A Pexider difference associated
to a Pexider quartic functional equation
in topological vector spaces

Saeid Ostadbashi, Abbas Najati,
Mahsa Solaimaninia and Themistocles M. Rassias
A Pexider difference associated to a Pexider quartic functional equation in topological vector spaces

Saeid Ostadbashi, Abbas Najati, Mahsa Solaimaninia and Themistocles M. Rassias

(Communicated by Martin Bohner)

Let \((G, +)\) be an Abelian group and \(X\) be a sequentially complete Hausdorff topological vector space over the field \(\mathbb{Q}\) of rational numbers. We deal with a Pexider difference

\[
2f(2x + y) + 2f(2x - y) - 2g(x + y) - 2g(x - y) - 12g(x) + 3g(y),
\]

where \(f\) and \(g\) are mappings defined on \(G\) and taking values in \(X\). We investigate the Hyers–Ulam stability of the Pexiderized quartic functional equation

\[
2f(2x + y) + 2f(2x - y) = 2g(x + y) + 2g(x - y) + 12g(x) - 3g(y)
\]

in topological vector spaces.

1. Introduction and preliminaries

The stability problem concerning the stability of group homomorphisms originated from a question of Ulam [1964] and was answered affirmatively by Hyers [1941] for Banach spaces. This result was generalized by Aoki [1950] for additive mappings and by Rassias [1978] for linear mappings by considering an unbounded Cauchy difference. The question of stability can be raised not only concerning the Cauchy functional equation but also in connection with other functional equations. For more concerning the stability results of functional equations, see [Czerwik 2002; 2003; Hyers et al. 1998; Jung 2001; Forti 1995; Hyers and Rassias 1992]. The stability of the quartic functional equation has been investigated in [Cădariu and Radu 2004; Chung and Sahoo 2003; Lee et al. 2005; Najati 2008].


MSC2010: primary 39B82; secondary 34K20, 54A20.
Keywords: Hyers–Ulam stability, quartic mapping, topological vector space.
paper, we prove that the Pexiderized quartic functional equation
\[ 2f(2x + y) + 2f(2x - y) = 2g(x + y) + 2g(x - y) + 12g(x) - 3g(y) \]
is stable for functions \( f, g \) defined on an Abelian group and taking values in a topological vector space.

Let \( G \) be an Abelian group and throughout this paper let \( X \) be a sequentially complete Hausdorff topological vector space over the field \( \mathbb{Q} \) of rational numbers. A mapping \( f : G \to X \) is quartic if it satisfies the functional equation
\[ f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y) \]
for all \( x, y \in G \). This equation is called the quartic functional equation. For a given \( f : G \to X \), we will use the notation
\[ Df(x, y) := f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) - 24f(x) + 6f(y). \]

For given sets \( A, B \subseteq X \) and a number \( k \in \mathbb{R} \), we define the well-known operations
\[ A + B := \{a + b : a \in A, b \in B\}, \quad kA := \{ka : a \in A\}. \]

We denote the convex hull of a set \( U \subseteq X \) by \( \text{conv}(U) \) and the sequential closure of \( U \) by \( \overline{U} \). Moreover it is well-known that:

(i) If \( A, B \subseteq X \) are bounded sets, then \( A + B \) and \( \overline{A} \) are bounded subsets of \( X \).

(ii) If \( A, B \subseteq X \) and \( \alpha, \beta \in \mathbb{R} \), then \( \alpha \text{conv}(A) + \beta \text{conv}(B) = \text{conv}(\alpha A + \beta B) \).

(iii) Let \( X_1 \) and \( X_2 \) be linear spaces over \( \mathbb{R} \). If \( f : X_1 \to X_2 \) is a quartic function, then \( f(rx) = r^4f(x) \) for all \( x \in X_1 \) and all \( r \in \mathbb{Q} \).

2. Main results

We start with the following lemma.

**Lemma 2.1.** Let \( G \) be an Abelian group and \( B \subseteq X \) be a nonempty set. If the even functions \( f, g : G \to X \) satisfy
\[ f(2x + y) + f(2x - y) - g(x + y) - g(x - y) - 6g(x) + \frac{3}{2}g(y) \in B \quad (2-1) \]
for all \( x, y \in G \), then
\[ Df(x, y) + 24f(0) \in 16 \text{conv}(B - B), \quad (2-2) \]
\[ Dg(x, y) + 24g(0) \in 4 \text{conv}(B - B) \quad (2-3) \]
for all \( x, y \in G \).
Proof. Putting \( x = 0 \) in (2-1), we get
\[
4f(y) - g(y) - 12g(0) \in 2B
\]
for all \( y \in G \). If we put \( x = y = 0 \) in (2-1), then we have
\[
4f(0) - 13g(0) \in 2B.
\]
(2-5)
It follows from (2-4) and (2-5) that, for all \( x, y \in G \),
\[
Df(x, y) + 24f(0) = [f(2x + y) + f(2x - y) - g(x + y) - g(x - y) - 6g(x) + \frac{3}{2}g(y)]
- [4f(x + y) - g(x + y) - 12g(0)] - [4f(x - y) - g(x - y) - 12g(0)]
- [24f(x) - 6g(x) - 72g(0)] + [6f(y) - \frac{3}{2}g(y) - 18g(0)] + [24f(0) - 78g(0)],
\]
which lies in \( 12 \text{conv}(B) + 12 \text{conv}(-B) = 16 \text{conv}(B - B) \). This proves (2-2).
Moreover, we have, for all \( x, y \in G \),
\[
Dg(x, y) + 24g(0) = [4f(2x + y) + 4f(2x - y) - 4g(x + y) - 4g(x - y) - 24g(x) + 6g(y)]
- [4f(2x + y) - g(2x + y) - 12g(0)] - [4f(2x - y) - g(2x - y) - 12g(0)]
\]
which lies in \( 4 \text{conv}(B) + 4 \text{conv}(-B) = 4 \text{conv}(B - B) \). Hence we get (2-3). \( \square \)

Theorem 2.2. Let \( G \) be an Abelian group and \( B \subseteq X \) be a nonempty bounded set. Suppose that the even functions \( f, g : G \rightarrow X \) satisfy (2-1) for all \( x, y \in G \). Then there exists exactly one quartic function \( \tilde{f} : G \rightarrow X \) such that
\[
\tilde{f}(x) - f(x) + f(0) \in \frac{8}{15} \text{conv}(B - B),
\]
\[
4\tilde{f}(x) - g(x) + g(0) \in \frac{2}{15} \text{conv}(B - B)
\]
for all \( x \in G \). Moreover, the function \( \tilde{f} \) is given by
\[
\tilde{f}(x) = \lim_{n \rightarrow \infty} \frac{1}{24^n} f(2^n x) = \lim_{n \rightarrow \infty} \frac{1}{24^n} g(2^n x)
\]
for \( x \in G \), and the convergence of the sequences are uniform on \( G \).

Proof. By Lemma 2.1, we have
\[
Df(x, y) \in -24f(0) + 16 \text{conv}(B - B)
\]
(2-6)
for all \( x, y \in G \). Setting \( y = 0 \) in (2-6), we get
\[
2f(2x) - 32f(x) \in -30f(0) + 16 \text{conv}(B - B)
\]
for all \( x \in G \). Therefore
\[
\frac{1}{24}f(2x) - f(x) \in \frac{1}{24} \tilde{B}
\]
(2-7)
As before, we can check that \( \tilde{B} := -15 f(0) + 8 \text{conv}(B - B) \). It is clear that \( \tilde{B} \) is convex. Replacing \( x \) by \( 2^n x \) in (2-7), we infer that
\[
\frac{1}{2^{4(n+1)}} f(2^{n+1} x) - \frac{1}{2^{4n}} f(2^n x) \in \frac{1}{2^{4(n+1)}} \tilde{B}
\]
for all \( x \in G \) and all integers \( n \geq 0 \). Therefore
\[
\frac{1}{2^{4n}} f(2^n x) - \frac{1}{2^{4m}} f(2^m x) \in \sum_{k=m}^{n-1} \frac{1}{2^{4(k+1)}} \left( f(2^{k+1} x) - \frac{1}{2^{4k}} f(2^k x) \right)
\]
\[
\subseteq \sum_{k=m}^{n-1} \frac{1}{2^{4(k+1)}} \tilde{B} \subseteq \frac{1}{15 \times 2^{4m}} \tilde{B}
\]
for all \( x \in G \) and all integers \( n > m \geq 0 \). Since \( B \) is bounded, we conclude that \( \tilde{B} \) is bounded. It follows from (2-8) and boundedness of the set \( \tilde{B} \) that the sequence \( \{(1/2^{4n}) f(2^n x)\} \) is (uniformly) Cauchy in \( X \) for all \( x \in G \). Since \( X \) is a sequential complete topological vector space, the sequence \( \{(1/2^{4n}) f(2^n x)\} \) converges for all \( x \in G \), and the convergence is uniform on \( G \). Define
\[
\mathcal{Q}_1 : G \to X, \quad \mathcal{Q}_1(x) := \lim_{n \to \infty} \frac{1}{2^{4n}} f(2^n x).
\]
Since \(-24 f(0) + 16 \text{conv}(B - B)\) is bounded, it follows from (2-6) that
\[
D\mathcal{Q}_1(x, y) = \lim_{n \to \infty} \frac{1}{2^{4n}} Df(2^n x, 2^n y) = 0
\]
for all \( x, y \in G \). So \( \mathcal{Q}_1 \) is quartic. Letting \( m = 0 \) and \( n \to \infty \) in (2-8), we get
\[
\mathcal{Q}_1(x) - f(x) + f(0) \in \frac{8}{15} \text{conv}(B - B)
\]
for all \( x \in G \). Applying (2-3) as before, we have
\[
\frac{1}{2^{4n}} g(2^n x) - \frac{1}{2^{4m}} g(2^m x) \in \sum_{k=m}^{n-1} \frac{1}{2^{4(k+1)}} \tilde{C} \subseteq \frac{1}{15 \times 2^{4m}} \tilde{C}
\]
for all \( x \in G \), where \( \tilde{C} := -15 g(0) + 2 \text{conv}(B - B) \). Then \( \{(1/2^{4n}) g(2^n x)\} \) is a (uniformly) Cauchy sequence in \( X \) for all \( x \in G \). Define
\[
\mathcal{Q}_2 : G \to X, \quad \mathcal{Q}_2(x) := \lim_{n \to \infty} \frac{1}{2^{4n}} g(2^n x).
\]
As before, we can check that \( \mathcal{Q}_2 \) is a quartic function satisfying
\[
\mathcal{Q}_2(x) - g(x) + g(0) \in \frac{2}{15} \text{conv}(B - B)
\]
for all \( x \in G \). To prove the equality \( 4\mathcal{Q}_1 = \mathcal{Q}_2 \), we have
\[
4\mathcal{Q}_1(x) - \mathcal{Q}_2(x) = [4\mathcal{Q}_1(x) - 4 f(x)] - [\mathcal{Q}_2(x) - g(x)] + [4 f(x) - g(x)]
\]
for all $x \in G$. Applying (2-4), (2-5), (2-9) and (2-11) in the above equation, we get
\[4\mathcal{Q}_1(x) - \mathcal{Q}_2(x) \in M := 2 \text{conv}(B - B) + 2(B - B) \quad (2-12)\]
for all $x \in G$. Replacing $x$ by $2^nx$ in (2-12), we get
\[4\mathcal{Q}_1(2^nx) - \mathcal{Q}_2(2^nx) \in M\]
for all $x \in G$ and all integers $n$. Since $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are quartic, we obtain
\[4\mathcal{Q}_1(x) - \mathcal{Q}_2(x) \in \frac{1}{2^4n} M \quad (2-13)\]
for all $x \in G$. Since $M$ is bounded, letting $n \to \infty$ in (2-13) we obtain $4\mathcal{Q}_1 = \mathcal{Q}_2$. Assuming $\mathcal{Q} := \mathcal{Q}_1$, we can see that the conditions of theorem are satisfied.

To prove uniqueness, suppose that there exists another quartic function $\mathcal{Q}': G \to X$ satisfying
\[\mathcal{Q}'(x) - f(x) + f(0) \in \frac{8}{15} \text{conv}(B - B)\]
for all $x \in G$. Then we have
\[\mathcal{Q}'(x) - \mathcal{Q}(x) = [\mathcal{Q}'(x) - f(x) + f(0)] - [\mathcal{Q}(x) - f(x) + f(0)] \in \frac{16}{15} \text{conv}(B - B)\]
for all $x \in G$. Applying the same method as before, we get $\mathcal{Q}' = \mathcal{Q}$. This completes the proof. \qed

Acknowledgements

The authors would like to thank the referee for useful comments.

References


Received: 2013-01-23 Accepted: 2013-01-28

s.ostadbashi@urmia.ac.ir Department of Mathematics, Urmia University, Urmia 57561-51818, Iran

a.najati@uma.ac.ir Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran

solaimaninia@urmia.ac.ir Department of Mathematics, Urmia University, Urmia 57561-51818, Iran

trassias@math.ntua.gr Department of Mathematics, Zografou Campus, National Technical University of Athens, 15780 Athens, Greece
Embeddedness for singly periodic Scherk surfaces with higher dihedral symmetry
Valmir Bucaj, Sarah Cannon, Michael Dorff, Jamal Lawson and Ryan Viertel

An elementary inequality about the Mahler measure
Konstantin Stulov and Rongwei Yang

Ecological systems, nonlinear boundary conditions, and Σ-shaped bifurcation curves
Kathryn Ashley, Victoria Sincavage and Jerome Goddard II

The probability of randomly generating finite abelian groups
Tyler Carrico

Free and very free morphisms into a Fermat hypersurface
Tabes Bridges, Rankeya Datta, Joseph Eddy, Michael Newman and John Yu

Irreducible divisor simplicial complexes
Nicholas R. Baeth and John J. Hobson

Smallest numbers beginning sequences of 14 and 15 consecutive happy numbers
Daniel E. Lyons

An orbit Cartan type decomposition of the inertia space of SO(2m) acting on ℝ2m
Christopher Seaton and John Wells

Optional unrelated-question randomized response models
Sat Gupta, Anna Tuck, Tracy Spears Gill and Mary Crowe

On the difference between an integer and the sum of its proper divisors
Nichole Davis, Dominic Klyve and Nicole Kraght

A Pexider difference associated to a Pexider quartic functional equation in topological vector spaces
Saeid Ostadbashi, Abbas Najati, Mahsa Solaimaninia and Themistocles M. Rassias