant 1. It follows that A must have determinant 1, and hence A can be any element of $SO(2a_0)$. Note that we may also conclude for arbitrary h that each $C_i \in SO(2\rho_i)$.

If $a_\pi \neq 0$ and $a_0 \neq 0$, then $A \in O(2a_0)$ and $B \in O(2a_\pi)$ can be any elements with the same determinant ± 1 , and the centralizer of h has two connected components.

If $a_\pi \neq 0$ and $a_0 = 0$, then as the determinant of each C_i is 1, B must also have determinant 1 and can be any element of $\text{SO}(2a_\pi)$.

The reader is cautioned that it is possible for elements of different types to have identical centralizers. For instance, for θ_1 and θ_2 generic, $\theta_1 \neq \pm\theta_2$, the centralizers of the elements $\text{diag}(R_0, R_{\theta_2})$, $\text{diag}(R_\pi, R_{\theta_2})$, and $\text{diag}(R_{\theta_1}, R_{\theta_2})$ coincide and are equal to the standard maximal torus $\text{SO}(2) \times \text{SO}(2) \leq \text{SO}(4)$, though these elements are in standard form of type $(1, 0, \{1\}, +)$, $(0, 1, \{1\}, +)$, and $(0, 0, \{1, 1\}, +)$, respectively. More generally, if ρ is any partition of $m - 1$, then elements of type $(1, 0, \rho, +)$, $(0, 1, \rho, +)$, and $(0, 0, \{1\} \cup \rho, +)$ in standard form have the same centralizer, as in either case, the first 2×2 -block is forced to be an element of $\text{SO}(2)$. If h is in standard form and of either of these types, then any element with the same centralizer as h is also in standard form.

However, if $a_0(h) = a_0(h') > 0$ for elements $h, h' \in \mathbb{T}_{2m}^{\text{st}}$ such that h is in standard form and both h and h' have centralizer H , then h and h' are in the same standard form. In particular, if H is connected, then $a_\pi(h) = a_\pi(h') = 0$, and if H is not connected, then the size of the second block in elements of H determines that $a_\pi(h) = a_\pi(h') > 0$. The size and structure of the later blocks in elements of H determine the values of ρ and $s = +$ for both h and h' as well as their form. Similarly, if $a_0(h) = a_0(h') = 0$ and $a_\pi(h) > 1$ with h in standard form, then the size and structure of the blocks in the centralizer again determine the form of h and hence h' .

Finally, suppose $a_0(h) = a_\pi(h) = 0$ with h in standard form. If $1 \notin \rho(h)$, then the size and structure of the blocks of H determine the form of h and hence h' so that h' is in the same standard form as h . Otherwise, h has type $(0, 0, \{1\} \cup \rho, s)$ for a partition ρ of $m - 1$ and is in standard form

$$h = \text{diag}(R_{\theta_1}, \underbrace{R_{\theta_2}, \dots, R_{\theta_2}}_{\rho_1}, \dots, \underbrace{R_{\theta_\ell}, \dots, R_{\pm\theta_\ell}}_{\rho_{\ell-1}})$$

so that $\theta_1, \dots, \theta_\ell$ are generic. If $a_0(h') = 0$, then either h' is in the same standard form as h or

$$h' = \text{diag}(R_\pi, \underbrace{R_{\phi_1}, \dots, R_{\phi_1}}_{\rho_1}, \dots, \underbrace{R_{\phi_\ell}, \dots, R_{\pm\phi_\ell}}_{\rho_{\ell-1}}),$$

which has type $(0, 1, \rho, +)$ and is not in standard form if $s(h) = -$.

We now summarize these observations.

Proposition 3.3. *Elements of $\mathbb{T}_{2m}^{\text{st}}$ in the same standard form have the same centralizer, and elements of $\text{SO}(2m)$ of the same type have conjugate centralizers. Conversely, if $h, h' \in \mathbb{T}_{2m}^{\text{st}}$ with h in standard form and $Z_{\text{SO}(2m)}(h) = Z_{\text{SO}(2m)}(h')$, then:*

- *If $a_0(h) = a_0(h') > 0$, then h and h' are in the same standard form.*

- If $a_0(h) = a_0(h') = 0$ and $a_\pi(h) > 1$, then h and h' are in the same standard form.
- If $a_0(h) = a_\pi(h) = 0$ and $1 \notin \rho(h)$, then h and h' are in the same standard form.
- If $a_0(h) = a_0(h') = 0$ and h has type $(0, 1, \rho, +)$ or $(0, 0, \{1\} \cup \rho, +)$, then h' is in standard form and has type $(0, 1, \rho, +)$ or $(0, 0, \{1\} \cup \rho, +)$.
- If $a_0(h) = a_0(h') = 0$ and h has type $(0, 0, \{1\} \cup \rho, -)$, then either h' is in standard form of type $(0, 0, \{1\} \cup \rho, -)$ or h' is of type $(0, 1, \rho, +)$ and is not in standard form.

Note that Proposition 3.3 does not exhaust all cases but considers those that we will need below.

Proof of Theorem 3.2. In this section, we demonstrate that the partition defined in Theorem 3.2 is indeed a decomposition that induces the orbit Cartan type stratification. First, we establish the following.

Lemma 3.4. *Let $h \in \mathbb{T}_{2m}^{\text{st}}$ be in standard form. Then there is a neighborhood U of h in $\mathbb{T}_{2m}^{\text{st}}$ small enough so that every $h' \in U$ of the same type as h is in the same standard form as h . If h has type $(0, 0, \{1\} \cup \rho, -)$, then we may choose U so that it contains no elements h' such that $a_\pi(h') > 0$.*

Proof. Let $h = \text{diag}(R_{\theta_1}, \dots, R_{\theta_m})$, where angles need not be distinct or generic. Choose $\epsilon > 0$ such that $(\theta_i - \epsilon, \theta_i + \epsilon)$ contains 0 (respectively π) if and only if $\theta_i = 0$ (respectively π), and, for $i \neq j$, the intersection $(\theta_i - \epsilon, \theta_i + \epsilon) \cap (\pm\theta_j - \epsilon, \pm\theta_j + \epsilon)$ is nonempty if and only if $\theta_i = \pm\theta_j$. Then for any $h' = \text{diag}(\phi_1, \dots, \phi_m)$ such that $|\phi_i - \theta_i| < \epsilon$ for each i , h' is of the same type as h if and only if it is in the same standard form. Moreover, if h has type $(0, 0, \{1\} \cup \rho, -)$, then as $\theta_i \neq \pi$ for each i , U contains no elements of type $(0, a_\pi, \sigma, +)$ for $a_\pi > 0$ and any partition σ . \square

Lemma 3.5. *Let $h \in \mathbb{T}_{2m}^{\text{st}}$ be an element of the maximal torus of $SO(2m)$.*

- A linear slice $V_{(h,0)}$ for the diagonal $SO(2m)$ -action on $SO(2m) \times \mathbb{R}^{2m}$ at $(h, 0)$ can be chosen such that $V_{(h,0)}$ contains $U_h \times U_0$, where U_h is a neighborhood of h in $\mathbb{T}_{2m}^{\text{st}}$ and U_0 is a neighborhood of 0 in \mathbb{R}^{2m} .*
- If $0 \neq x \in \mathbb{R}^{2m}$ such that $hx = x$, then a linear slice $V_{(h,x)}$ for the diagonal $SO(2m)$ -action on $SO(2m) \times \mathbb{R}^{2m}$ at (h, x) can be chosen such that $V_{(h,x)}$ contains $U_h \times U_x$ where U_h is a neighborhood of h in $\mathbb{T}_{2m}^{\text{st}}$ and U_x is a connected neighborhood of x in the span $\langle x \rangle$ of x in \mathbb{R}^{2m} .*

Proof. Fix the standard ($SO(2m)$ -invariant) Riemannian metric on \mathbb{R}^{2m} , choose a bi-invariant metric on $SO(2m)$, and let $SO(2m) \times \mathbb{R}^{2m}$ carry the product metric. Recall that (h) denotes the $SO(2m)$ -conjugacy class of h . By [Duistermaat and Kolk

2000, Proposition 3.1.1], the only slice at h for the $SO(2m)$ -action on $SO(2m)$ by conjugation is given by a neighborhood S_h of h in the centralizer $Z_{SO(2m)}(h)$, where the linear structure is inherited from the Lie algebra \mathfrak{z}_h of $Z_{SO(2m)}(h)$ via a logarithmic chart. Because the orthogonal complement of $T_h(h)$ in $T_hSO(2m)$ with respect to the metric is mapped to a slice by the exponential map (see [Duis-termaat and Kolk 2000, Theorem 2.3.3]), it follows that $T_hS_h = T_hZ_{SO(2m)}(h)$ is the orthogonal complement of $T_h(h)$ in $T_hSO(2m)$.

As $SO(2m)(h, 0) = (h) \times \{0\} \subset SO(2m) \times \mathbb{R}^{2m}$, using the isometry

$$T_{(h,0)}(SO(2m) \times \mathbb{R}^{2m}) \rightarrow T_hSO(2m) \oplus T_0\mathbb{R}^{2m},$$

we have that

$$\begin{aligned} T_{(h,0)}(Z_{SO(2m)}(h) \times \mathbb{R}^{2m}) &\cong T_hZ_{SO(2m)}(h) \oplus T_0\mathbb{R}^{2m} \\ &\cong (T_{(h,0)}SO(2m)(h, 0))^\perp. \end{aligned}$$

Hence, a slice for the $SO(2m)$ -action on $SO(2m) \times \mathbb{R}^{2m}$ may be chosen to be a suitably small neighborhood of $(h, 0)$ in $Z_{SO(2m)}(h) \times \mathbb{R}^{2m}$. Clearly $\mathbb{T}_{2m}^{\text{st}} \leq Z_{SO(2m)}(h)$, proving (i).

To prove (ii), note that the orbit $SO(2m)x$ of x is given by the sphere of radius $\|x\|$, so that in $T_x\mathbb{R}^{2m}$, $(T_xSO(2m)x)^\perp = T_x\langle x \rangle$. Then as

$$T_{(h,x)}SO(2m)(h, x) \subseteq T_h(h) \oplus T_xSO(2m)x,$$

we have

$$\begin{aligned} T_hZ_{SO(2m)}(h) \oplus T_x\langle x \rangle &= (T_h(h))^\perp \oplus (T_xSO(2m)x)^\perp \\ &\subseteq (T_h(h) \times T_xSO(2m)x)^\perp \\ &\subseteq (T_{(h,x)}SO(2m)(h, x))^\perp. \end{aligned}$$

It follows that we may choose a slice $V_{(h,x)}$ at (h, x) such that

$$T_{(h,x)}V_{(h,x)} = (T_{(h,x)}SO(2m)(h, x))^\perp,$$

and hence an open neighborhood of (h, x) in $Z_{SO(2m)}(h) \times \langle x \rangle$ is contained in $V_{(h,x)}$. □

Proof of Theorem 3.2. Given an arbitrary element $(k, x) \in \Lambda\mathbb{R}^{2m}$, as k is conjugate to an element of $\mathbb{T}_{2m}^{\text{st}}$, the type of k is defined. Moreover, as the type is conjugation invariant, it is well defined, so that the pieces defined in I, II, and III clearly form a partition of $\Lambda\mathbb{R}^{2m}$. Moreover, as the number of types is finite, the partition is finite and hence trivially locally finite.

For each element (k, y) of a piece P , we now demonstrate that for some (h, x) in the orbit of (k, y) and appropriate choices of slice and maximal torus,

there is an open, $SO(2m)$ -invariant neighborhood of (h, x) within which the set $P \cap SO(2m)V_{(h,x)}$ coincides with the set defined in (2-1). This implies that the decomposition induces the orbit Cartan type stratification. Moreover, as the germs defining the stratification are germs of locally closed, smooth manifolds, it follows that each piece P is a locally closed, smooth submanifold of $SO(2m) \times \mathbb{R}^{2m}$. With this, we will need only show that the pieces satisfy the frontier condition.

I. Suppose $(k, 0)$ is of type $T = (a_0, a_\pi, \rho, s)$ with $a_0 > 0$, $a_\pi > 1$, $s = -$, or $a_\pi = 0$ and $1 \notin \rho$. Choose an element $h \in (k) \cap \mathbb{T}_{2m}^{st}$ in standard form and a slice $V_{(h,0)}$ at $(h, 0)$ for the $SO(2m)$ -action on $SO(2m) \times \mathbb{R}^{2m}$ with $U_h \times U_0 \subseteq V_{(h,0)}$ as in Lemma 3.5. Applying Lemma 3.4 and shrinking $V_{(h,0)}$ if necessary, we assume that if $(h', x) \in V_{(h,0)}$ with $h' \in \mathbb{T}_{2m}^{st}$ of the same type as h , then h' is in the same standard form as h . Moreover, if h has type $(0, 0, \{1\} \cup \rho, -)$, we assume that $V_{(h,0)}$ contains no elements of the form (h', x) such that $a_\pi(h') > 0$. Let $H = SO(2m)_{(h,0)} = Z_{SO(2m)}(h)$, and define the set

$$Q_{(h,0)} := V_{(h,0)}^H \cap ((\mathbb{T}_{2m}^{st})_{(h,0)}^* \times \mathbb{R}^{2m}).$$

That is, the $SO(2m)$ -saturation $SO(2m)Q_{(h,0)}$ is the set that defines the germ of the stratum containing $(h, 0)$ in (2-1). Note that as H contains \mathbb{T}_{2m}^{st} , which only fixes the origin in \mathbb{R}^{2m} , any element of $V_{(h,0)}^H$ is of the form $(h', 0)$ for $h' \in SO(2m)$. Moreover, as $h \in H$, it must be that for any $(h', 0) \in V_{(h,0)}^H$, the element h' commutes with h .

Let $(h', 0) \in Q_{(h,0)}$ be arbitrary. Then $h' \in (\mathbb{T}_{2m}^{st})_{(h,0)}^*$, implying that the h and h' fix the same subset of $SO(2m)V_{(h,0)}$. In particular, as $\{h\} \times U_0 \subseteq V_{(h,0)}$, with U_0 a neighborhood of the origin in \mathbb{R}^{2m} , and as h' commutes with h , it follows that $(\mathbb{R}^{2m})^h = (\mathbb{R}^{2m})^{h'}$, so $a_0(h) = a_0(h')$. Additionally, by the definition of slice, every point in $V_{(h,0)}$ has isotropy group contained in H , so $V_{(h,0)}^H$ consists only of points with isotropy group equal to H . Hence $Z_{SO(2m)}(h) = Z_{SO(2m)}(h')$, so by Proposition 3.3 and the choice of slice, h and h' are in the same standard form. It follows that the orbit of any element of $Q_{(h,0)}$ is contained in $P_{T,0}$ and hence $SO(2m)Q_{(h,0)} \subseteq P_{T,0}$.

Conversely, if $(k', 0) \in P_{T,0} \cap SO(2m)V_{(h,0)}$ so that k' is of the same type as h , then by the choice of $V_{(h,0)}$, there is an $(h', 0) \in V_{(h,0)} \cap SO(2m)(k', 0)$ such that h' is in the same standard form as h . Then h and h' have the same centralizer by Proposition 3.3 so that $(h', 0) \in V_{(h,0)}^H$. Moreover, because $Z_{SO(2m)}(h) = Z_{SO(2m)}(h')$ and the angle 0 occurs in the same positions in both, h and h' fix the same elements of $SO(2m) \times \mathbb{R}^{2m}$ so that clearly $(SO(2m)V_{(h,0)})^h = (SO(2m)V_{(h,0)})^{h'}$. Hence $(h', 0) \in Q_{(h,0)}$. Therefore, we have that $SO(2m)Q_{(h,0)} = P_{T,0} \cap SO(2m)V_{(h,0)}$, so that $SO(2m)Q_{(h,0)}$ and $P_{T,0}$ define the same germ at $(h, 0)$.

II. The argument in this case is similar to I above. Choosing a representative $(h, 0)$ of the orbit of an arbitrary point with $h \in \mathbb{T}_{2m}^{\text{st}}$ in standard form, for any $h' \in (\mathbb{T}_{2m}^{\text{st}})_{(h,0)}^*$, as h and h' have the same fixed point set in \mathbb{R}^{2m} , $a_0(h) = 0$ implies $a_0(h') = 0$. In this case, however, while elements $h, h' \in \mathbb{T}_{2m}^{\text{st}}$ of the same type have the same centralizer, the centralizers do not distinguish between group elements in standard form of type $(0, 1, \rho, +)$ and $(0, 0, \{1\} \cup \rho, +)$ by Proposition 3.3. Moreover, any neighborhood of an element in standard form of type $(0, 1, \rho, +)$ clearly contains elements in standard form of type $(0, 0, \{1\} \cup \rho, +)$. As the fixed-point sets of such elements in $\text{SO}(2m) \times \mathbb{R}^{2m}$ coincide, the argument is identical to that of I combining these two types.

III. Let $(k, x) \in \Lambda \mathbb{R}^{2m}$ and let T be the type of k . As the $\text{SO}(2m)$ -action on \mathbb{R}^{2m} is transitive on spheres about the origin, we may assume that x has coordinates $(\|x\|, 0, \dots, 0)$, and hence $\text{SO}(2m)_x = \{\text{diag}(1, A) : A \in \text{SO}(2m-1)\} \cong \text{SO}(2m-1)$. As any element of $\text{SO}(2m)_x$ is conjugate to an element of the standard maximal torus $\mathbb{T}_{2m-1}^{\text{st}}$ via an element of $\text{SO}(2m)_x$, we may choose an element (h, x) in the orbit $\text{SO}(2m)_x(k, x)$ such that $h \in \mathbb{T}_{2m-1}^{\text{st}}$ is in standard form. Note that as h fixes x , we have $a_0(h) > 0$.

Choose a slice $V_{(h,x)}$ at (h, x) that contains $U_h \times U_x$ as in Lemma 3.5, and shrink $V_{(h,x)}$ if necessary so that $V_{(h,x)} \cap (\text{SO}(2m) \times -U_x) = \emptyset$. We again assume by Lemma 3.4 and shrinking $V_{(h,x)}$ that for any $(h', y) \in V_{(h,x)}$ such that $h' \in \mathbb{T}_{2m}^{\text{st}}$ has the same type as h , h' must also have the same form.

It will be convenient to restrict to a smaller open neighborhood of (h, x) in $\text{SO}(2m) \times \mathbb{R}^{2m}$. To do so, recall that the Weil group $N_{\text{SO}(2m)}(\mathbb{T}_{2m}^{\text{st}})/\mathbb{T}_{2m}^{\text{st}}$ is finite. Hence, by [tom Dieck 1987, Proposition 3.23], we may shrink U_h to assume that for $g \in N_{\text{SO}(2m)}(\mathbb{T}_{2m}^{\text{st}})$, we have $gU_h = U_h$ if $g \in Z_{\text{SO}(2m)}(h)$ and $U_h \cap gU_h = \emptyset$ otherwise. Moreover, letting $\text{SO}(2m)_*$ denote the set of conjugacy classes in $\text{SO}(2m)$ equipped with its natural quotient topology, we may assume that the quotient of U_h by $N_{\text{SO}(2m)}(\mathbb{T}_{2m}^{\text{st}})/Z_{\text{SO}(2m)}(h)$ is homeomorphic to an open subset of $\text{SO}(2m)_*$ containing (h) . In particular, as the quotient map $\text{SO}(2m) \rightarrow \text{SO}(2m)_*$ is continuous, $\text{SO}(2m)U_h$ is open in $\text{SO}(2m)$. Let $W = (\text{SO}(2m)U_h) \times (\text{SO}(2m)U_x)$, and then as $\text{SO}(2m)U_x = \{z \in \mathbb{R}^{2m} : \epsilon_1 < \|z\| < \epsilon_2\}$ for some $0 < \epsilon_1 < \epsilon_2$, $\text{SO}(2m)U_x$ is open in \mathbb{R}^{2m} . Hence W is an open, $\text{SO}(2m)$ -invariant neighborhood of (h, x) in $\text{SO}(2m) \times \mathbb{R}^{2m}$. Finally, we further shrink $V_{(h,x)}$ if necessary to assume that it does not intersect $gU_h \times \mathbb{R}^{2m}$ for any of the finite translates of U_h by $g \in N_{\text{SO}(2m)}(\mathbb{T}_{2m}^{\text{st}})$ such that $g \notin Z_{\text{SO}(2m)}(h)$. We will show that the piece $P_{T,1}$ coincides with the set given in (2-1) when intersected with W .

Let $H = \text{SO}(2m)_{(h,x)} = Z_{\text{SO}(2m-1)}(h)$ so that H consists of those elements of $Z_{\text{SO}(2m)}(h)$ whose first row and column are that of the identity. Define the set

$$Q_{(h,x)} := V_{(h,x)}^H \cap ((\mathbb{T}_{2m-1}^{\text{st}})_{(h,x)}^* \times \mathbb{R}^{2m}) \cap W.$$

Fix $(h', y) \in \mathcal{Q}_{(h,x)}$ so that $h' \in (\mathbb{T}_{2m-1}^{st})^*_{(h,x)}$. Therefore, as any neighborhood of (h, x) contains points (h, y') where any coordinate of y' except the first may be chosen to be zero or nonzero, and as $h \in H$ so that h and h' commute, we have that h and h' must have 0 occur as an angle with the same multiplicity in the same positions. Therefore, $a_0(h') = a_0(h) > 0$. Note that $(h', y) \in V_{(h,x)}^H$ so that $SO(2m)_{(h',y)} = Z_{SO(2m),y}(h') = H$. In particular, as $(h', y) \in W \cap V_{(h,x)}$ and $h' \in \mathbb{T}_{2m-1}^{st} \leq \mathbb{T}_{2m}^{st}$, we may conclude h' is in U_h . We consider two cases:

If $a_0(h) > 1$ or $a_\pi(h) > 0$, then H contains \mathbb{T}_{2m-1}^{st} as well as the element $g = \text{diag}(1, -1, -1, 1, I_{2m-4})$. The fixed point set in \mathbb{R}^{2m} of the group generated by g and \mathbb{T}_{2m-1}^{st} is $\langle x \rangle$, so that $y \in \langle x \rangle$. Then as $a_0(h') = a_0(h)$, connectedness of H determines whether $a_\pi(h)$, and hence $a_\pi(h')$, vanish. If not, the second block of elements of H indicates that $a_\pi(h') = a_\pi(h)$, and the following blocks further indicate that h and h' have the same type. Therefore, $(h', y) \in P_{T,1}$.

If $a_0(h) = 1$ and $a_\pi(h) = 0$, then every element of H , and in particular h' , is given by $\text{diag}(I_2, D)$ for a $(2m-2) \times (2m-2)$ matrix D . As H contains \mathbb{T}_{2m-1}^{st} which then must fix y , it follows that $y = (a, b, 0, \dots, 0)$ for some $a, b \in \mathbb{R}$. Then there is a $\bar{g} = \text{diag}(R_\theta, I_{2m-2})$ such that $\bar{g}y = (\|y\|, 0, \dots, 0)$. Moreover, as $h' = \text{diag}(I_2, D)$ for some D , we have $\bar{g}h'\bar{g}^{-1} = h'$, and $\bar{g}(h', y) = (h', (\|y\|, 0, \dots, 0))$. However, as $y \in SO(2m)U_x$, and $\bar{g}y \in \langle x \rangle$ has positive first coordinate, it follows that $\bar{g}y \in U_x$. Moreover, as $h' \in U_h$, we have $\bar{g}(h', y) \in U_h \times U_x \subseteq V_{(h,x)}$, so that as $(h', y) \in V_{(h,x)}$, it follows from the definition of slice that $\bar{g} \in H$. Then as elements of H fix y , we have that $y = (\|y\|, 0, \dots, 0)$ to begin with.

With this, the element $g = \text{diag}(1, -1, -1, 1, I_{2m-4})$ fixes y and hence, as it is not an element of H , cannot commute with h' . It follows that $a_\pi(h') = 0$, and then the structure of blocks of elements of H imply that h and h' have the same type. We again have $(h', y) \in P_{T,1}$, and hence $SO(2m)\mathcal{Q}_{(h,x)} \subseteq P_{T,1}$, since $P_{T,1}$ is $SO(2m)$ -invariant.

Conversely, if $(k, y) \in P_{(T,1)} \cap SO(2m)V_{(h,x)} \cap W$, then (k, y) is in the orbit of an element $(h', y') \in V_{(h,x)}$. Then as h' has the same type as h , it must have the same standard form as h . This implies that h' and h have the same centralizer, and moreover that $a_0(h') = a_0(h) > 0$. Noting that h' fixes y' , and hence that y' has nonzero coordinates only in the first $2a_0(h)$ positions, there is an element $\bar{g} = \text{diag}(D, I_{2(m-a_0(h))}) \leq SO(2m)$ for some $D \in SO(2a_0(h))$ such that $\bar{g}y' = (\|y'\|, 0, \dots, 0)$. As $(h', y') \in W \cap V_{(h,x)}$ and $h' \in \mathbb{T}_{2m}^{st}$, $h' \in U_h$. Hence, as \bar{g} commutes with h' , $\bar{g}(h', y') = (h', (\|y'\|, 0, \dots, 0)) \in U_h \times U_x \subseteq V_{(h,x)}$. That h and h' have the same centralizer and $\bar{g}y' \in \langle x \rangle$ implies $\bar{g}(h', y') \in V_{(h,x)}^H$. In addition, that h and h' have the same type implies $h' \in (\mathbb{T}_{2m-1}^{st})^*_{(h,x)}$. It follows that $\bar{g}(h', y') \in \mathcal{Q}_{(h,x)} \cap W$ so that $(k, y) \in SO(2m)\mathcal{Q}_{(h,x)} \cap W$, completing the proof that $SO(2m)\mathcal{Q}_{(h,x)} \cap W = P_{(T,1)} \cap SO(2m)V_{(h,x)} \cap W$.

The frontier condition. To show that the pieces defined in Theorem 3.2 satisfy the frontier condition, we first claim that $k \in \text{SO}(2m)$ is in the closure in $\text{SO}(2m)$ of the set of elements of type T if and only if some conjugate gkg^{-1} of k is in the closure in $\mathbb{T}_{2m}^{\text{st}}$ of the set of elements of type T in standard form. Note that gkg^{-1} itself need not be in standard form.

Let $\{k_i\}_{i \in \mathbb{N}}$ be a convergent sequence of elements of $\text{SO}(2m)$ that are all of the same type $T = (a_0, a_\pi, \rho, s)$, and let $k = \lim_{i \rightarrow \infty} k_i \in \text{SO}(2m)$. Then for each i , there is a g_i such that $g_i k_i g_i^{-1} \in \mathbb{T}_{2m}^{\text{st}}$ is of standard form. By compactness of $\text{SO}(2m)$, we may assume by passing to a subsequence that the g_i converge to some $g \in \text{SO}(2m)$. Then by continuity of the action by conjugation and as $\mathbb{T}_{2m}^{\text{st}}$ is closed, we have

$$gkg^{-1} = \lim_{i \rightarrow \infty} g_i k_i g_i^{-1} \in \mathbb{T}_{2m}^{\text{st}}.$$

Conversely, if k is conjugate to some $gkg^{-1} \in \mathbb{T}_{2m}^{\text{st}}$, where gkg^{-1} is the limit of a sequence $\{h_i\}_{i \in \mathbb{N}}$ of elements in $\mathbb{T}_{2m}^{\text{st}}$ of the same type T in standard form, then $g^{-1}h_i g$ is a sequence of elements of type T that converges to k .

Now, for a type $T = \{a_0, a_\pi, \rho, s\}$ with $\rho = \{\rho_1, \dots, \rho_\ell\}$, let $\mathbb{T}_{2m}^{\text{st}}(T)$ denote the set of elements in $\mathbb{T}_{2m}^{\text{st}}$ in standard form of type T . Suppose $h \in \overline{\mathbb{T}_{2m}^{\text{st}}(T)}$ so that there is a sequence $\{h_i\}_{i \in \mathbb{N}} \subseteq \mathbb{T}_{2m}^{\text{st}}(T)$ such that $h_i \rightarrow h$. Recall that if $s = -$, then the sign discrepancy in the angles of the h_i is taken to be in the final position, corresponding to ρ_ℓ . As each h_i has I and $-I$ in the first a_0 and a_π positions, respectively, it follows that h must as well. Similarly, letting $\theta_{j,i}$ denote the angle in the ρ_j position of h_i for $j = 1, \dots, \ell$, we have that $\lim_{i \rightarrow \infty} \theta_{j,i}$ exists and is given by θ_j , the angle in the corresponding position of h , which can have any value. Let $J = \{j \in \{1, \dots, \ell\} : \theta_j = 0\}$, and let $J' = \{j \in \{1, \dots, \ell\} : \theta_j = \pi\}$. As it may be the case that the θ_j are not distinct, let σ denote the partition formed from $\rho \setminus \{\rho_j : j \in I \cup J\}$ by summing elements ρ_j and $\rho_{j'}$ when $\theta_j = \theta_{j'}$. Then if $s = +$ or θ_ℓ is generic, h has type

$$\left(a_0 + \sum_{j \in J} \rho_j, a_\pi + \sum_{j \in J'} \rho_j, \sigma, s \right),$$

while if $s = -$ and $\theta_\ell \in \{0, \pi\}$, h has type

$$\left(a_0 + \sum_{j \in I} \rho_j, a_\pi + \sum_{j \in J} \rho_j, \sigma, + \right).$$

Given an arbitrary element h' of $\mathbb{T}_{2m}^{\text{st}}$ of the same form as h , it is easy to see that one can define a sequence $\{h'_i\}_{i \in \mathbb{N}}$ of elements of type T such that $h'_i \rightarrow h'$ simply by redefining the angles in the h_i corresponding to $j \notin J \cup J'$ to converge to those of h' , choosing distinct sequences when $\theta_j = \theta_{j'}$ for $j \neq j'$ as above. It follows that

if $h \in \overline{\mathbb{T}_{2m}^{\text{st}}(T)}$, then every element of $\mathbb{T}_{2m}^{\text{st}}$ of the same form of h is contained in $\overline{\mathbb{T}_{2m}^{\text{st}}(T)}$. However, by applying the Weil group to this sequence, it then follows that every element of $\mathbb{T}_{2m}^{\text{st}}$ of the same type as h is contained in the closure of elements of type T in $\mathbb{T}_{2m}^{\text{st}}$. This claim extends by conjugation to all of $SO(2m)$ as above, so we conclude that the partition of $SO(2m)$ into types satisfies the frontier condition.

Finally, note that this partition still satisfies frontier if we combine types of the form $(0, 1, \rho, s)$ and $(0, 0, \{1\} \cup \rho, s)$. If the set of elements of type T contains points of type $(0, 1, \sigma, s)$ in its closure, then T must itself be of the form either $(0, 1, \rho, s)$ or $(0, 0, \{1\} \cup \rho, s)$, where σ is formed from ρ or $\{1\} \cup \rho$ by summing elements as above. As these types are also combined, the resulting set must contain all elements of type $(0, 1, \sigma, s)$ and $(0, 0, \{1\} \cup \sigma, s)$ in its closure.

With this, we need only note that as the closure of $\mathbb{R}^{2m} \setminus \{0\}$ is clearly \mathbb{R}^{2m} , by inspection, the pieces of type I, II, and III satisfy the frontier condition. Hence, by $SO(2m)$ -invariance of these pieces, frontier is satisfied in the quotient as well. \square

It is of interest to note that the sets of type III form a decomposition of the loop space of the $SO(2m)$ -space $\mathbb{R}^{2m} \setminus \{0\}$. Because each point in $\Lambda(\mathbb{R}^{2m} \setminus \{0\})$ is contained in an $SO(2m)$ -invariant neighborhood in $\Lambda\mathbb{R}^{2m}$ that does not intersect $SO(2m) \times \{0\}$, it follows that this decomposition induces the orbit Cartan type stratification of the inertia space $\Lambda(SO(2m) \setminus (\mathbb{R}^{2m} \setminus \{0\}))$.

The loop space $\Lambda(\mathbb{R}^{2m} \setminus \{0\})$ is the loop space of a $SO(2m)$ -manifold with a single isotropy type and hence is a smooth manifold by [Farsi et al. 2012, Proposition 4.4]. Given an element $(h, x) \in \Lambda(\mathbb{R}^{2m} \setminus \{0\})$, where we may assume up to conjugation that $x = (\|x\|, 0, \dots, 0)$ and $h \in \mathbb{T}_{2m-1}^{\text{st}}$ is in standard form as above, it must be that $a_0(h) > 0$. Hence, as the types of such elements are determined by their centralizers by Proposition 3.3, the decomposition of $\Lambda(\mathbb{R}^{2m} \setminus \{0\})$, and hence the associated inertia space, corresponds to the decomposition into isotropy types, demonstrating that the orbit Cartan type stratification of this $SO(2m)$ -manifold coincides with its stratification by isotropy types. This is not generally true for the odd case, as it fails in the case of $SO(3)$ acting on $\mathbb{R}^3 \setminus \{0\}$ described in [Farsi et al. 2012, Section 4.2.6].

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References

- [Adem et al. 2007] A. Adem, J. Leida, and Y. Ruan, *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics **171**, Cambridge University Press, Cambridge, 2007. MR 2009a:57044 Zbl 1157.57001
- [Bredon 1972] G. E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics **46**, Academic Press, New York, 1972. MR 54 #1265 Zbl 0246.57017
- [Bröcker and tom Dieck 1995] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics **98**, Springer, New York, 1995. MR 97i:22005 Zbl 0874.22001
- [Brylinski 1987] J.-L. Brylinski, “Cyclic homology and equivariant theories”, *Ann. Inst. Fourier (Grenoble)* **37**:4 (1987), 15–28. MR 89j:55008 Zbl 0625.55003
- [tom Dieck 1987] T. tom Dieck, *Transformation groups*, de Gruyter Studies in Mathematics **8**, De Gruyter, Berlin, 1987. MR 89c:57048 Zbl 0611.57002
- [Duistermaat and Kolk 2000] J. J. Duistermaat and J. A. C. Kolk, *Lie groups*, Springer, Berlin, 2000. MR 2001j:22008 Zbl 0955.22001
- [Farsi et al. 2012] C. Farsi, M. Pflaum, and C. Seaton, “Inertia spaces of proper Lie group actions and their topological properties”, preprint, 2012. arXiv 1207.0595v1
- [Humphreys 1978] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics **9**, Springer, New York, 1978. MR 81b:17007 Zbl 0447.17001
- [Koszul 1953] J. L. Koszul, “Sur certains groupes de transformations de Lie”, *Colloques Int. Centre Nat. Rech. Sci.* **52** (1953), 137–141. MR 15,600g Zbl 0101.16201
- [Navarro González and Sancho de Salas 2003] J. A. Navarro González and J. B. Sancho de Salas, *C^∞ -differentiable spaces*, Lecture Notes in Mathematics **1824**, Springer, Berlin, 2003. MR 2005c:58008 Zbl 1039.58001
- [Pflaum 2001] M. J. Pflaum, *Analytic and geometric study of stratified spaces*, Lecture Notes in Mathematics **1768**, Springer, Berlin, 2001. MR 2002m:58007 Zbl 0988.58003
- [Tapp 2005] K. Tapp, *Matrix groups for undergraduates*, Student Mathematical Library **29**, American Mathematical Society, Providence, RI, 2005. MR 2006d:20001 Zbl 1089.20001

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
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