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Discrete time optimal control applied to pest control problems

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We apply discrete time optimal control theory to the mathematical modeling of pest control. Two scenarios: biological control and the combination of pesticide and biological control are considered. The goal is maximizing the "valuable" population, minimizing the pest population and the cost to apply the control strategies. Using the extension of Pontryagin's maximum principle to discrete system, the adjoint systems and the characterization of the optimal pest controls are derived. Numerical simulations of various cases are provided to show the effectiveness of our methods.

1. Introduction

Pesticides and biological control are two popular ways of pest control. One of the conventional applications of control uses pesticides. The detrimental effects to local ecologies of overuse of pesticides has been widely documented, therefore, the conservation, introduction, and restocking of a pest's natural enemies has become increasingly popular. Biological control is the use of living organisms to suppress the population of a specific pest organism, making it less abundant or less damaging than it would otherwise be [Eilenberg et al. 2001]. It is an environmentally sound and effective means of reducing or mitigating pests and pest effects through the use of natural enemies, and biological control has successfully contributed to the protection of the flora and fauna of many natural ecosystems [Driesche et al. 2010; Driesche 1994].

This study will focus on developing and analyzing two mathematical models for pest control using biocontrol and the combination of the pesticide and the biocontrol, while finding the optimal pest control strategies.

But biological control is both powerful and risky. Biological control agents may negatively affect native species directly or indirectly. Historically biological

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control introductions were not regulated the way they are today, and some horrible mistakes were made in the name of biological control (e.g., cane toads in Australia). Hawkins and Cornell [1999] gathered together recent theoretical developments and provide a guide to the critical issues that need to be considered in applying theory to biological control, they pointed out by developing theories based on fundamental population principles and the biological characteristics of the pest and agent, we can gain a much better understanding of when and how to use biological control.

A lot of studies done in this field have focused on the continuous predator-prey models, which are based on the assumption that population changes are always occurring. While this may be true for humans (births and deaths are fairly well distributed over time), many species have well-defined cycles of reproductions (births and deaths generally occur over a season or period of a few weeks or months). This fact causes us to focus on a discrete model over a continuous one for these biological systems.

The efficacy with which one is able to reduce a pest population is always subject to the amount of resources available to control that population. Due to cost and environmental consideration, it may be more appropriate to release a smaller amount of the predator population into the ecosystem, and then add to that amount incrementally for a given time frame to reduce the pest population more gradually. The costs involved will be substantially less in this case because the predator population will grow on its own, thus reducing the need to introduce more predators artificially, and it will be beneficial to the natural ecosystems.

Optimal control theory for discrete systems is well developed [Clark 1990; Sethi and Thompson 2006], but there are very few applications in pest control problems. Tang and Cheke [2008] studied integrated pest control problems using both continuous and discrete host-parasitoid models. Jang and Yu [2012] proposed a simple discrete time host-parasitoid model and derived an optimal control model using a chemical as a control for the hosts. They conclude that applying a chemical to eliminate the hosts directly may be a more effective control strategy than using the parasitoids to indirectly suppress the hosts. Whittle et al. [2007] use a discrete-time optimal control model to provide management for an invasive species consisting of a large main focus and several smaller outlier populations. Dabbs [2010] presents discrete time pest control models using three different growth functions: logistic, Beverton–Holt and Ricker spawner-recruit functions and compares the optimal control strategies respectively. Berryman [Hawkins and Cornell 1999] provides a review of the historic development of the ecological theory that relates to biological control, focusing on discrete time models that best describe systems in which the insects reproduce seasonally. He presents the control theory and the theory of predator-prey dynamics which are the key elements of the theory of biological control.

The paper is organized as follows: in Section 2 we present the optimal biological control problem, derive the adjoint equations and the characterization of the control, and give numerical results. In Section 3, we formulate the optimal dual control problems, derive the necessary conditions of optimal control and give some numerical results.

2. Optimal control using biological control

2.1. ***The biological control problem.*** Biological control of pests has been practiced in greenhouse as well as in field crops. For example *E. formosa* and *P. persimilis* have been used as biological control agents to reduce parasites over different crops such as tomatoes and cucumbers [van Lenteren and Woets 1988]. In our model, the valuable population, pest population and the predator (biological control agent) population are represented by

$$x = (x_0, x_1, \ldots, x_T), \quad y = (y_0, y_1, \ldots, y_T), \quad z = (z_0, z_1, \ldots, z_T),$$

respectively, where the subscripts represent the time steps. The control satisfies

$$U_1 = \{u = (u_0, u_1, \ldots, u_{T-1}) \in \mathbb{R}^T \mid 0 \le u_k \le M,\ k = 0, 1, \ldots, T-1\},$$

with M the maximum control effort.

The model is, for $k = 0, 1, \ldots, T-1$ and given x_0, y_0, z_0,

$$\begin{aligned} x_{k+1} &= x_k + r x_k(1 - x_k) - c_1 x_k y_k, \\ y_{k+1} &= d y_k + c_2 x_k y_k - c_3 y_k z_k, \\ z_{k+1} &= z_k - m z_k + c_4 y_k z_k + u_k z_k, \end{aligned} \tag{2-1}$$

where r and d are the intrinsic growth rates for the valuable population and pest population respectively, m is the death rate of the predator (biological control agent), the constants c_i, $i = 1, \ldots, 4$ are the interaction coefficients between the species. We apply the control u_k to increase the growth rate of the predator at each time step, for example, we can import the natural enemies of the pest or supplement the existing predators.

The goal is to maximize

$$\sum_{k=0}^{T-1} B_1 x_k - B_2 y_k - B_3 z_k - \tfrac{1}{2} A u_k^2 \tag{2-2}$$

over $u \in U_1$, with $A > 0$, $B_i > 0$, $i = 1, 2, 3$ constants; that is, we want to maximize the valuable population while minimizing the pest population and the cost of applying the biological control, we also minimize the predator population for environmental consideration over the entire time period. We choose a quadratic cost for simplicity and other forms could be treated.

We will use the extension of Pontryagin's maximum principle (PMP) [Lenhart and Workman 2007; Pontryagin et al. 1962; Sethi and Thompson 2006] for the optimal control of discrete system. The technique involves the use of adjoint functions, which append the discrete system (2-1) to the maximization of the objective functional (2-2). PMP gives the optimality system of difference equations consisting of the state and adjoint difference equations coupled with the control characterization. Note that the adjoint equations have final time boundary conditions while the state equations have initial conditions. The key idea is that the adjoint method provides us with the gradient of the cost function needed for the maximization procedure. We note that an optimal control exists due to the finite dimensional structure of this system.

Applying the extension of Pontryagin's maximum principle for discrete systems [Lenhart and Workman 2007; Pontryagin et al. 1962; Sethi and Thompson 2006], we form the Hamiltonian:

$$\begin{aligned} H_k = B_1 x_k - B_2 y_k - B_3 z_k - \tfrac{1}{2} A u_k^2 + \lambda_{1,k+1}\big(x_k + r x_k(1-x_k) - c_1 x_k y_k\big) \\ + \lambda_{2,k+1}(d y_k + c_2 x_k y_k - c_3 y_k z_k) + \lambda_{3,k+1}(z_k - m z_k + c_4 y_k z_k + u_k z_k), \end{aligned} \tag{2-3}$$

which is used to derive the necessary conditions in the next theorem.

Theorem 2.1. *Given an optimal control* $u^* \in U_1$ *and the corresponding states* x^*, y^*, z^* *from* (2-1), *there exist adjoint functions* $\lambda_i,\ i = 1, 2, 3$ *satisfying*:

$$\begin{aligned} \lambda_{1,k} &= B_1 + \lambda_{1,k+1}\big(1 + r - 2r x_k^* - c_1 y_k^*\big) + \lambda_{2,k+1} c_2 y_k^*, \\ \lambda_{2,k} &= -B_2 - \lambda_{1,k+1} c_1 x_k^* + \lambda_{2,k+1}\big(d + c_2 x_k^* - c_3 z_k^*\big) + \lambda_{3,k+1} c_4 z_k^*, \\ \lambda_{3,k} &= -B_3 - \lambda_{2,k+1} c_3 y_k^* + \lambda_{3,k+1}\big(1 - m + c_4 y_k^* + u_k^*\big), \\ \lambda_{1,T} &= \lambda_{2,T} = \lambda_{3,T} = 0. \end{aligned} \tag{2-4}$$

Furthermore, the characterization of u_k^* *is*

$$u_k^* = \min\big\{\max\{\lambda_{3,k+1} z_k^*/A, 0\}, M\big\}. \tag{2-5}$$

Proof. Using the extension of Pontryagin's maximum principle for discrete systems [Lenhart and Workman 2007; Pontryagin et al. 1962; Sethi and Thompson 2006], we have

$$\begin{aligned} \lambda_{1,k} &= \frac{\partial H_k}{\partial x_k} = B_1 + \lambda_{1,k+1}(1 + r - 2r x_k^* - c_1 y_k^*) + \lambda_{2,k+1} c_2 y_k^*, \\ \lambda_{2,k} &= \frac{\partial H_k}{\partial y_k} = -B_2 - \lambda_{1,k+1} c_1 x_k^* + \lambda_{2,k+1}(d + c_2 x_k^* - c_3 z_k^*) + \lambda_{3,k+1} c_4 z_k^*, \\ \lambda_{3,k} &= \frac{\partial H_k}{\partial z_k} = -B_3 - \lambda_{2,k+1} c_3 y_k^* + \lambda_{3,k+1}(1 - m + c_4 y_k^* + u_k^*). \end{aligned} \tag{2-6}$$

In addition, the transversality conditions are

$$\lambda_{1,T} = \lambda_{2,T} = \lambda_{3,T} = 0.$$

Using

$$\frac{\partial H_k}{\partial u_k} = -Au_k + \lambda_{3,k+1} z_k,$$

and $\partial H_k / \partial u_k = 0$ at u^* on the interior of the control set, we have the control characterization

$$u_k^* = \min\{\max\{\lambda_{3,k+1} z_k^* / A, 0\}, M\}. \tag{2-7}$$

This concludes the proof. □

The optimality system consists of the state equations (2-1) with initial conditions and adjoint equations (2-4) with the final time conditions and with the characterization of the optimal control (2-5).

2.2. ***Numerical results.*** To solve the optimal biological control problem numerically, due to the boundary conditions being at the initial time for the states and at the final time for adjoints, an iterative method is used to solve this optimality system. Given initial guesses for the control and the state equations, the state system (2-1) is solved forward in time, and the adjoint system (2-4) is solved backward in time. The control is updated using the characterization (2-7) with the newly found state and adjoint values, and the iteration repeats until convergence occurs. See [Lenhart and Workman 2007] for details of this method.

Figure 1 (left) gives the valuable, pest and predator populations without the application of the control. Without the control, the pest population increases, killing off the valuable population, for $r = d = 1.1$, $m = 0.05$, $c_1 = 1$, $c_2 = c_3 = 1.2$,

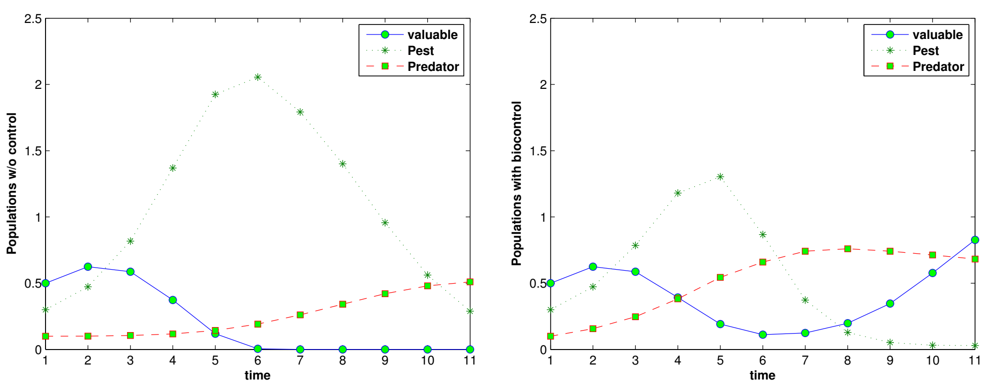


Figure 1. Valuable, pest, predator populations without biological control (left) and with biological control (right; $A = 5$).

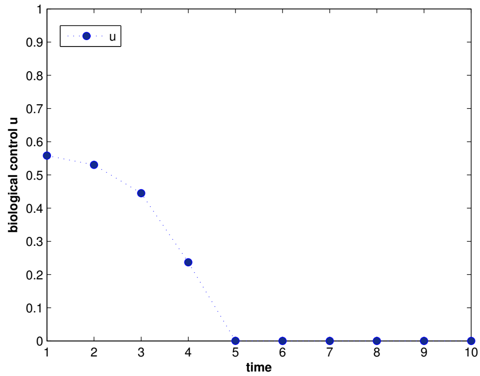


Figure 2. Optimal biological control, $A = 5$.

$c_4 = 0.2$. In contrast, with the biological control, the growth of the pest population decreases, allowing the valuable population to grow quickly; see Figure 1 (right). Figure 2 gives the optimal biocontrol result for $A = 5$, $M = 1$, $B_i = 1$, $i = 1, 2, 3$. We see the optimal biological control effort is gradually decreasing and we don't apply it after time step 5.

3. Optimal control using dual control

3.1. ***The dual control problem.*** Now we attempt to control the pest population using both biological control and pesticide at the same time.

The controls satisfy

$$U_2 = \big\{(u_{i,0}, u_{i,1}, \dots, u_{i,T-1}) \in \mathbb{R}^T \mid 0 \le u_{i,k} \le M,\ i = 1, 2,\ k = 0, 1, \dots, T-1\big\},$$

with M the maximum control effort.

The model is, for $k = 0, 1, \dots, T-1$ and given x_0, y_0, z_0,

$$\begin{aligned} x_{k+1} &= x_k + r x_k(1 - x_k) - c_1 x_k y_k, \\ y_{k+1} &= d y_k + c_2 x_k y_k - c_3 y_k z_k - u_{1,k} y_k, \\ z_{k+1} &= z_k - m z_k + c_4 y_k z_k + u_{2,k} z_k, \end{aligned} \tag{3-1}$$

where the control $u_{1,k}$ is the pesticide and we also apply the control $u_{2,k}$ to increase the growth rate of the predator (biological control agent) at each time step.

The goal is to maximize

$$\sum_{k=0}^{T-1} B_1 x_k - B_2 y_k - B_3 z_k - \tfrac{1}{2} A_1 u_{1,k}^2 - \tfrac{1}{2} A_2 u_{2,k}^2, \tag{3-2}$$

with $B_i > 0$, $i = 1, 2, 3$, $A_j > 0$, $i = 1, 2$; that is, we want to maximize the valuable population while minimizing the pest population and the cost of applying the pesticide and biological control, we also minimize the predator population for environmental purpose over the entire time period.

Applying the extension of Pontryagin's maximum principle for discrete systems [Lenhart and Workman 2007; Pontryagin et al. 1962; Sethi and Thompson 2006], we form the Hamiltonian:

$$\begin{aligned} H_k = B_1 x_k - B_2 y_k - B_3 z_k - \tfrac{1}{2} A_1 u_{1,k}^2 - \tfrac{1}{2} A_2 u_{2,k}^2 \\ + \lambda_{1,k+1}\big(x_k + r x_k(1-x_k) - c_1 x_k y_k\big) \\ + \lambda_{2,k+1}(d y_k + c_2 x_k y_k - c_3 y_k z_k - u_{1,k} y_k) \\ + \lambda_{3,k+1}(z_k - m z_k + c_4 y_k z_k + u_{2,k} z_k), \end{aligned} \tag{3-3}$$

which is used to derive the necessary conditions in the next theorem.

Theorem 3.1. *Given optimal controls* $u_i^* \in U_2, i = 1, 2$ *and the corresponding states* $x^*,\ y^*,\ z^*$ *from* (3-1), *there exist adjoint functions* $\lambda_i,\ i = 1, 2, 3$ *satisfying*

$$\begin{aligned} &\lambda_{1,k} = B_1 + \lambda_{1,k+1}\big(1 + r - 2r x_k^* - c_1 y_k^*\big) + \lambda_{2,k+1} c_2 y_k^*, \\ &\lambda_{2,k} = -B_2 - \lambda_{1,k+1} c_1 x_k^* + \lambda_{2,k+1}\big(d + c_2 x_k^* - c_3 z_k^* - u_{1,k}^*\big) + \lambda_{3,k+1} c_4 z_k^*, \\ &\lambda_{3,k} = -B_3 - \lambda_{2,k+1} c_3 y_k^* + \lambda_{3,k+1}\big(1 - m + c_4 y_k^* + u_{2,k}^*\big), \\ &\lambda_{1,T} = \lambda_{2,T} = \lambda_{3,T} = 0. \end{aligned} \tag{3-4}$$

Furthermore, the characterizations of $u_{1,k}^*, u_{2,k}^*$ *are*

$$\begin{aligned} u_{1,k}^* &= \min\big\{\max\{-\lambda_{2,k+1} y_k^*/A_1, 0\}, M\big\}, \\ u_{2,k}^* &= \min\big\{\max\{\lambda_{3,k+1} z_k^*/A_2, 0\}, M\big\}. \end{aligned} \tag{3-5}$$

Proof. Using the extension of Pontryagin's maximum principle for discrete systems [Lenhart and Workman 2007; Pontryagin et al. 1962; Sethi and Thompson 2006], we have

$$\begin{aligned} \lambda_{1,k} &= \frac{\partial H_k}{\partial x_k} = B_1 + \lambda_{1,k+1}\big(1 + r - 2r x_k^* - c_1 y_k^*\big) + \lambda_{2,k+1} c_2 y_k^*, \\ \lambda_{2,k} &= \frac{\partial H_k}{\partial y_k} = -B_2 - \lambda_{1,k+1} c_1 x_k^* + \lambda_{2,k+1}(d + c_2 x_k^* - c_3 z_k^* - u_{1,k}^*) + \lambda_{3,k+1} c_4 z_k^*, \\ \lambda_{3,k} &= \frac{\partial H_k}{\partial z_k} = -B_3 - \lambda_{2,k+1} c_3 y_k^* + \lambda_{3,k+1}(1 - m + c_4 y_k^* + u_{2,k}^*). \end{aligned}$$

In addition, the transversality conditions are

$$\lambda_{1,T} = \lambda_{2,T} = \lambda_{3,T} = 0. \tag{3-6}$$

Using

$$\frac{\partial H_k}{\partial u_{1,k}} = -A_1 u_{1,k} - \lambda_{2,k+1} y_k,$$

and $\partial H_k / \partial u_{1,k} = 0$ at u_1^* on the interior of the control set, we have the control characterization

$$u_{1,k}^* = \min\bigl\{\max\{-\lambda_{2,k+1} y_k^*/A_1, 0\}, M\bigr\}. \tag{3-7}$$

And using

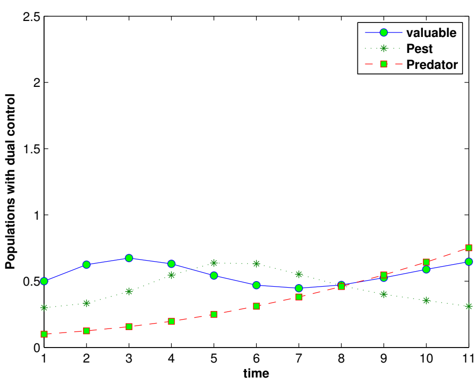
$$\frac{\partial H_k}{\partial u_{2,k}} = -A_2 u_{2,k} + \lambda_{3,k+1} z_k,$$

and $\partial H_k / \partial u_{2,k} = 0$ at u_2^* on the interior of the control set, we have the control characterization

$$u_{2,k}^* = \min\bigl\{\max\{\lambda_{3,k+1} z_k^*/A_2, 0\}, M\bigr\}. \tag{3-8}$$

This concludes the proof. □

3.2. ***Numerical results and conclusion.*** In this section, we apply dual control, combining pesticide and biological control. Figure 3 shows the significant increase in valuable population and decrease in pest population after applying dual control, and maintaining the predator at a reasonable level. Figure 3 (left) gives the result for $A_1 = A_2 = 5$ and Figure 3 (right) gives the result for $A_1 = A_2 = 1$, with all the other parameters kept the same with Section 2.2. We vary the cost coefficients A_1, A_2 to see the effect on the populations and the optimal control strategy. With the lower cost coefficients A_i, $i = 1, 2$, we can apply more pesticide and biological control, and the pest population can be reduced to a lower level; see Figures 3 and



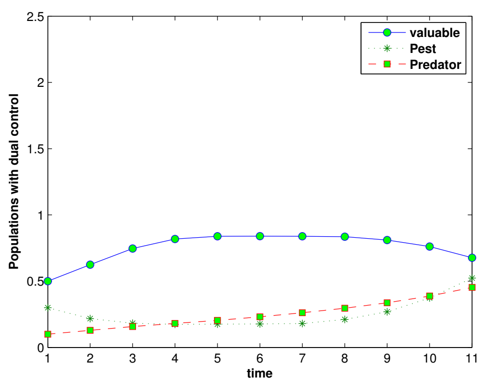


Figure 3. Valuable, pest, predator populations with dual control. Left: $A_1 = 5$, $A_2 = 5$. Right: $A_1 = 1$, $A_2 = 1$.

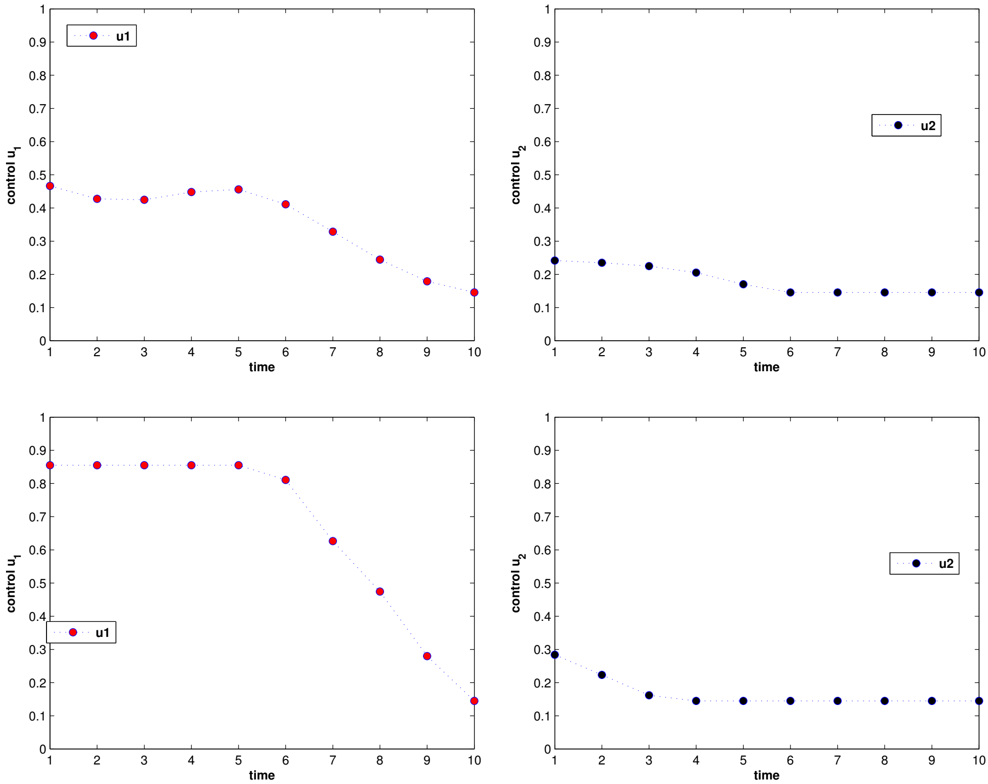


Figure 4. Optimal dual control, different A_1 and A_2. Top: pesticide, $A_1 = 5,\ A_2 = 5$ (left); biological control, $A_1 = 5,\ A_2 = 5$ (right). Bottom: pesticide, $A_1 = 1,\ A_2 = 1$ (left); biological control, $A_1 = 1,\ A_2 = 1$ (right).

4. We note that lowering the cost of both controls provides a substantial increase in pesticide use and only a modest increase in the use of biological control.

We also compare the result of biological control and dual control. We see from Figures 1 (right) and 3 that dual control gives better results for maintaining the valuable population and reducing the pest population, while keeping the predator at a low level.

In summary, we give a theoretical framework using discrete time optimal control theory for pest control problems and provide the numerical results. We apply the biological control and the combination of the pesticide and biological control (dual control) to find the optimal strategy. The results provide suggestions in the design of appropriate control strategies and assist management decision-making.

We should note that in our models (2-1) and (3-1), the order of events is that population growth occurs first, then it is increased/decreased by interactions with other species or through human intervention. We can explore other order of events since Bodine et al. [2012] point out for discrete models different order of events can lead to qualitatively different optimal control strategies.

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References

[Bodine et al. 2012] E. N. Bodine, L. J. Gross, and S. Lenhart, "Order of events matter: comparing discrete models for optimal control of species augmentation", *J. Biol. Dyn.* **6**:suppl. 2 (2012), 31–49. MR 2994278

[Clark 1990] C. W. Clark, *Mathematical bioeconomics: the optimal management of renewable resources*, 2nd ed., Wiley, New York, 1990. MR 91c:90037

[Dabbs 2010] K. Dabbs, *Optimal control in discrete pest control models*, Ph.D. thesis, University of Tennessee, 2010, Available at http://goo.gl/2Cp8uI.

[Driesche 1994] R. G. V. Driesche, "Classical biological control of environmental pests", *Florida Entomologist* **77** (1994), 20–33.

[Driesche et al. 2010] R. G. V. Driesche, R. I. Carruthers, T. Center, M. S. Hoddle, J. Hough-Goldstein, L. Morin, L. Smith, D. L. Wagner, B. Blossey, V. Brancatini, R. Casagrande, C. E. Causton, J. A. Coetzee, J. Cuda, J. Ding, S. V. Fowler, J. H. Frank, R. Fuester, J. Goolsby, M. Grodowitz, T. A. Heard, M. P. Hill, J. H. Hoffmann, J. Hubert, M. Julien, M. T. K. Kairo, M. Kenis, P. Mason, J. Medal, R. Messing, R. Miller, A. Moore, P. Neuenschwander, R. Newman, H. Norambuena, W. A. Palmer, R. Pemberton, A. P. Panduro, P. D. Pratt, M. Rayamajhi, S. Salom, D. Sands, S. Schooler, M. Schwarzländer, A. Sheppard, R. Shaw, P. W. Tipping, and R. van Klinken, "Classical biological control for the protection of natural ecosystems", *Biological Control* **54** (2010), S2–S33.

[Eilenberg et al. 2001] J. Eilenberg, A. Hajek, and C. Lomer, "Suggestions for unifying the terminology in biological control", *BioControl* **46** (2001), 387–400.

[Hawkins and Cornell 1999] B. Hawkins and H. Cornell, *Theoretical approaches to biological control*, Cambridge University Press, 1999.

[Jang and Yu 2012] S. Jang and J. Yu, "Discrete-time host-parasitoid models with pest control", *Journal of Biological Dynamics* **6**:2 (2012), 718–739.

[Lenhart and Workman 2007] S. Lenhart and J. T. Workman, *Optimal control applied to biological models*, Chapman & Hall/CRC, Boca Raton, FL, 2007. MR 2008f:49001

[van Lenteren and Woets 1988] J. C. van Lenteren and J. Woets, "Biological and integrated pest control in greenhouses", *Ann. Rev. Entomol* **33** (1988), 239–269.

[Pontryagin et al. 1962] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The mathematical theory of optimal processes*, Wiley Interscience, New York, 1962. MR 29 #3316b

[Sethi and Thompson 2006] S. P. Sethi and G. L. Thompson, *Optimal control theory: applications to management science and economics*, 2nd ed., Kluwer, Boston, 2006. MR 84g:49002

[Tang and Cheke 2008] S. Tang and R. A. Cheke, "Models for integrated pest control and their biological implications", *Math. Biosci.* **215**:1 (2008), 115–125. MR 2009i:92099

[Whittle et al. 2007] A. J. Whittle, S. Lenhart, and L. J. Gross, "Optimal control for management of an invasive plant species", *Math. Biosci. Eng.* **4**:1 (2007), 101–112. MR 2007h:92094

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