An interesting proof of the nonexistence of a continuous bijection between $\mathbb{R}^n$ and $\mathbb{R}^2$ for $n \neq 2$

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We show that there is no continuous bijection from $\mathbb{R}^n$ onto $\mathbb{R}^2$ for $n \neq 2$ by an elementary method. This proof is based on showing that for any cardinal number $\beta \leq 2^{\aleph_0}$, there is a partition of $\mathbb{R}^n$ ($n \geq 3$) into $\beta$ arcwise connected dense subsets.

1. Introduction

In 1877 Cantor discovered a bijection of $\mathbb{R}$ onto $\mathbb{R}^n$ for any $n \in \mathbb{N}$. Cantor’s map was discontinuous, but the discovery of the Peano curve in 1890 showed that there existed continuous (although not injective) maps of $\mathbb{R}$ onto $\mathbb{R}^n$. Between then and 1910, several mathematicians showed that there does not exist a bicontinuous bijection (homeomorphism) from $\mathbb{R}^m$ onto $\mathbb{R}^n$ for the cases $m = 2$ and $m = 3$ and $n > m$. Finally in 1911, Brouwer showed that there does not exist a homeomorphism between $\mathbb{R}^m$ and $\mathbb{R}^n$ for $n \neq m$ (for a modern treatment, see [Munkres 1984, p. 109]). The present paper proves the nonexistence of a continuous bijection from $\mathbb{R}^n$ onto $\mathbb{R}^2$ for $n \neq 2$ by an elementary method.

Rudin [1963] showed that for any countable cardinal $\alpha > 2$, we cannot partition the plane into $\alpha$ arcwise connected dense subsets. In this paper we show that for any cardinal number $\beta \leq 2^{\aleph_0}$, there is a partition of $\mathbb{R}^n$ ($n \geq 3$) into $\beta$ arcwise connected dense subsets; then by using this we show that there is no continuous bijection from $\mathbb{R}^n$ onto $\mathbb{R}^2$ for $n \neq 2$.

Lemma 1. There is a partition of $\mathbb{R}^+ \cup [0, \infty)$ into $2^{\aleph_0}$ dense subsets.

Proof. Consider the additive group $(\mathbb{R}, +)$. The quotient group $\mathbb{R}/\mathbb{Q}$ has $2^{\aleph_0}$ elements which are dense subsets of $\mathbb{R}$. Intersect them with $\mathbb{R}^+$. □

Theorem 1. There is a partition of $\mathbb{R}^3$ into $2^{\aleph_0}$ arcwise connected dense subsets.

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Proof. Let \( \{S_i \mid i \in I\} \) be a partition of \( \mathbb{R}^+ \) into \( 2^{\aleph_0} \) dense subsets. The set \( I \) is just an index set, so we may suppose that \( I = (0, 1) \). Define \( L_i = \{(t, it, 0) \mid t > 0\} \) and \( M = \bigcup_{i \in I} L_i \) and let \( A_i \) be the union of all spheres with center at the origin and radius from \( S_i \), that is, \( A_i = \{x \in \mathbb{R}^3 \mid \|x\| \in S_i\} \). Let \( B_i = (A_i \setminus M) \cup L_i \). If \( S \) is a sphere centered at the origin, then \( S \setminus M \) is a sphere with a small arc removed. Therefore \( A_i \setminus M \) is the union of some arcwise connected punctured spheres. Open half-line \( L_i \) pastes these punctured spheres together, so \( B_i \) is arcwise connected. It is obvious that \( \{B_i \mid i \in I\} \) is a partition of \( \mathbb{R}^3 \) with size \( 2^{\aleph_0} \). Since \( S_i \) is dense in \( \mathbb{R}^+ \), \( A_i \) and consequently \( B_i \) are dense in \( \mathbb{R}^3 \).

Corollary 1. There is a partition of \( \mathbb{R}^n \) into \( 2^{\aleph_0} \) arcwise connected dense subsets for \( n \geq 3 \).

Proof. It is enough to set \( B_i^{(n)} = B_i \times \mathbb{R}^{n-3} \), in which \( B_i \) is as above. The collection \( \{B_i^{(n)} \mid i \in I\} \) is a partition of \( \mathbb{R}^n \) satisfying the claim.

Note that the union of any number of the sets \( B_i^{(n)} \) is an arcwise connected dense subset of \( \mathbb{R}^n \), hence:

Corollary 2. For any cardinal number \( \beta \leq 2^{\aleph_0} \), there is a partition of \( \mathbb{R}^n \) (\( n \geq 3 \)) into \( \beta \) arcwise connected dense subsets.

Theorem 2. For any countable cardinal \( \alpha > 2 \), we cannot partition the plane into \( \alpha \) arcwise connected dense subsets.

Proof. This statement is proved in [Rudin 1963].

Lemma 2. Let \( X, Y \) be metric spaces and \( T : X \to Y \) be a continuous map.

(a) If \( A \) is dense in \( X \) and \( T \) is surjective, then \( T(A) \) is dense in \( Y \).

(b) If \( B \subseteq X \) is arcwise connected, then \( T(B) \) is also arcwise connected.

Theorem 3. There is no continuous bijection from \( \mathbb{R} \) onto \( \mathbb{R}^m \) for \( m \neq 1 \).

Proof. Suppose the contrary: Let \( g : \mathbb{R} \to \mathbb{R}^m \) be a continuous bijective map. We put \( B_n = [-n, n] \), and so we have \( \mathbb{R}^m = g\left( \bigcup_{n=1}^{\infty} B_n \right) = \bigcup_{n=1}^{\infty} g(B_n) \). Since \( \mathbb{R}^m \) is not in the first category, at least one of the \( g(B_n) \), for example \( g(B_k) \), has nonempty interior in \( \mathbb{R}^m \). Suppose \( B(x, r) \subseteq g(B_k) \). Since \( B_k \) is compact, \( f : B_k \to g(B_k) \) is a homeomorphism. It follows that \( B(x, r) \) is homeomorphic with an interval in \( \mathbb{R} \). This is a contradiction, because if we remove 3 points from \( B(x, r) \) it remains connected, but this is not the case for the intervals in \( \mathbb{R} \).

Theorem 4. There is no continuous bijection from \( \mathbb{R}^n \) onto \( \mathbb{R}^2 \) for \( n \neq 2 \).

Proof. Suppose the contrary:

(a) If \( n > 2 \), then according to Corollary 2 and Lemma 2 we can partition \( \mathbb{R}^2 \) into 3 arcwise connected dense subsets, and this contradicts Theorem 2.

(b) If \( n = 1 \), then this contradicts Theorem 3.
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References


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