involve
a journal of mathematics

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msp

2013 vol. 6, no. 4

INVOLVE 6:4 (2013)
dx.doi.org/10.2140/involve.2013.6.493

On the difference between an integer and the sum of its proper divisors

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(Communicated by Kenneth S. Berenhaut)

Let $\sigma(n)$ be the sum of the divisors of n. Although much attention has been paid to the possible values of $\sigma(n) - n$ (the sum of proper divisors), comparatively little work has been done on the possible values of $e(n) := \sigma(n) - 2n$. Here we present some theoretical and computational results on these values. In particular, we exhibit some infinite and possibly infinite families of integers that appear in the image of $e(n)$. We also find computationally all values of $n < 10^{20}$ for which $e(n)$ is odd, and we present some data from our computations. At the end of this paper, we present some conjectures suggested by our computational work.

1. Introduction and background

Let $s(n)$ be the sum of the proper divisors of n, so that $s(n) = \sigma(n) - n$, where $\sigma(n)$ represents the standard sum-of-divisors function. We shall refer to the value by which the sum of the proper divisors of an integer n exceeds n as the *excedent* of n, which we denote by $e(n)$, so that $e(n) := s(n) - n$, or $e(n) := \sigma(n) - 2n$. In a sense, values of $e(n)$ have been studied since antiquity. The Pythagoreans, for example, were especially interested in finding those n for which $e(n) = 0$. These are the *perfect* numbers. Today we also use the ancient Greek descriptors *deficient* and *abundant* to refer to those integers n for which $e(n) < 0$ and $e(n) > 0$, respectively.

More recently, some particular values of the excedent of n have been studied in the literature. Most noteworthy is the case where $e(n) = 1$. An integer for which $e(n) = 1$ is said to be *quasiperfect*. Quasiperfect numbers were first studied by Cattaneo [1951], who referred to $e(n)$ as the *eccedenza* of n, partly inspiring our choice of the English word *excedent*. Cattaneo showed that a quasiperfect number must be an odd square, and that if it is relatively prime to 3, it must have at least seven distinct prime factors. These results have since been improved. We now know, for example, that if n is a quasiperfect number:

MSC2010: 11A25, 11Y70.
Keywords: sigma function, sum of divisors, excedents, computational mathematics.

(1) $n = k^2$, where k is odd [Cattaneo 1951];

(2) if m is a proper divisor of n, then $\sigma(m) < 2n$ [Cattaneo 1951];

(3) if $r \mid \sigma(n)$ then $r \equiv 1$ or $3 \pmod 8$ [Cattaneo 1951];

(4) n has at least seven prime factors [Hagis and Cohen 1982];

(5) $n > 10^{35}$ [Hagis and Cohen 1982].

Despite this impressive list, however, the biggest question concerning quasiperfect numbers, namely, *do quasiperfect numbers exist?*, remains unanswered. In the language of this paper, we could say that we still don't know whether there are integers n for which $e(n) = 1$.

There seems to have been only one attempt to pursue more general questions of this sort. In his Ph.D. thesis (see [Cohen 1982] for a summary), Cohen considered a generalization of quasiperfect numbers. According to his definition, a *k-quasiperfect number* is an integer n for which $s(n) - n = k^2$ for a positive integer k relatively prime to n. He proved, among other things, that if such numbers exist, they must be larger than 10^{20} and have at least four distinct prime factors.

In this work, we wish to broaden Cohen's definition of a k-quasiperfect number to allow for any integer value of k. Then a 0-*quasiperfect number* is just a perfect number, a 1-*quasiperfect number* is the integer normally defined as a quasiperfect number, and we could denote the integers which Cohen considered simply as k-*quasiperfect numbers for square* k.

Our primary goal is to classify those integers m that are in the image of the excedent function. We call these integers *excedents*. Integers not in the image of $e(n)$ we call *nonexcedents*. The general problem of how to determine whether a given integer is an excedent seems very hard, however, and we are far from a complete classification. We do, however, give a few results concerning infinite and two potentially infinite families of excedents. We also give a conjecture, based on extensive computational evidence, about which small values of m are excedents, and which are nonexcedents.

2. Related work

It is worth noting that although references to values of $s(n) - n$ in the literature are fairly rare, some work has been done on values of $\sigma(n) - n$. Erdős [1973] showed that there are infinitely many numbers m for which $\sigma(n) - n = m$ has no solution, and furthermore that these m have positive lower density. Chen and Zhao [2011] have recently improved this to show that the density of these m is at least 0.06. Pomerance [1975] has considered a more general case, considering the set

$$S(a) = \{n : \sigma(n) \equiv a \pmod n\}.$$

He showed that for all a, the set $S(a)$ has at least two elements.

More recently, there has been an increase in interest in topics related to the values of $s(n)-n$. Anavi, Pollack and Pomerance [Anavi et al. 2013] show that the number of elements not greater than x in $S(a)$ (not counting those in a certain "obvious" set involving multiples of perfect and multiply perfect numbers) is bounded by $x^{1/2+o(1)}$ for each $|a| \le x^{1/4}$. Since $\sigma(n) \equiv e(n) \pmod{n}$, this immediately gives an upper bound on the number of n up to x for which $e(n) \equiv a \pmod{n}$ as well. One conclusion is that there can be no more than $x^{1/2+o(1)}$ k-quasiperfect integers up to x (outside of the obvious set) for any $k \le x^{1/4}$.

is studied in [Pollack and Shevelev 2012]. These are integers whose excedent is equal to one of the divisors. Finally, it is shown in [Pollack and Pomerance 2013] that for odd k, the number of k-quasiperfect numbers that are $\le x$ is at most $x^{1/4+o(1)}$ as $x \to \infty$.

Somewhat disappointingly, a close study of the references in this paper, including several suggested by the referee, show that the first three theorems in this paper have already appeared in some form in the literature. We shall still give our statements (and in one case, our proof) of these theorems in the hope that they may offer two things. First, we present and prove our theorems in an elementary manner. Second, our independent discovery of these results play an important role in our story, and help to motivate much of the computational work in the latter parts of the paper.

3. Computational experiments

Most computations for this work were conducted using PARI/GP. Initially, we computed $e(n)$ for all n in the range $[1, 10^{10}]$. We then recorded the number of times an integer m occurred as a value of $e(n)$ in this range. Let $N_m(x)$ be the number of integers $n \le x$ for which $e(n) = m$. Values for some small m from our computation are given in Table 1. It is worth noting that there are several methods which can speed up the computation of many values of $e(n)$. We worked primarily by isolating the values in which we were especially interested. A clever method for finding all numbers not in the image of $s(n)$ up to a given bound has recently been described in [Pomerance and Yang 2012].

In looking at this data, a few things immediately stand out. The most obvious is that there are many integers whose excedent is 12. Slightly less obvious, perhaps, is what seems to be a bias toward even values of the excedent function. These observations would guide our initial work.

It is clear that 12 is in the image of $e(n)$ quite often, leading us to ask immediately if there are other values which appear very often. Extending our search, we found a few other values of m for which there are a large number of integers n with $e(n) = m$, namely $m = 56$ and $m = 992$. A bit of consideration reveals that the

m	$N_m(10^{10})$	m	$N_m(10^{10})$
1	0	−1	32
2	9	−2	4
3	1	−3	0
4	10	−4	14
5	0	−5	1
6	3	−6	8
7	1	−7	1
8	25	−8	15
9	0	−9	0
10	3	−10	9
11	0	−11	1
12	78505339	−12	7
13	0	−13	0
14	6	−14	4
15	0	−15	0
16	20	−16	35
17	1	−17	0
18	10	−18	5
19	1	−19	2
20	20	−20	15

Table 1. Number of integers $n \leq 10^{10}$ for which $e(n) = m$ for small m.

numbers $12, 56, 992$ are precisely double the first three perfect numbers, $6, 28, 496$, leading us to suspect that integers that are exactly double perfect numbers may come up unusually often. We proved that each of these numbers in fact occurs infinitely often (see Theorem 2), and we note that these numbers are a special case of the set described in [Anavi et al. 2013]. The "obvious" set mentioned above contains multiples of both perfect and multiply perfect numbers — it's clear that these values of the excedent function behave quite differently than do other values. Anavi et al. [2013] refer to these as *regular* solutions of $\sigma(n) \equiv a \pmod{n}$, as opposed to the other, *sporadic* solutions.

Similarly, the observation that odd numbers occur in the image of $e(n)$ infrequently led us to seek a classification for those n with $e(n)$ odd. We succeeded in completely classifying these values; see Theorem 1.

A final observation we made was that, among the odd values of $s(n)$, many integers that are one less than a power of 2 seemed to appear. An inquiry into these numbers led us to the discovery that every Mersenne prime is the excedent of at least one positive integer. This is proven in Theorem 3.

m	rank of apparition of m	m	rank of apparition of m
−2	3	2	20
−4	5	4	12
−6	7	6	8925
−8	22	8	56
−10	11	10	40
−12	13	12	24
−14	27	14	272
−16	17	16	550
−18	19	18	208
−20	46	20	176

Table 2. The smallest n for which $e(n) = m$ for small even m.

4. Ranks of apparition

If we wish to decide whether an integer is an excedent, it would be helpful to know how far we ought to search via brute force before believing that an integer which has not yet appeared as an excedent will never appear. We ask then, for a given excedent m, what is the smallest integer n for which $e(n) = m$? We shall refer to this n as the *rank of apparition* of m. If all m that are excedents have small rank of apparition, we may trust that for all m, either m is the excedent of a small integer, or it is never the excedent of any.

Table 2 gives the rank of apparition of all even integers m with $|m| \le 20$. It suggest that if m is an even excedent of any integer, it is likely the excedent of a rather small integer. Indeed the rank of apparition of all even m in the range $-20 \le m \le 20$ is under 10,000.

For odd excedents, the situation is quite different. Recall that for some odd values m (including 1), we do not know whether m is ever an excedent. Table 3 lists every small odd integer which is the excedent of some $n < 10^{20}$, together with its rank of apparition.

The fact that some values, say $m = -11$, don't appear until over 200,000 makes us hesitate to claim that values which don't come up early will never appear. In fact, the situation is even worse than this. The smallest integer we found (in absolute value) whose rank of apparition is more than 10^6 is 127, for which $g(127) = 1032256$. Similarly, $g(1529) = 66324736$ is the smallest known m for which the rank of apparition is greater than 10^7. If we want any hope of putting together a list of excedents and nonexcedents, then, we shall clearly have to extend our search beyond these small values.

m	rank of apparition of m	m	rank of apparition of m
−1	1	3	18
−5	9	7	196
−7	50	17	100
−11	244036	19	36
−19	25	31	15376
−25	98	39	162
−47	484		

Table 3. The smallest n for which $e(n) = m$ for small odd m.

5. Results

As described above, most of our results were motivated by a careful observation of a large amount of data. We here state and prove the three theorems briefly mentioned above, which constitute the primary theoretical results of our research.

Before proceeding, we wish to remind the reader of some basic facts about the function $\sigma(n)$. There are two properties of $\sigma(n)$ which we shall need. First, for a prime power p^k, we have

$$\sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}.$$

Second, $\sigma(n)$ is multiplicative. That is, if a and b are relatively prime, then $\sigma(ab) = \sigma(a)\sigma(b)$. It seems likely that this property, which is so useful in a large number of applications, is the primary reason that so much more attention has been paid to $\sigma(n)$ than to $s(n)$. From these two facts, it is fairly straightforward to show the following theorem.

Theorem 1. *The excedent of n, $e(n)$, is odd if and only if $n = k^2$ or $n = 2k^2$ for some integer k.*

This theorem is enormously useful. Since we have already determined (computationally) that odd excedents are rare, we wish to extend our search for these numbers. Thanks to Theorem 1, if we wish to look for odd excedents, we now know that we need to consider only squares and numbers that are double a square. We will use this to great effect for our computations in Section 7. While Theorem 1 is crucial in our work below, it is not original. It is similar to one often encountered in number theory courses; see [Burton 1976, Chapter 6, Exercise 7], for example.

Our second theorem concerns one family of numbers (probably infinite), all of which appear in the image of the excedent function. Although this theorem was new to us, we have learned that this is not the first time it has appeared in the literature. The referee pointed out that this theorem appears in more general form in [Pomerance 1975], from which we learned that the first appearance of Theorem 2 was in a note by Mąkowski [1960].

Theorem 2. *If N is a perfect number, then $2N$ will be the excedent of infinitely many integers m. In particular, if p is a prime not dividing $2N$, then $2N$ is the excedent of $2pN$.*

Although the proof of this theorem can be found with a literature search, the reader is encouraged to try to prove it herself. It takes only straightforward calculation. Indeed, this result also can now be found in some elementary number theory texts; Theorem 2 appears, for example, as Exercise 21 of [Robbins 2006].

Somewhat disappointingly, despite the fact that our third theorem was new when we proved it, a proof appeared in a paper by Pollack and Shevelev [2012] after our work was submitted to *Involve*. We discovered this work while reading references recommended by the referee during revisions.

Theorem 3. *Let $M_p = 2^p - 1$ be a Mersenne prime. Then $2^p - 1$ will be in the image of the excedent function. In particular,*

$$e(2^{p-1}M_p^2) = M_p.$$

Proof. Let $M_p = 2^p - 1$ be a Mersenne prime, and let $n = 2^{p-1}M_p^2$. We wish to show that $e(n) = \sigma(n) - 2n = M_p$. Because n is already written as a power of 2 multiplied by an odd prime, we can use the multiplicativity of $\sigma(n)$ to write

$$\begin{aligned}
\sigma(2^{p-1}M_p^2) &= \sigma(2^{p-1})\sigma(M_p^2) \\
&= (2^p - 1)(M_p^2 + M_p + 1) \\
&= (2^p - 1)(M_p^2 + 2^p) \\
&= 2^p M_p^2 + 2^{2p} - M_p^2 - 2^p \\
&= 2^p M_p^2 + 2^p M_p - M_p^2 \\
&= 2^p M_p^2 + M_p(2^p - M_p) \\
&= 2^p M_p^2 + M_p.
\end{aligned}$$

Then, since $n = 2^{p-1}M_p^2$ and $\sigma(n) = 2^p M_p^2 + M_p$, we have that the excedent of n, $\sigma(n) - 2n$, is

$$e(n) = (2^p M_p^2 + M_p) - 2(2^{p-1}M_p^2) = M_p,$$

as desired. □

6. Arithmetic progressions

As we noted above, the set of Mersenne primes is a (probably) infinite family of values of the excedent function. We might then ask: are there any provably infinite families of excedents? A bit of thought reveals the answer to be in the affirmative. For example, $e(p) = -(p-1)$ for any prime p, so any integer of the form $-(p-1)$ is certainly an excedent. Indeed, we could find several other infinite families of excedents in terms of their prime factorization as well. Rather than pursue this avenue of study, however, we would like to turn our attention to one more idea — looking for excedents in arithmetic progressions.

To this end, we present one more theorem, and the result of one intriguing computation. The demonstration of the following theorem relies on the Goldbach conjecture. The Goldbach conjecture, as it is usually stated, is that every even integer greater than 2 is the sum of two primes. Although the problem remains open, van der Corput [1936; 1938], Estermann [1938] and Chudakov [1937] each proved independently that *almost* every even number is the sum of two primes — that is, every even number is the sum of two primes, except possibly for a set of density zero.

We should note that this implies a related fact which will prove useful to us. Since the density of integers of the form $2p$ for prime p has density zero, we can also say that almost every even number is the sum of two *distinct* primes. This fact will allow us to prove the following.

Theorem 4. *Every integer* $n \equiv 12 \pmod{24}$ *is contained in the image of the excedent function, except perhaps for a set of density* 0.

Proof. Let $n = pq$, with p and q both prime. Then $s(n) = p + q + 1$. Since, by the discussion above, we know that almost all even integers can be written in the form $p + q$ for distinct p and q, it follows that almost odd integers can be written in the form $p + q + 1$. Thus, we have that almost all odd integers are in the image of $s(n)$.

Now let m be any integer relatively prime to 6, so that $m \equiv 1$ or $5 \pmod 6$. For such an m, $e(12m)$ has an interesting form. We see this by writing

$$\begin{aligned} e(6m) &= \sigma(6m) - 2(6m) = \sigma(6)\sigma(m) - 12m \\ &= 12(m + s(m)) - 12m = 12s(m). \end{aligned}$$

Since numbers relatively prime to 6 are odd, they can almost all be written as $s(m)$ for some m, and therefore almost all numbers of the form $12(2k+1)$ lie in the image of $e(n)$, from which the theorem follows. ☐

Finding this arithmetic progression of excedents raises the obvious question of whether there are other arithmetic progressions that are (almost) all contained in the image of the excedent function. Preliminary computations show that this may be a

m	residue class $k \pmod{m}$
8	4
12	4, 8
16	4, 8, 12
18	6
20	4, 8, 12, 16
24	4, 8, 12, 16, 20
26	2
28	4, 8, 12, 14, 16, 20, 24
30	12, 14, 18, 26
32	4, 8, 12, 16, 20, 24, 28
34	2, 10, 12, 22, 26
36	4, 6, 8, 12, 16, 18, 20, 24, 28, 32
38	8, 12, 20, 22, 30
40	4, 8, 12, 16, 20, 24, 28, 32, 34, 36
42	2, 6, 12, 14, 18, 24, 28, 32, 34, 36, 38
44	4, 8, 12, 16, 20, 24, 28, 32, 36, 40
46	2, 10, 14, 18, 22, 26, 30, 40
48	4, 8, 12, 14, 16, 20, 24, 28, 30, 32, 36, 40, 42, 44
50	4, 6, 12, 16, 20, 22, 24, 32, 38, 46
52	2, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48
54	2, 6, 8, 12, 14, 18, 20, 24, 32, 36, 42, 52
56	4, 8, 10, 12, 14, 16, 20, 24, 28, 32, 34, 36, 40 42, 44, 48, 52, 54
58	26, 34, 38, 40, 42, 44, 46, 50, 52, 54
60	4, 8, 12, 14, 16, 18, 20, 24, 26, 28, 30, 32, 36, 40, 42, 44, 48, 52

Table 4. Every integer up to 10,000 lying in one of the residue classes listed here is contained in the image of the excedent function.

promising line of inquiry. We searched for arithmetic progressions all of whose members up to 10,000 are contained in the excedents we have found. They are listed in Table 4.

Of all the residue classes in Table 4, we have succeeded in explaining only the class $12 \pmod{24}$. We encourage others to use the ideas above to see if more of these classes can be proven to lie entirely (or almost entirely) in the image of $e(n)$.

7. Computational results (redux)

By Theorem 1, we know that all odd excedents are the image under $e(n)$ of an integer of the form k^2 or $2k^2$. Therefore, if we wish to search just for odd excedents, we need only look at numbers of this specialized form. We therefore revised our earlier search to consider only squares and double squares, and were able to extend

Bound on n	$e(n)$ even $0 < e(n)$	$e(n) < 0$	$e(n)$ odd $0 < e(n)$	$e(n) < 0$	Total $-10^4 \le e(n) \le 10^4$
10^4	0.6126	0.2202	0.0166	0.0134	0.2157
10^5	0.9378	0.5888	0.0320	0.0240	0.3956
10^6	0.9722	0.6922	0.0370	0.0310	0.4330
10^7	0.9832	0.7618	0.0400	0.0328	0.4544
10^8	0.9894	0.8390	0.0408	0.0334	0.4756
10^9	0.9894	0.8390	0.0408	0.0334	0.4756
10^{10}	0.9894	0.8390	0.0408	0.0334	0.4756

Table 5. The proportion of integers m with $|e(m)| \le 10^4$ in various classes that are excedents of a number less than the given bound.

our preliminary computation by several orders of magnitude.

In the end, we computed the value of $e(n)$ for $n = k^2$ and $n = 2k^2$ for all n up to 10^{20}. Despite searching to this large value, we find that of the fifty odd values of m with $-50 < m < 50$, thirty-two of them are never in the image of the excedent function. The values that never occur are

$$-49, -45, -43, -39, -35, -33, -31, -29, -27, -23, -21, -17, -15, -13, -9, -3, \\ 1, 5, 9, 11, 13, 15, 21, 23, 25, 27, 29, 33, 35, 37, 43, 45. \quad (1)$$

Among the positive nonexcedents are those studied by Cohen — all the odd squares appear on the list. There are, however, many other odd values that never appear. We cannot explain these values or find any way to classify them, nor do they appear in Sloane's *Online encyclopedia of integer sequences*.

We can, however, use our data to speculate about the density of integers that are excedents. We recorded all excedents of integers up to 10^{10} with absolute value less than 10,000, and we shall use these to get an idea about the density of integers that are excedents. In the table below, we give the proportion of integers that are excedents. Because even excedents behave differently than odd excedents, and because the sign of an integer also seems to affect its probability of being an excedent, we first break integers into four groups (by parity and sign) and consider these proportions separately.

Based on this (admittedly limited) data, it seems reasonable to conjecture that most positive integers are excedents.

8. Conjectures and future work

The theorems above represent observations we made based on our data, and which we have been able to prove. We have also made other observations which we have been unable to prove. Among these are:

Conjecture 5. Every even number is the excedent of at least one positive integer.

Up to 10^8, our computational data show that every even integer n satisfying

$$-480 < n < 130$$

is the excedent of some integer, and we see no reason to expect that any even number will not appear on the list of excedents at some point. We saw in Table 2 that it seems to be the case that if m is an even excedent of any integer, it is likely the excedent of a rather small integer, but when we extend to m beyond the range of Table 2, we actually do find some even numbers k appear in the image of $e(n)$ only for fairly large n. For example, the smallest n such that $e(n) = -384$ is $n = 99413968$.

Conjecture 6. The values given in (1), giving integers that are not in the image of the excedent function for any $n \leq 10^{20}$, are in fact nonexcedents, and will never be in the image of this function.

This conjecture seems less certain. Since we know that there are some k which appear in the image of $e(n)$ only for large n, it is certainly possible that one (or more!) of the values in (1) may yet appear. However, we find no new excedents with absolute value less than 100 appear for any integers greater than 10^9 — we believe that these exceptional values are unlikely to appear after 10^{20}.

There are several other open questions. Can one find an infinite family of integers all of which are nonexcedents? Possibly easier: do the excedents have a density in the integers? If so, what is it? It is striking that more than 2500 years after the concept was first considered by the Pythagoreans, questions about the excedent of an integer continue to beguile and challenge us. It is our hope that these preliminary investigations may serve as a catalyst for further research on the excedent function.

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Received: 2012-12-07 Revised: 2013-05-18 Accepted: 2013-05-20

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2013 vol. 6 no. 4

1944-4176(2013)6:4;1-7