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Analysis of a Sudoku variation using partially ordered sets and equivalence relations

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Sudoku is a popular game of logic, and there are many variations of the standard puzzle. We investigate a variation of Sudoku that uses inequalities between cells rather than numerical clues. We begin with an overview of the rules and strategies of the game. We then examine the solvability of an individual $m \times n$ block with the use of partially ordered sets, and combine 2×2 blocks to form 4×4 puzzles.

1. Introduction

The basic concepts behind the popular Sudoku number puzzles may be familiar from the newspaper, the internet, or any variety of puzzle books. A *Sudoku board* is a 9×9 grid in which the entries 1 through 9 appear exactly once in each row, column, and 3×3 *block*. A *Sudoku puzzle* is created from a board by strategically removing some of the entries, leaving only select clues from which the player must try to reconstruct the original board. In order to be a valid puzzle, the clues must lead to a unique solution. In this paper, we will refer to this game as *standard Sudoku* (see Figure 1, left).

One variation on the basic puzzle is *Greater Than Sudoku*. A *Greater Than Sudoku board* (Figure 1, right) meets the same criteria as the standard board, but has an additional condition: within each block, every pair of adjacent entries, both horizontal and vertical, must satisfy the inequality which separates them. While the standard puzzle begins with some entries filled in, providing the player with numerical clues which will lead to a unique solution, a Greater Than Sudoku puzzle gives the player only the inequalities on an empty grid. Furthermore, the inequalities must be arranged in such a way that a unique solution exists.

The primary focus of this paper is a smaller version called *Greater Than Shidoku* (Figure 2, left) consisting of a 4×4 grid partitioned into four 2×2 blocks, which is played with the entries 1 through 4. Many results are also extended to blocks of

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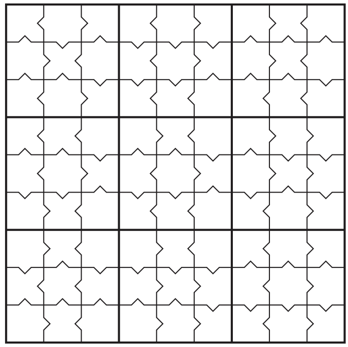


Figure 1. Left: standard Sudoku puzzle [Mepham 2011]; right: Greater Than Sudoku puzzle [Sudoku 2006].

larger variations, including *Greater Than Rokudoku* (Figure 2, right), which has six 2×3 blocks, and Greater Than Sudoku.

2. Playing the game

Solving a Greater Than puzzle of any size requires a slightly different approach than that used to play standard Sudoku, and this approach will also prove to be instrumental in the analysis of Greater Than puzzles of any size. In particular, the player identifies the minimal and maximal cells as well as using the conditions placed on rows, columns, and blocks. A *minimal* cell of a block is any unfilled cell whose inequalities all point inward from adjacent unfilled cells. Similarly, a *maximal* cell is any unfilled cell with all inequalities pointing outward into adjacent unfilled cells. Since these properties depend upon the cells that have not yet been filled, the maximal and minimal cells will change as the game is played. In Figure 3, the unfilled Greater Than Shidoku block contains one minimal cell, identified by $\bullet$, and one maximal cell, identified by $\bigcirc$.

Our first step in solving the Greater Than Shidoku puzzle in Figure 2 is to identify where to place the 1 entries. (The solutions to the other puzzles are at the end of

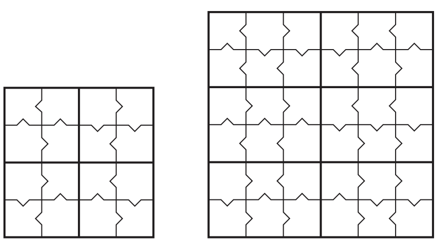


Figure 2. Smaller puzzles. Left: Greater Than Shidoku; right: Greater Than Rokudoku.

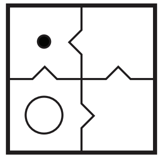


Figure 3. Minimal and maximal cells.

this article.) Since 1 is the smallest element we use, each 1 must be placed in a minimal cell in an unfilled block. For example, in Figure 4, the top two blocks each contain only one (shaded) minimal cell, so we know those cells must contain 1. The bottom two blocks, however, each contain two cells that are minimal. In such cases where the inequalities do not determine unique placement of each 1 entry, the next step is to consider any information provided by the rows and columns. Thus, while there are two possible placements of 1 entries in each block of the lower half of the board, by using the columns it is possible to uniquely determine their proper placement.

Next we will determine where to place the 2 entries by considering the minimal cells among those that remain unfilled. If necessary, the rows and columns may again be used to determine the correct placements. Similarly, a 3 entry must have inequalities pointing inward from each adjacent cell not containing a 1 or a 2, and so on. The player may also begin with the largest entry and work backwards by looking for maximal cells. A 4 must be placed in a maximal cell, where the inequalities all point outward. A 3 would have inequalities pointing out into any cell not containing a 4, and so on.

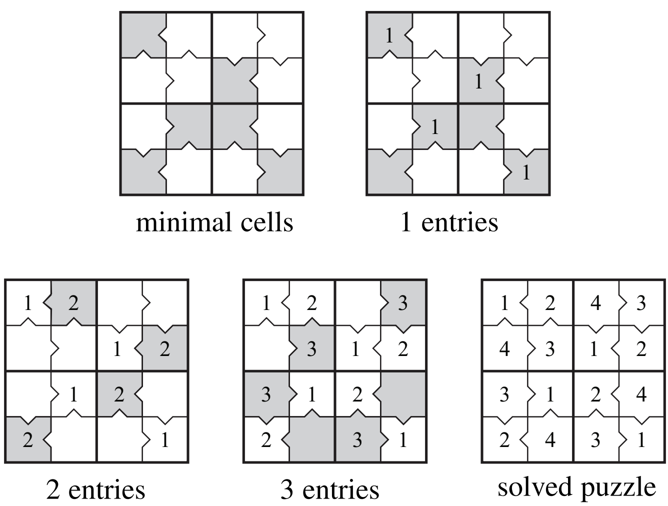


Figure 4. Playing Greater Than Shidoku.

3. Inequality blocks and cycles

A Greater Than puzzle contains inequalities that compare adjacent entries; however, only entries within the same block are considered. The player must begin by examining the ways in which individual blocks can be filled before moving on to the puzzle as a whole. Similarly, we begin our investigation of Greater Than puzzles by considering individual blocks.

Definition 1. An *inequality block* is an $m \times n$ grid, with $m, n \in \mathbb{N}$, in which an inequality separates each pair of horizontally or vertically adjacent cells.

In one block of Greater Than Shidoku there are four inequalities, and each can be oriented in one of two directions. Thus there are $2^4 = 16$ possible 2×2 inequality blocks. There are four cells in each block, so without considering the inequalities there are $4! = 24$ ways of permuting the entries. Similarly, we can count the number of inequality blocks and permutations of any size block. Greater Than Rokudoku blocks have $2^7 = 128$ ways of arranging the inequalities and $6! = 720$ permutations of entries. For Greater Than Sudoku, we have $2^{12} = 4096$ inequality blocks and $9! = 362{,}880$ ways of permuting the entries.

Definition 2. An inequality block is *solvable* if there exists at least one permutation of entries satisfying all inequalities in that block. A block is *unsolvable* if no such permutation exists.

Note that for each size block, there are many more ways to permute the entries than there are ways to arrange the inequalities. Each permutation of entries corresponds to one arrangement of the inequalities because, given any filled block, we can insert the inequalities accordingly. However, since there are significantly fewer inequality arrangements than permutations, some inequality arrangements must correspond to more than one permutation. In other words, without considering the other blocks in a puzzle, many inequality blocks have more than one solution. This leads to two natural questions: are all inequality blocks solvable, and for those that are, how many solutions exist? We can only use solvable blocks to create Greater Than puzzles, so our first goal is to determine criteria for deciding which blocks are solvable.

A *path* in an inequality block is a sequence of adjacent cells where the inequalities are always increasing or always decreasing. If a path includes any cell more than once, that path contains a *cycle*. A cycle of cells is impossible to fill with entries without contradicting at least one of the inequalities; thus any inequality block containing a cycle is unsolvable. In a 2×2 inequality block, there are two inequality arrangements that produce a cycle of the four cells, shown in Figure 5. These two cycles correspond to two unsolvable inequality blocks, leaving us with 14 that are acyclic.

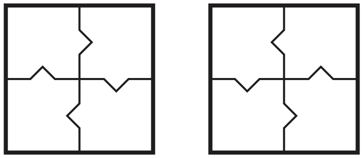


Figure 5. Unsolvable 2×2 blocks.

In Greater Than Rokudoku, the 2×3 inequality blocks may also contain cycles, but more than two unsolvable blocks result from such arrangements. There are six different cycles that may appear in a 2×3 block, shown in Figure 6.

A block that contains a cycle is unsolvable, but some inequality arrangements may contain more than one cycle, so to count the number of blocks with cycles, we use the principle of inclusion-exclusion. Let C_i be the set of all blocks containing cycle i for $1 \leq i \leq 6$. Blocks from sets C_1 through C_4 each have three inequalities that are not involved in the given cycle, so $|C_i| = 2^3$ for $1 \leq i \leq 4$. Sets C_5 and C_6 consist of blocks with only one inequality not involved in the cycle, thus $|C_5| = |C_6| = 2$, and consequently $\sum_{i=1}^{6} |C_i| = 36$. However, some blocks will be counted in two sets; for example, if the remaining inequality in a block from set C_5 is pointing down, that block also contains cycle 1, thus that block is included in set C_1. If the inequality is pointing up, that block is included in set C_3. Similarly, one block in C_6 is also contained in set C_2, while the other is contained in set C_4. There is one block containing both cycles 1 and 4, and another containing cycles 2 and 3. There are no 2×3 blocks that contain 3 different cycles. Thus we have double counted 6 blocks that are in two sets, and so we subtract this from our previous tally, resulting in a total of 30 2×3 inequality blocks with at least one cycle. These 30 blocks are unsolvable, so we eliminate them from the number of inequality blocks we need to consider. This leaves us with 98 acyclic 2×3 inequality blocks.

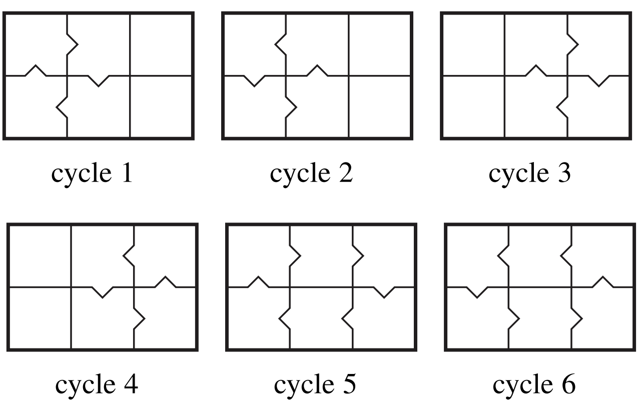


Figure 6. Cycles in 2×3 blocks.

We employ the same strategy with 3×3 Sudoku blocks, but now we have twenty-six possible cycles that can be formed among the inequalities (see if you can find them all!). Many inequality arrangements contain multiple cycles. To count the number of blocks that contain at least one cycle, we again use inclusion-exclusion. There are 1698 such puzzle blocks that are impossible to fill in, so of the 4096 3×3 inequality blocks, 2398 are acyclic. The results of this section are summarized in the following theorem.

Theorem 1. *There are* 14 *acyclic* 2×2 *inequality blocks*, 98 *acyclic* 2×3 *inequality blocks*, *and* 2398 *acyclic* 3×3 *inequality blocks.*

4. Posets and solvable blocks

We have shown that every inequality block containing a cycle is unsolvable; however, it remains to be seen that every acyclic inequality block *is* solvable. While playing the game, we compared cells using the inequalities and identified minimal cells, but we found that minimal cells were not always unique. This suggests considering an acyclic block as a *partially ordered set* and leads us to another way of describing solutions of the block.

Definition 3. A *partial order* $\preceq$ on a set A is a binary relation that is reflexive, antisymmetric, and transitive. A *partially ordered set*, or *poset*, is a pair $(A, \preceq)$, where $\preceq$ is a partial order on the set A.

We now define a relation on inequality blocks of arbitrary size and show that it satisfies the above definition:

Definition 4. Let $A = \{a_1, a_2, \dots, a_{mn}\}$ be the set of cells of an $m \times n$ acyclic inequality block. For all $a_i, a_j \in A$, we define a relation $\preceq$ on A such that $a_i \preceq a_j$ if $a_i = a_j$ or if a_i precedes a_j in an increasing path.

Theorem 2. *With A and $\preceq$ as defined above*, $(A, \preceq)$ *is a partially ordered set.*

Proof. Let $a_i \in A$. Since $a_i = a_i$, then $a_i \preceq a_i$ and consequently $\preceq$ is reflexive. Now let $a_i, a_j \in A$, where $a_i \preceq a_j$ and $a_j \preceq a_i$, and assume that $a_i \neq a_j$. Then a_i precedes a_j in an increasing path, and a_j precedes a_i in an increasing path. The concatenation of these two increasing paths will contain a cycle, which contradicts our assumption that the block is acyclic. Thus $a_i = a_j$, and $\preceq$ is antisymmetric. Finally, let $a_i, a_j, a_k \in A$, where $a_i \preceq a_j$ and $a_j \preceq a_k$. If $a_i = a_j$ or $a_j = a_k$, it is clear that $a_i \preceq a_k$, so let us consider the case where $a_i \neq a_j$ and $a_j \neq a_k$. This means that a_i precedes a_j in an increasing path, and a_j precedes a_k in an increasing path. The concatenation of these paths forms an increasing path in which a_i precedes a_k. Thus $a_i \preceq a_k$, and $\preceq$ is transitive. Therefore, $\preceq$ is a partial order on A, and $(A, \preceq)$ is a poset. □

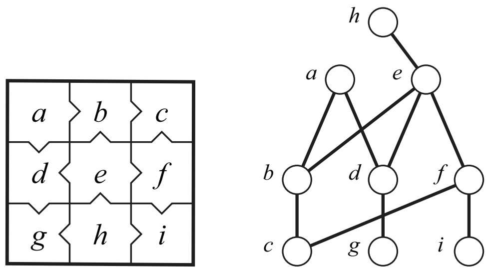


Figure 7. Inequality block (left) and Hasse diagram (right) of poset of cells.

We may visualize a poset by creating a *Hasse diagram* of the set. In a Hasse diagram, the vertices represent the elements of the set. If $x \preceq y$, then the vertex for x is placed below the vertex for y. If $x \neq y$, $x \preceq y$, and there is no intermediate element $z \neq x, y$ such that $x \preceq z \preceq y$, then we say that y *covers* x, and an edge is drawn connecting the two elements. However, if there is such an element z, an edge from x to z, and one from z to y, then $x \preceq y$ by transitivity. Figure 7 shows an acyclic 3×3 inequality block as well as the corresponding Hasse diagram.

This type of relation is called a *partial* order because it may not be possible to use the relation to compare all of the elements in the set.

Definition 5. Let $\preceq$ be a partial order on a set A. Elements a and b are called *comparable* if and only if either $a \preceq b$ or $b \preceq a$. Otherwise, a and b are *incomparable*.

Even though cells c and h are not adjacent in the above inequality block, there is an increasing path c, b, e, h; therefore $c \preceq h$ and we know any solution for this block must have a smaller element in cell c than in cell h. On the other hand, there is no such increasing path between b and g, so those two cells are incomparable and we cannot predict which cell will contain the larger entry. A useful fact about posets is that any finite, nonempty poset has a minimal element, and furthermore, any subset of a poset is also a poset [Epp 2004]. This means that if we remove a minimal element from a poset, we will always have at least one minimal element among the remaining cells.

This brings us back to our technique for solving the Greater Than Shidoku puzzle by identifying minimal cells in each block. When we placed the 1 entries, we effectively removed those cells from the posets for each block, then we identified the minimal cells in the resulting posets in order to place the 2 entries. Previously proven results about posets give us another way to view our solution to the puzzle.

Definition 6. If $\preceq$ is a relation on a set A, and for any two elements a and b in A either $a \preceq b$ or $b \preceq a$, then $\preceq$ is a *total order* on A. A *linear extension* is obtained by putting a total order on a poset $(A, \preceq)$ which preserves the partial order $\preceq$.

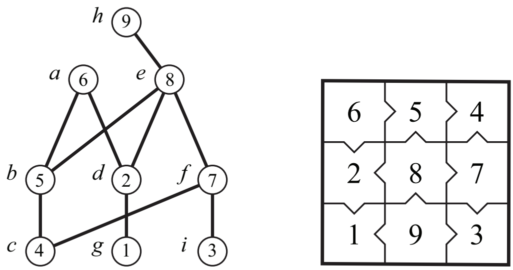


Figure 8. Creating a linear extension to solve a block.

Theorem 3. *Every partial order may be extended to a total order.*

This theorem was first proven by Szpilrajn in [1930], and from it we may conclude that we can put a total order on the poset of cells of any acyclic inequality block. Furthermore, creating a linear extension of the poset of cells is the same as finding a solution for the block, which leads us directly to the following corollary.

Corollary 4. *Every acyclic $m \times n$ inequality block is solvable.*

To demonstrate, we will create a linear extension of our poset in Figure 8. First, we pick a minimal cell g and label it with 1. Once g has been labeled and thus removed from future comparisons, our new set of minimal cells consists of c, d, and i. We arbitrarily choose d and label it 2. The minimal cells are now c and i; we choose cell i and label it 3. We continue to choose and label minimal elements until all are labeled. Figure 8 shows one example of how the remaining entries can be labeled, and the corresponding solution of the inequality block. However, at each step in the process, there were often multiple minimal cells to choose from, so the solution in the figure is only one of the many solutions we could have chosen.

Now that we can find a solution of any acyclic inequality block, the next step is to find a method of counting the number of solutions of any such block. This is essential because some blocks have a large number of solutions, so it is often tedious to attack this task by hand. Many researchers have studied the question of creating and counting linear extensions of posets; we used *A Maple Package for Posets*, created by John R. Stembridge [2009] of the University of Michigan. This package includes a command to count the number of linear extensions of any given poset, and when we applied it to the 3×3 block in Figure 7, we found that there are actually 261 solutions to the block. Surprising as this might seem, it is far from the highest number of solutions of a 3×3 block. After testing all possible inequality combinations on a block, we find that there are 34 3×3 blocks that have over 1000 solutions. In fact, the block in Figure 9 has 4800 solutions!

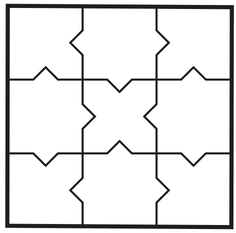


Figure 9. Block with 4800 solutions.

5. Equivalent Shidoku blocks

There are 14 acyclic, and therefore solvable, inequality blocks that we can use to create a Greater Than Shidoku puzzle. Each 4×4 grid is comprised of four blocks, which means that there are $14^4 = 38{,}416$ possible combinations of inequality blocks to choose from, although we shall see that the number of Greater Than Shidoku boards and puzzles is considerably smaller. For Greater Than Rokudoku and Sudoku, the number of possible combinations are $98^6 \approx 8.9 \times 10^{11}$ and $2398^9 \approx 2.6 \times 10^{30}$, respectively. For this reason, we are interested in grouping similar blocks together, thereby reducing the number of possible combinations to a more manageable size; thus we will define a method of grouping inequality blocks based on the positions of minimal and maximal cells in an unfilled block.

In Greater Than Shidoku, each block has four entries, and we want to find all combinations of maximal and minimal cells. Recall that every inequality block must have at least one maximal and at least one minimal cell. Further note that, given any two adjacent cells, the entry in one must be larger than that of the other, thus it is not possible to have two adjacent minimal cells nor two adjacent maximal cells. Finally, we recognize that if we have two maximal cells diagonal from each other, the inequalities of the remaining two cells are determined, and those cells are forced to be minimal. Consequently there are two cases for the number of minimal and maximal cells: we may either have one minimal and one maximal cell, or we may have two of each. To take all the different arrangements into account, we define the following relation.

Definition 7. Let $S_{2,2}$ be the set of all solvable 2×2 inequality blocks. We define a relation $\sim$ as follows. Let $A, B \in S_{2,2}$. Then $A \sim B$ if and only if A can be transformed into B using some sequence of reflections across the vertical, horizontal, and diagonal axes of the block.

Theorem 5. *The relation $\sim$ is an equivalence relation on $S_{2,2}$.*

Proof. To prove $\sim$ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive. Let $A \in S_{2,2}$. Clearly $A \sim A$ because no transformation

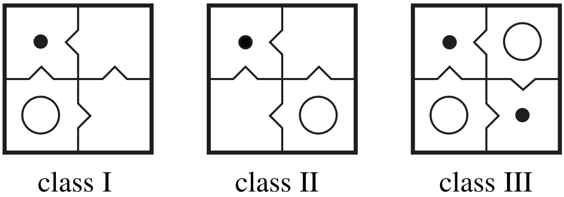


Figure 10. Representatives of the three equivalence classes.

is necessary, and thus $\sim$ is reflexive. Now let $A, B \in S_{2,2}$ such that $A \sim B$. Then A can be transformed into B by some sequence of reflections. Applying these reflections to B in reverse order transforms B into A. Thus $B \sim A$, and $\sim$ is symmetric. Finally, let $A, B, C \in S_{2,2}$, with $A \sim B$ and $B \sim C$. Then there is a sequence of reflections that will transform A into B, and another sequence which will transform B into C. The concatenation of these sequences yields a sequence of reflections that will transform A into C. Thus the relation $\sim$ is transitive, and $\sim$ is an equivalence relation. $\square$

Using equivalence relation $\sim$, the set of solvable blocks can be partitioned into equivalence classes. Blocks within a class are *equivalent*, and those from different classes are said to be *distinct*.

Theorem 6. *There are three equivalence classes of* 2×2 *inequality blocks.*

This is easily verified by checking the 14 blocks in $S_{2,2}$. An example from each class is shown in Figure 10. We next consider the number of solutions of each block. For the following section, we will also find it helpful to observe that within an equivalence class, the same entry is always placed diagonally from 1 in the block.

Consider a block from class I in Figure 11. When solving the block, we see there is only one possible position for the 1 entry, and similarly, only one way to place the 4 entry. The 2 and 3 entries are adjacent to one another, and so the 3 must go in the greater of these two cells. There is only one way to fill in this block, and reflections do not change the number of solutions. In fact, blocks of class I correspond to all blocks with unique solutions. We further note that in each block, entries 1 and 3 will be placed diagonally in the block.

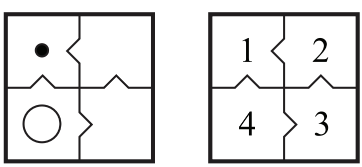


Figure 11. Class I blocks have 1 solution.

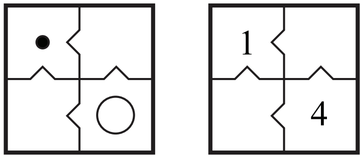


Figure 12. Class II blocks have 2 solutions.

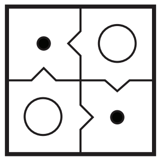


Figure 13. Class III blocks have 4 solutions.

Following the same procedure with blocks from class II, we can see in Figure 12 that the 1 and 4 entries are uniquely placed. However, examining the remaining two cells, we see that the entries in each must be greater than the 1 entry and less than the 4 entry. It is not possible from this arrangement to uniquely determine placement of the remaining two entries. Thus, blocks from class II correspond to blocks with two solutions, and in each solution 1 and 4 will be placed diagonally.

In class III blocks, however, we have two possible positions for the 1 entry. Similarly, there are two possible placements of the 4 entry. Once 1 and 4 are placed in the cells, there is only one way to place 2 and 3. Each puzzle block from this class has four solutions as shown in Figure 13; entries 1 and 2 will be in the minimal cells, which are placed diagonally.

6. Greater Than Shidoku puzzles

Now that we have a better understanding of the different types of inequality blocks, we are able to examine ways in which they can be combined to form puzzles. Recall that a Greater Than board is an $mn \times mn$ grid, where $m, n \in \mathbb{N}$, in which the numbers 1 through mn must satisfy the inequalities between adjacent cells and appear exactly once in each row, column, and $m \times n$ block. If when the numerical entries are removed there is a unique solution to the board, the unfilled board is a *Greater Than puzzle.*

It is important to note that, by definition, *every* Greater Than board is solvable when the entries are removed. It is not necessarily the case, however, that each board has a unique solution and is therefore a puzzle. In this section, we will first find a way to create Greater Than Shidoku boards, then determine whether the unfilled boards are puzzles. Previously, we saw that each of the three equivalence

a	b		
c	d		

Figure 14. Initial block.

classes of blocks may be identified by the entry that is diagonal from 1 when a block is solved. This is a useful tool in proving Theorem 7, which states a rule for combining blocks to create boards. Although the proof begins by considering only entries without inequalities, a Greater Than board can be formed from the standard board by inserting the appropriate inequalities between adjacent cells within each block.

Theorem 7. *Every block of a Greater Than Shidoku board must be horizontally or vertically adjacent to another block from the same equivalence class.*

Proof. Assume, to the contrary, that a block need not be adjacent to another block from the same equivalence class. Without loss of generality, consider the filled block in Figure 14.

To complete the top row, we place c and d in one of two ways. Once these are placed, we then position a and b in the second row to ensure that the top two blocks are from different classes. Although we don't know which cell will contain the 1 entry, it is sufficient to ensure that the blocks do not contain any common diagonal. We use similar logic on the first two columns to fill in the bottom-left block in one of two ways, again ensuring that it is not equivalent to the first block. This gives us the four cases shown in Figure 15. In each case we attempt to complete the board by filling in the last block. There is only one cell where we can place the a entry, but then we find that we are unable to place the d entry without violating the condition that an entry may only appear once in each row and column, leading

a	b	c	d
c	d	b	a
b	c	a	
d	a		

Case 1

a	b	c	d
c	d	b	a
d	a		
b	c	a	

Case 2

a	b	d	c
c	d	a	b
b	c		a
d	a		

Case 3

a	b	d	c
c	d	a	b
d	a		
b	c		a

Case 4

Figure 15. Each case leads to a contradiction.

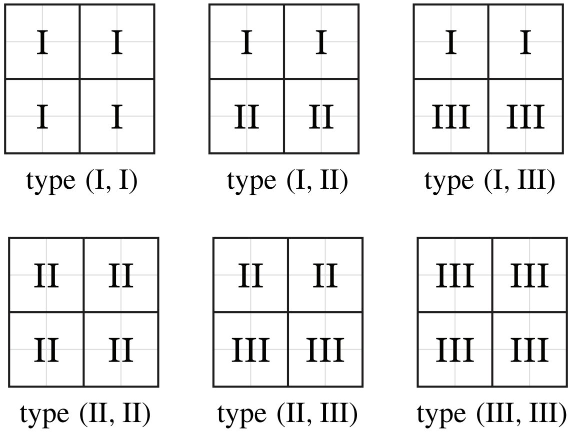


Figure 16. Six types of Greater Than Shidoku boards.

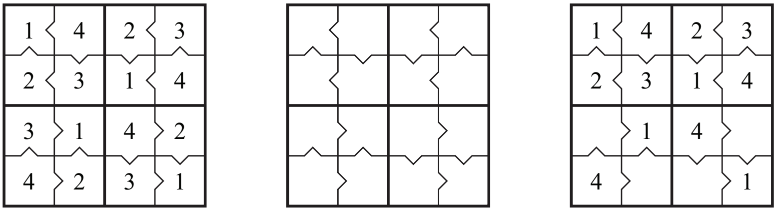
to our desired contradiction. Thus, every block must be adjacent to at least one equivalent block to form a Greater Than Shidoku board. □

Corollary 8. *There are six types of Greater Than Shidoku boards.*

This corollary follows directly from counting the possible combinations of our three equivalence classes, shown in Figure 16. A board of type (I, II), for example, is comprised of two blocks from class I and two from class II. Note that boards comprised of two different block classes may be written in four different ways, taking rotations of 90°, 180°, and 270° into consideration. Each of these types may be used to form Greater Than Shidoku boards, so our next goal is to determine which of these boards have unique solutions when the entries are removed, and are therefore Greater Than Shidoku puzzles. In the following lemmas, we will see that 4 of these 6 board types correspond to puzzles, and we will count the number of puzzles of each type.

Lemma 9. *Every board of type* (I, I) *has a unique solution when the entries are removed, and therefore corresponds to a Greater Than Shidoku puzzle. There are* 32 *puzzles of type* (I, I).

Proof. Consider any board of type (I, I) and remove all entries, leaving only inequalities. This board consists of Greater Than blocks from class I, and each of these blocks has a unique solution, so there is only one way to fill in entries on the entire board. Thus every board of type (I, I) corresponds to a puzzle. To create a board, there are eight ways to order the first block, since there are four cells in which to place the 1 entry, two ways to place 4 adjacent to 1, and then the cells containing 2 and 3 are uniquely determined. The second and third blocks can each be arranged in two ways, similar to the argument in the proof of Theorem 7, and the fourth block is uniquely determined by the first three. We then place the



board type (I, II) puzzle type (I, II) initial placements

Figure 17. Type (I, II) example.

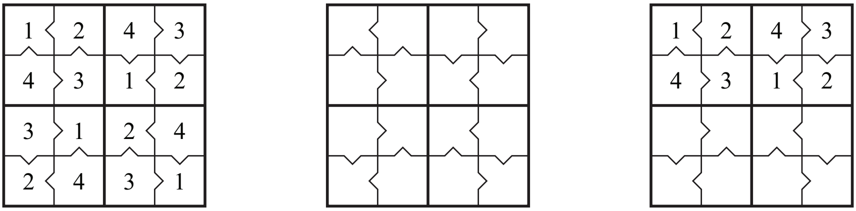
appropriate inequalities to finish the board. Consequently, there are $(8)(2)(2) = 32$ puzzles of type (I, I). ☐

Lemma 10. *Each of the* 64 *type* (I, II) *boards corresponds to a puzzle.*

Proof. Consider a board of type (I, II) such as that in Figure 17, and remove the entries. Without loss of generality, suppose the top two blocks are from class I and the lower two from class II. Since blocks from class I can only be filled in one way and blocks from class II have uniquely determined 1 and 4 entries, the only entries not uniquely determined by inequalities are the 2 and 3 entries on the blocks from class II. However, these entries are placed diagonally in their block, so each column has only one unfilled cell. Thus, by standard Shidoku rules, there is only one possible entry that can be placed in each unfilled cell, leading to a unique solution for the unfilled board. To count these puzzles, we will start by counting boards with blocks placed as in Figure 17. In the top-left block, there are four ways to place the 1 entry, then the 3 entry must be diagonal from 1. There are two ways to place the remaining 2 and 4 entries. In both the top-right block (class I) and the class II block on the bottom-left, we have two choices for placing 1 so that it isn't in the same row or column as the 1 in the first block. Once those choices are made, the placement of the other entries in those blocks is uniquely determined. All of the entries in the last block are uniquely determined, and once again we finish by writing in the inequalities. Thus there are $(4)(2)(2) = 16$ puzzles of type (I, II) in the form described, however, since each of these puzzles may be rotated 90°, 180°, or 270° to create new puzzles, there are 64 puzzles of this type. ☐

Lemma 11. *There are* 64 *type* (I, III) *puzzles.*

Proof. As in the previous case, we will consider a board of type (I, III) such as that in Figure 18 with class I blocks on top and class III below. Again, the class I blocks have a unique solution. The blocks from class III all have the entry 2 placed diagonally from 1, with 3 and 4 on the other diagonal. Since entries 1 and 2 must be in different columns in the class III blocks, the two blocks from class I must be oriented so entries 1 and 2 are also in different columns to avoid contradiction with



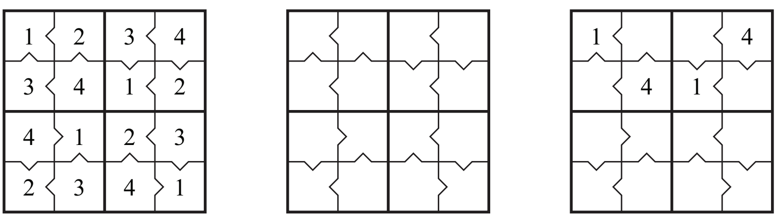
board type (I, III) puzzle type (I, III) 1, 2 in different columns

Figure 18. Type (I, III) example.

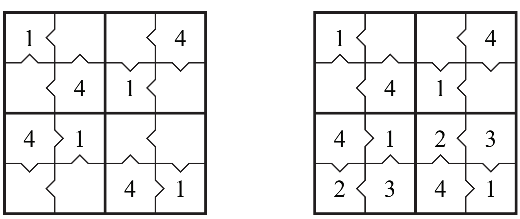
the class III blocks. Furthermore, we recall that each column in a class III block contains both a maximal and minimal cell. Each column of the board contains either a 1 or a 2 in the top two blocks; the other is placed in the minimal cell in that column. Similarly, each column already contains either a 3 or a 4; the other must go in the remaining cell, which is a maximal cell. Thus each cell is filled uniquely, and the board corresponds to a puzzle of type (I, III). We count the puzzles as in the previous lemma: four ways to fill the first block, two ways to fill blocks to right and below, then one way to complete the last block. Including rotations, there are 64 type (I,III) puzzles. □

Lemma 12. *There are* 64 *type* (II, III) *puzzles.*

Proof. Suppose our board has class II blocks on top and class III blocks below, such as in Figure 19. The entries are uniquely placed in the top blocks. As argued in the proof of Lemma 11, there is a maximal and minimal cell in each column of the



board type (II, III) puzzle type (II, III) initial placements



using rows and columns 2, 3 in bottom blocks

Figure 19. Type (II, III) example.

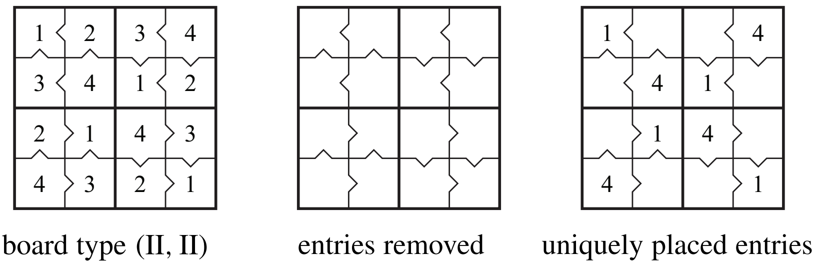


Figure 20. Type (II, II) example.

bottom blocks. Since the 1 entries have already been placed in two of the columns of the puzzle, the minimal entries in each of the remaining columns on the lower band must contain 1 entries. Similarly, the 4 entries have already been placed in two columns; the maximal entries in each of the remaining columns on the lower band must contain 4 entries as well. There are now two remaining cells in each bottom block. These cells are adjacent, and thus the 2 entry is placed in the lesser of the cells, while the 3 is placed in the greater of the two. So the bottom blocks are uniquely filled in. Now, returning to the top blocks we see that there is only one remaining unfilled cell in each column. Therefore, there is only one possible entry for each cell, which completes the unique solution, so the type (II, III) board corresponds to a puzzle.

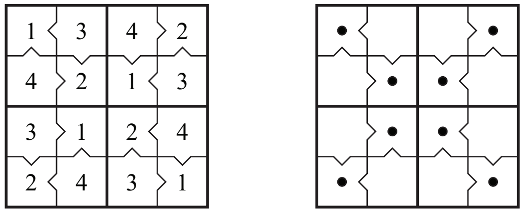
There are four choices in placing the 1 in the top-left class II block; after that the 4 must be placed diagonally from 1. The 1 and 2 entries cannot be in the same column, since 1 and 2 must be in different columns in the class III block below it, so the placement of the 2 and 3 entries in the first block is uniquely determined. There are two choices for orienting each of the blocks adjacent to the top-left block, and one way to complete the remaining block. Thus there are $(4)(2)(2) = 16$ ways to fill in the board, and taking into consideration the 4 possible rotations there are 64 puzzles. □

Lemma 13. *Boards of type* (II, II) *do not correspond to puzzles.*

Proof. Consider a block from class II. The 1 and 4 elements are uniquely determined, but there are two remaining cells which contain precisely the same inequality set. Thus, given any class II block, it is not possible to identify a unique placement of either the 2 or the 3 entries. As we see in Figure 20, even when the 1 and 4 entries are placed in a type (II, II) board we still have two choices for placing 2 and 3, and therefore removing the entries does not create a puzzle. □

Lemma 14. *Boards of type* (III, III) *do not correspond to puzzles.*

Proof. Class III blocks each have two minimal and two maximal cells. Regardless of how we orient the blocks in a type (III, III) board, when we remove the entries



board type (III, III) no unique placements

Figure 21. Type (III, III) example.

the unfilled board will have two minimal cells in each row and column. As seen in Figure 21, we have two choices for placing the 1 entry in the first block, which then determines the placement of the remaining 1 entries as well as the 2 entries. Similarly, having two maximal cells in each row and column will result in two different ways to place the set of 3 and 4 entries. Thus the unfilled board has 4 solutions and is not a valid puzzle. □

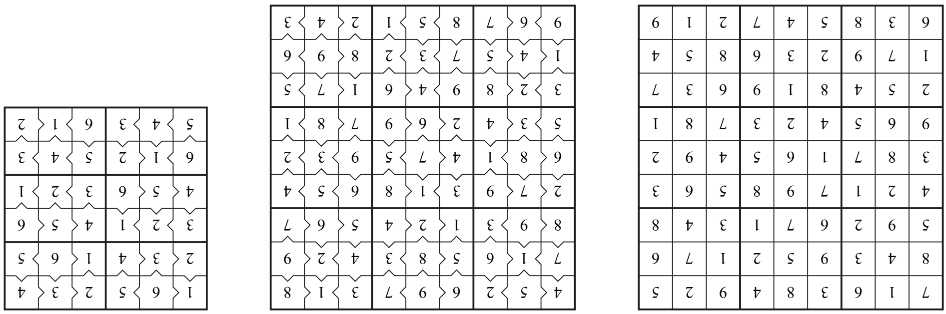
Combining these six lemmas, we can now make a statement about Greater Than Shidoku puzzles.

Theorem 15. *There are* 224 *Greater Than Shidoku puzzles.*

7. Further explorations

When creating Greater Than puzzles of any size, we must avoid the use of blocks that contain cycles because these blocks are unsolvable. We have shown that every acyclic inequality block is solvable, thus the acyclic Greater Than Shidoku, Rokudoku, and Sudoku blocks that were counted in Section 3 may all potentially be used to create Greater Than puzzles. Nevertheless, we have also seen that not all types of solvable blocks may be used together to form a valid puzzle. The task of combining blocks to form puzzles becomes increasingly complex as the size of the puzzle increases, and research related to standard Sudoku is not always directly applicable. For example, Rosenhouse and Taalman [2011] showed that there are 288 standard Shidoku boards, but those corresponding to types (II, II) and (III, III) do not have a unique solution when the numbers are removed and only the inequalities remain, so there are fewer Greater Than Shidoku puzzles than boards. The equivalence relation defined on 2×2 acyclic blocks can also be applied to other sets of $m \times n$ acyclic blocks (excluding reflection across the diagonals if $m \neq n$). This will allow us to partition the sets of acyclic blocks into equivalence classes that may facilitate our investigation, but Felgenhauer and Jarvis' computer-aided calculation [2006] of 6,670,903,752,021,072,936,960 standard 9×9 Sudoku boards hints at the challenge presented by task.

Puzzle solutions



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