

# involve

a journal of mathematics

Infinite cardinalities in the Hausdorff  
metric geometry

Alexander Zupan





# Infinite cardinalities in the Hausdorff metric geometry

Alexander Zupan

(Communicated by Józef H. Przytycki)

The Hausdorff metric measures the distance between nonempty compact sets in  $\mathbb{R}^n$ , the collection of which is denoted  $\mathcal{H}(\mathbb{R}^n)$ . Betweenness in  $\mathcal{H}(\mathbb{R}^n)$  can be defined in the same manner as betweenness in Euclidean geometry. But unlike betweenness in  $\mathbb{R}^n$ , for some elements  $A$  and  $B$  of  $\mathcal{H}(\mathbb{R}^n)$  there can be many elements between  $A$  and  $B$  at a fixed distance from  $A$ . Blackburn et al. (“A missing prime configuration in the Hausdorff metric geometry”, *J. Geom.*, **92**:1–2 (2009), pp. 28–59) demonstrate that there are infinitely many positive integers  $k$  such that there exist elements  $A$  and  $B$  having exactly  $k$  different elements between  $A$  and  $B$  at each distance from  $A$  while proving the surprising result that no such  $A$  and  $B$  exist for  $k = 19$ . In this vein, we prove that there do not exist elements  $A$  and  $B$  with exactly a countably infinite number of elements at any location between  $A$  and  $B$ .

## 1. Introduction

The Hausdorff metric provides a means to measure distance in the family  $\mathcal{H}(\mathbb{R}^n)$  of nonempty compact sets in  $n$ -dimensional Euclidean space. There is a natural embedding of  $\mathbb{R}^n$  into  $\mathcal{H}(\mathbb{R}^n)$  that takes  $x \in \mathbb{R}^n$  to  $\{x\} \in \mathcal{H}(\mathbb{R}^n)$ . The notion of betweenness in  $\mathbb{R}^n$  extends naturally to  $\mathcal{H}(\mathbb{R}^n)$ . However, in Euclidean space, there is a unique point between  $a$  and  $b$  at a given distance less than  $d(a, b)$  from  $a$ , while in  $\mathcal{H}(\mathbb{R}^n)$  there can be many distinct elements between elements  $A$  and  $B$  at a given distance from  $A$ . For instance, for infinitely many numbers  $k$  we can find  $A$  and  $B$  with exactly  $k$  elements between  $A$  and  $B$  at a given distance from  $A$ , and we can also find  $A$  and  $B$  such that this number of elements between  $A$  and  $B$  is infinite. Blackburn et al. [2009] proved the surprising result that there exist no two elements  $A$  and  $B$  in  $\mathcal{H}(\mathbb{R}^n)$  with the property that  $A$  and  $B$  have exactly 19 elements of  $\mathcal{H}(\mathbb{R}^n)$  between them at a given distance from  $A$ . In this paper, we will prove that there is another cardinality that is missing; namely, there exist no

---

*MSC2010*: primary 51F99; secondary 54B20.

*Keywords*: Hausdorff metric, betweenness, metric geometry.

two elements  $A, B \in \mathcal{H}(\mathbb{R}^n)$  with exactly a countably infinite number of elements between them at any location. The argument uses a different approach than that of [Blackburn et al. 2009]: the proof there is exhaustive, and while it is succinct enough to prove the absence of 19, the method may be too unwieldy to show the existence of larger conjectured missing numbers. It is our hope that the idea of too many removable points forcing a larger cardinality of sets between  $A$  and  $B$  might be adapted to the finite case.

## 2. Preliminaries

Let  $\mathcal{H}(\mathbb{R}^n)$  denote the collection of nonempty compact subsets of  $\mathbb{R}^n$ . We will refer to these compact sets as “elements” of  $\mathcal{H}(\mathbb{R}^n)$ . For any  $a \in \mathbb{R}^n$  and  $B \in \mathcal{H}(\mathbb{R}^n)$ , let  $d(a, B) = \min_{b \in B} d_E(a, b)$ , where  $d_E$  denotes the Euclidean metric on  $\mathbb{R}^n$ . The Hausdorff metric is then defined as follows:

**Definition 2.1.** Let  $A, B \in \mathcal{H}(\mathbb{R}^n)$ . The Hausdorff distance between  $A$  and  $B$  is given by

$$h(A, B) = \max\{d(A, B), d(B, A)\},$$

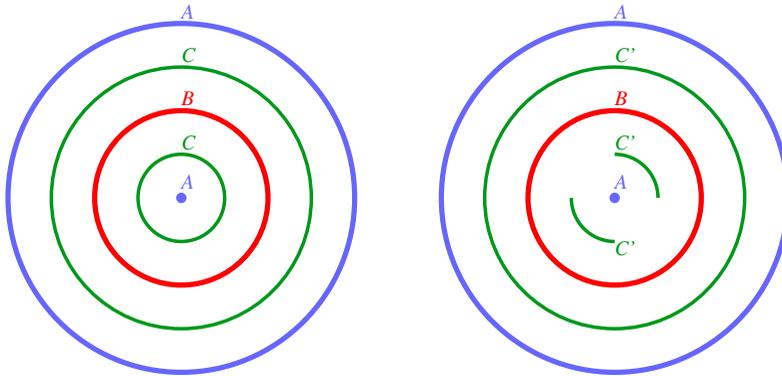
where  $d(A, B) = \max_{a \in A} d(a, B)$ .

In other words, the distance from  $A$  to  $B$  is the maximum of the distances between points in  $A$  to the set  $B$ , and the Hausdorff distance between  $A$  and  $B$  is the maximum of the distance from  $A$  to  $B$  and the distance from  $B$  to  $A$ . Note the maximum and minimum in the definitions above are well-defined since both  $A$  and  $B$  are compact sets. To verify that the Hausdorff distance defines a metric on  $\mathcal{H}(\mathbb{R}^n)$ , see, for instance, [Edgar 1990].

**Example 2.2.** Let  $n = 2$  and consider the sets shown on the left in Figure 1. Let  $S^1(r)$  denote the circle of radius  $r$  centered at the origin, so that  $A = \{(0, 0)\} \cup S^1(4)$ ,  $B = S^1(2)$ , and  $C = S^1(1) \cup S^1(3)$ . Then, for any  $a \in A$  and  $b \in B$ , we have  $d_E(a, b) \geq 2$ . Further, for such  $a$  there exists a point  $b \in B$  such that  $d(a, b) = 2$ , which implies that  $d(a, B) = 2$  for all  $a \in A$ ; hence,  $d(A, B) = 2$ . Similarly, for every  $b \in B$ , we have  $d_E(b, a_0) = 2$  where  $a_0$  is the origin, so  $d(b, A) = 2$  for all  $b \in B$ , which shows  $d(B, A) = 2$  as well. It follows that  $h(A, B) = 2$ . A similar verification shows that  $h(A, C) = h(B, C) = 1$ .

The set  $C'$  pictured on the right in Figure 1 is a compact subset of  $C$ . Here we see that for every  $a \in A$  and  $c \in C'$ ,  $d_E(a, c) \geq 1$ . Additionally, for every  $a \in A$ , there exists  $c \in C'$  such that  $d_E(a, c) = 1$ , so  $d(a, C') = 1$  for all such  $A$  and  $d(A, C') = 1$ . Likewise, for all  $c \in C'$ , there is some  $a \in A$  such that  $d_E(c, a) = 1$ , so  $d(C', A) = 1$  and  $h(A, C') = 1$ . A similar computation shows that  $d(B, C') = 1$ .

In  $\mathbb{R}^n$  we say that  $c$  is between  $a$  and  $b$  at a distance  $t \in \mathbb{R}$  from  $a$  (where  $0 < t < d_E(a, b)$ ) if  $d_E(a, b) = d_E(a, c) + d_E(c, b)$  and  $d_E(a, c) = t$ . If  $\{a\}, \{b\} \in \mathcal{H}(\mathbb{R}^n)$



**Figure 1.** Two elements  $C$  and  $C'$  between sets  $A$  and  $B$ .

are single point sets, it is easy to see that  $d_E(a, b) = h(\{a\}, \{b\})$ . Thus, we can naturally extend betweenness in  $\mathbb{R}^n$  to  $\mathcal{H}(\mathbb{R}^n)$ .

**Definition 2.3.** Let  $A, B, C \in \mathcal{H}(\mathbb{R}^n)$  and  $0 < t < h(A, B)$ . We say that  $C$  is between  $A$  and  $B$  at a distance  $t$  from  $A$  if

$$h(A, B) = h(A, C) + h(C, B) \quad \text{and} \quad h(A, C) = t.$$

Thus, betweenness in  $\mathbb{R}^n$  is preserved under the natural embedding of  $\mathbb{R}^n$  into  $\mathcal{H}(\mathbb{R}^n)$ . To see that there can be multiple elements at some location between compact sets  $A$  and  $B$ , recall the previous example:

**Example 2.4.** As computed above, we have

$$h(A, B) = 2 = 1 + 1 = h(A, C) + h(C, B).$$

However, we also have

$$h(A, B) = 2 = 1 + 1 = h(A, C') + h(C', B)$$

for the same sets  $A$  and  $B$ , so  $C'$  and  $C$  lie between  $A$  and  $B$  one unit from  $A$ . In fact, if  $C''$  is the union of  $S^1(3)$  and any nonempty compact subset of  $S^1(1)$ ,  $C''$  is also between  $A$  and  $B$  one unit from  $A$ , so for this example there are uncountably many elements of  $\mathcal{H}(\mathbb{R}^2)$  between  $A$  and  $B$  at a distance one from  $A$ .

Lemma 2.5 of [Blackburn et al. 2009] says that if  $A, B \in \mathcal{H}(\mathbb{R}^n)$  and there exists some  $a \in A$  or  $b \in B$  such that  $d(a, B) \neq h(A, B)$  or  $d(b, A) \neq h(A, B)$ , then there are infinitely many elements  $C \in \mathcal{H}(\mathbb{R}^n)$  between  $A$  and  $B$  at any location. We can improve this result. Under the hypotheses, the authors find an injective map from the open interval  $(0, \epsilon)$  to the collection of elements in  $\mathcal{H}(\mathbb{R}^n)$  that lie between  $A$  and  $B$  at a given location to conclude that there are infinitely many such elements, but this implies further that under the assumptions of the lemma, then there are, in

fact, uncountably many elements of  $\mathcal{H}(\mathbb{R}^n)$  between  $A$  and  $B$  at any location, since  $(0, \epsilon)$  is uncountable.

In light of this observation, we employ the following definition:

**Definition 2.5** [Blackburn et al. 2009]. A configuration  $[A, B]$  is a pair of sets  $A, B \in \mathcal{H}(\mathbb{R}^n)$  with  $A \neq B$  such that

$$h(A, B) = d(b, A) = d(a, B) \quad \text{for all } a \in A \text{ and } b \in B.$$

It follows that if the pair  $A, B \in \mathcal{H}(\mathbb{R}^n)$  is not a configuration, then the number of elements at any location between  $A$  and  $B$  is uncountable. Hence, if there are countably many elements at each location between  $A$  and  $B$ , then the pair must be a configuration  $[A, B]$ . Note that the elements  $A$  and  $B$  described in the example above constitute a configuration, and so being a configuration is necessary but not sufficient for there to be countably many elements of  $\mathcal{H}(\mathbb{R}^n)$  at a given location between  $A$  and  $B$ .

We adopt the notation  $\#[A, B]_t$  to represent the cardinality of the collection of elements between  $A$  and  $B$  in a configuration at a distance  $0 < t < h(A, B)$  from  $A$ . Blackburn et al. [2009] demonstrated that when  $A$  and  $B$  are finite sets,  $\#[A, B]_t$  is finite and  $\#[A, B]_s = \#[A, B]_t$  for every  $s, t$  satisfying  $0 < s, t < h(A, B)$ . In this case,  $\#[A, B]_t$  is simply denoted  $\#[A, B]$ .

### 3. An alternative characterization of $\#[A, B]_t$

Let  $(A)_t$  denote the dilation of  $A \in \mathcal{H}(\mathbb{R}^n)$  by  $t$ ; that is,  $(A)_t = \{x \in \mathbb{R}^n : d(x, A) \leq t\}$ . In addition, for elements  $A, B \in \mathcal{H}(\mathbb{R}^n)$  with  $t + s = h(A, B)$ ,  $t, s > 0$ , define  $C(t)$  to be the set  $(A)_t \cap (B)_s$ . Lemma 3.6 of [Bogdewicz 2000] shows that for such  $A, B, t$ , and  $s$ , the set  $C(t)$  is between  $A$  and  $B$  at a distance  $t$  from  $A$ . In [Braun et al. 2005] it is shown that any element  $C \in \mathcal{H}(\mathbb{R}^n)$  with  $h(A, C) = t$  satisfies  $C \subset (A)_t$ . It follows that if  $C$  is any element between  $A$  and  $B$  at a distance  $t$  from  $A$ , then  $C \subset C(t)$ . Thus, we can think of  $C(t)$  as the largest element between  $A$  and  $B$  at a distance  $t$  from  $A$ . From this point forward, we set the convention that for any configuration  $[A, B]$  with  $0 < t < h(A, B)$ , we have  $s = h(A, B) - t$  and  $C(t) = (A)_t \cap (B)_s$ .

**Example 3.1.** Using our previous example with  $t = s = 1$ , we can see that  $(A)_t$  is the union of the unit disk with an annulus with inner and outer radii of three and five, while  $(B)_s$  is an annulus with inner and outer radii of one and three, so that  $C(t) = (A)_t \cap (B)_s = S^1(1) \cup S^1(3) = C$ , where  $C$  is the set pictured on the left side of Figure 1.

In general, one way to determine  $\#[A, B]_t$  is to count the number of elements of  $\mathcal{H}(\mathbb{R}^n)$  at a location  $t$  from  $A$  between  $A$  and  $B$ . Alternatively, we can count the

number of ways to remove subsets  $U \subset \mathbb{R}^n$  from the largest set  $C(t)$  between  $A$  and  $B$  to get another element  $C(t) \setminus U \in \mathcal{H}(\mathbb{R}^n)$  between  $A$  and  $B$  at a distance  $t$  from  $A$ . We note immediately that if  $C(t) \setminus U$  is to be compact,  $U$  must be open in  $C(t)$ .

Recall that in a configuration  $[A, B]$ , we have that  $h(A, B) = d(a, B) = d(b, A)$  for every  $a \in A$  and  $b \in B$ . Thus, by the compactness of  $B$ , for every  $a \in A$  there must be at least one  $b \in B$  such that  $d_E(a, b) = h(A, B)$ , and likewise for each  $b \in B$ . This relation between pairs of points in  $A$  and  $B$  proves to be especially important, and so we make the following definition:

**Definition 3.2.** Let  $A, B \in \mathcal{H}(\mathbb{R}^n)$ . We say that  $a \in A$  and  $b \in B$  are *adjacent*, and write  $a \rightleftharpoons b$ , if  $d_E(a, b) = h(A, B)$ . The adjacency set of  $a$  in  $B$ ,  $[a]_B$ , is defined to be  $[a]_B = \{b \in B : a \rightleftharpoons b\}$ .

Note that under the definition of adjacency, it is not necessary for the sets  $A$  and  $B$  to form a configuration, but for a configuration  $[A, B]$ , we have that for any  $a \in A$  and  $b \in B$ , both  $[a]_B$  and  $[b]_A$  are nonempty. Referring back to our original example, we have for the origin  $a_0 \in A$  that  $[a_0]_B = B$ , since every point  $b \in B$  satisfies  $d_E(a_0, b) = 2$ . On the other hand, for any  $b \in B$ , we can write  $b = 2e^{i\theta}$ , and  $[b]_A$  consists of the origin  $a_0$  and  $4e^{i\theta}$ .

Suppose a configuration  $[A, B]$  has largest set  $C(t)$  between  $A$  and  $B$ . Lemma 3.1 of [Blackburn et al. 2009] says that for every  $c \in C(t)$ , there is precisely one  $a \in A$  and  $b \in B$  such that  $c \rightleftharpoons a$  and  $c \rightleftharpoons b$ . Also,  $[a]_{C(t)}$  and  $[b]_{C(t)}$  are nonempty. Thus, the functions  $q_A : C(t) \rightarrow A$  and  $q_B : C(t) \rightarrow B$  that map  $c$  to these unique points  $a$  and  $b$ , respectively, are both well-defined and onto.

Now, we return to the idea of deciding which sets  $U$  we can remove from  $C(t)$  to get some element  $C(t) \setminus U$  between  $A$  and  $B$  at the same location. Observe that if we remove every point in  $[a]_{C(t)}$  for some  $a \in A$ , then  $d(a, C(t) \setminus U) > d(a, C(t)) = t$ , and thus  $C(t) \setminus U$  cannot be between  $A$  and  $B$  at the same location. Similarly, we cannot remove every point in  $C(t)$  adjacent to some  $b \in B$ . Thus, we define a new collection of sets  $\Upsilon_t$ , which will turn out to be the collection of removable sets described above:

Given  $[A, B]$  and  $0 < t < h(A, B)$ , define  $\Upsilon_t$  to be the collection of sets  $U$  open in  $C(t)$  such that

- for every  $a \in q_A(U)$ ,  $[a]_{C(t) \setminus U} \neq \emptyset$ , and
- for every  $b \in q_B(U)$ ,  $[b]_{C(t) \setminus U} \neq \emptyset$ .

These two conditions ensure that for any  $U \in \Upsilon_t$ , we have  $C(t) \setminus U$  is between  $A$  and  $B$  at a distance  $t$  from  $A$ . Note that  $\emptyset$  is always an element of  $\Upsilon_t$ , and  $C(t)$  is never such an element. We set one more convention, that if  $[A, B]$  is a configuration and  $0 < t < h(A, B)$ , then  $\mathcal{K}_t$  is the collection of all elements of  $\mathcal{H}(\mathbb{R}^n)$  between  $A$  and  $B$  at a distance  $t$  from  $A$ . More precisely, we have:

**Theorem 3.3** [Blackburn et al. 2009]. *For any configuration  $[A, B]$  and any  $t$  satisfying  $0 < t < h(A, B)$ , the function  $f : \Upsilon_t \rightarrow \mathcal{H}_t$  defined by  $f(U) = C(t) \setminus U$  is a bijection.*

From the theorem it follows that  $\#[A, B]_t = |\mathcal{H}_t| = |\Upsilon_t|$ . This is the exact tool we will need to show that no configuration  $[A, B]$  and  $0 < t < h(A, B)$  satisfies  $\#[A, B]_t = |\mathbb{Z}|$ . In our example, with  $t = 1$ ,  $\Upsilon_1$  is the collection of all sets  $U \neq S^1(1)$  that are open in  $S^1(1)$ . We observe that in this case  $\Upsilon_1$  is uncountable, verifying that there are uncountably many elements of  $\mathcal{H}(\mathbb{R}^n)$  between  $A$  and  $B$  at a distance one from  $A$ .

#### 4. Orders of infinity between sets in a configuration

Recall from above that if two elements  $A, B$  do not form a configuration, then there are uncountably many elements between  $A$  and  $B$  at every location. Thus, we may restrict our search for pairs of sets  $A, B \in \mathcal{H}(\mathbb{R}^n)$  with countably many such elements to configurations. We use the fact that if  $[A, B]$  is a configuration with  $0 < t < h(A, B)$ , then as stated above,  $\#[A, B]_t = |\Upsilon_t|$ .

We will need a definition and two lemmas to prove our main result.

**Definition 4.1.** A point  $w$  contained in a set  $W$  is a cluster point of  $W$  if for every  $\epsilon > 0$ ,  $B_\epsilon(w) \cap (W \setminus \{w\}) \neq \emptyset$ . If  $w$  is not a cluster point, it is isolated.

The first lemma follows directly from our definition of  $\Upsilon_t$ .

**Lemma 4.2.** *For a configuration  $[A, B]$  with  $0 < t < h(A, B)$ , let  $U \in \Upsilon_t$ . If  $V \subset U$  such that  $V$  is open in  $C_t$ , then  $V \in \Upsilon_t$  as well.*

*Proof.* By definition of  $\Upsilon_t$ , we have that

- for all  $a \in q_A(U)$  there exists  $c \in [a]_{C(t)}$  such that  $c \notin U$  and
- for all  $b \in q_B(U)$  there exists  $c \in [b]_{C(t)}$  such that  $c \notin U$ .

This means that for all  $a \in q_A(V)$ , we must have that  $a \in q_A(U)$  as  $V \subset U$ . Thus, there exists  $c \in [a]_{C(t)}$  such that  $c \notin U$ , and so  $c \notin V$ . Similarly, for every  $b \in q_B(V)$ , there exists  $c \in [b]_{C(t)}$  such that  $b \notin V$ . It follows that  $V \in \Upsilon_t$ .  $\square$

In other words, if we can remove some set of points  $U$  from  $C(t)$  to get an element of  $\mathcal{H}_t$ , then certainly we can remove some relatively open subset  $V$  of  $U$  from  $C(t)$  to get another element of  $\mathcal{H}_t$ . The next lemma takes a bit more work and lies at the core of our argument.

**Lemma 4.3.** *For a configuration  $[A, B]$  with  $0 < t < h(A, B)$ , if  $\#[A, B]_t = \infty$ , then there exists  $W \in \Upsilon_t$  such that  $|W| = \infty$ .*

*Proof.* Suppose by way of contradiction that  $|U|$  is finite for every  $U \in \Upsilon_t$ . Let  $U \in \Upsilon_t$ , and choose a point  $x \in U$ . As  $|U|$  is finite, we can find some  $\epsilon_x > 0$  such that  $B_{\epsilon_x}(x) \cap U = \{x\}$ . Further, since  $U$  is relatively open in  $C(t)$ , we can choose  $\epsilon_x$  to be small enough so that  $B_{\epsilon_x}(x) \cap C \subset U$ . Hence, if  $V_x = B_{\epsilon_x}(x) \cap C$ , then  $V_x = \{x\}$  and certainly  $V_x$  is open in  $C$  and  $V_x \subset U$ , so by Lemma 4.2,  $V_x \in \Upsilon_t$ . Since  $|\Upsilon_t| = \infty$  and every element  $U \in \Upsilon_t$  can be written as a union of sets  $V_x$ , we must have infinitely many such singleton point sets as well.

Define

$$V = \bigcup_{\substack{x \in U \\ U \in \Upsilon_t}} V_x.$$

We split the proof into two cases. Suppose first that there exists some  $a \in q_A(V)$  such that  $|[a]_V| = \infty$ . This means that  $a$  is adjacent to infinitely many points in  $V$ . Note that if  $v_1, v_2 \in V$  satisfy  $v_1 \approx a_0, v_2 \approx a_0, v_1 \approx b_0$ , and  $v_2 \approx b_0$  for some  $a_0 \in A$  and  $b_0 \in B$ , then  $v_1 = v_2$  by the uniqueness of betweenness in Euclidean geometry. Thus, every pair of distinct points  $v_1$  and  $v_2$  in  $[a]_V$  must be adjacent to distinct points in  $B$ , or equivalently,  $q_B$  is injective on  $[a]_V$ . Fix a point  $v^* \in [a]_V$ , and let  $W = [a]_V \setminus \{v^*\}$ .

It is clear that  $|W| = \infty$ . We claim that  $W \in \Upsilon_t$ . First, note that  $W$  is the union of singleton point sets, each of which is open in  $C(t)$ , so  $W$  is open in  $C$ . We have established that  $q_A(W) = \{a\}$ , where  $v^* \in [a]_{C(t)}$  but  $v^* \notin W$ . Now, let  $b \in q_B(W)$ . By the argument above, we have that  $b$  is adjacent to exactly one point  $w \in W$ . Further,  $\{w\} \in \Upsilon$ , so there exists some  $c \in [b]_C$  such that  $c \notin \{w\}$ ; that is,  $c \neq w$ . Since  $b$  is adjacent to no other points in  $W$ , it follows that  $c \notin W$ , as desired. We conclude that  $W \in \Upsilon$ , which is a contradiction to the assumption that every set in  $\Upsilon_t$  is finite. A similar proof holds if there exists some  $b \in q_B(V)$  such that  $|[b]_V| = \infty$ .

In the second case suppose that  $[a]_V$  and  $[b]_V$  are finite for every  $a \in q_A(V)$  and  $b \in q_B(V)$ . Choose a point  $v_1 \in V$  and let  $a_1 = q_A(v_1)$  and  $b_1 = q_B(v_1)$ . Since  $[a_1]_V$  and  $[b_1]_V$  are finite while  $V$  is infinite, we can choose  $v_2 \neq v_1 \in V$  such that  $v_2$  is adjacent to neither  $a_1$  nor  $b_1$ . Continuing in this manner, we can construct three infinite sequences of distinct points  $\{a_i\}_i, \{b_i\}_i$ , and  $\{v_i\}_i$  such that  $v_m$  is adjacent to  $a_m$  and  $b_m$  but  $v_m$  is not adjacent to any of the points  $a_1, \dots, a_{m-1}, b_1, \dots, b_{m-1}$ .

Let  $W = \bigcup v_i$ . Then certainly  $|W| = \infty$  and  $W$  is open in  $C$  as the union of open singleton sets. We claim that  $W \in \Upsilon_t$ . For any  $a \in q_A(W)$ , we know that  $a = a_m$  for some integer  $m$ . Now  $a_m \approx v_m$ , and since  $\{v_m\} \in \Upsilon_t$ , we know that there exists some  $c \in [a]_C$  such that  $c \neq v_m$ . But by definition of the sequence  $\{v_i\}_i$ ,  $c \neq v_i$  for any  $i \neq m$ , and thus  $c \notin W$ . A similar argument shows that for every  $b \in q_B(W)$ , there exists  $c \in [b]_C$  such that  $c \notin W$ . We conclude that  $W \in \Upsilon_t$ , which is a contradiction, completing the proof. □

Finally, we are in a position to prove our main theorem.

**Theorem 4.4.** *There exist no two sets  $A$  and  $B$  that have a countably infinite number of elements at any given location between  $A$  and  $B$ .*

*Proof.* Suppose by way of contradiction there exists a configuration  $[A, B]$  and some  $t$ ,  $0 < t < h(A, B)$ , such that  $\#([A, B])_t = |\Upsilon_t| = |\mathbb{Z}|$ . Thus, by Lemma 4.3 there exists some element  $W \in \Upsilon_t$  such that  $|W| = \infty$ . We will find an infinite family of nonempty disjoint open subsets of  $C$  contained in  $W$ . There are two cases to consider. Suppose first that  $W$  contains infinitely many points which are isolated in  $W$ , and call a countably infinite subset of these points  $\{w_i\}_i$ . By definition  $w_m \in W$  is isolated if there exists a ball  $B_{\epsilon_m}(w_m)$  such that  $B_{\epsilon_m}(w_m) \cap W = \{w_m\}$ , and by choosing  $\epsilon$  small enough, we can guarantee that  $B_{\epsilon_m}(w_m) \cap C(t) \subset W$ . Thus, if  $W_m = B_{\epsilon_m}(w_m) \cap C = \{w_m\}$ , then  $\{W_i\}_i$  is a family of infinite disjoint open subsets of  $C(t)$  contained in  $W$ .

In the second case, suppose that  $W$  contains finitely many isolated points, so that  $W$  contains infinitely many cluster points. Choose some cluster point  $w_1 \in W$ . Since there are finitely many isolated points in  $W$ , we can choose  $\epsilon_1 > 0$  such that  $B_{\epsilon_1}(w_1) \cap W$  contains only cluster points. Since  $w_1$  is itself a cluster point in  $W$ , we know  $|B_{\epsilon_1}(w_1) \cap W|$  must be infinite, and so if we shrink  $\epsilon_1$  further we can find a cluster point  $w_2 \in W$  such that  $w_2 \notin \overline{B_{\epsilon_1}(w_1)}$ . Now, we choose  $\epsilon_2$  such that  $B_{\epsilon_2}(w_2) \cap W$  consists of infinitely many cluster points and  $B_{\epsilon_2}(w_2) \cap B_{\epsilon_1}(w_1) = \emptyset$ . Shrinking  $\epsilon_2$  slightly yields a cluster point  $w_3 \in W$  such that  $w_3$  is in neither  $\overline{B_{\epsilon_1}(w_1)}$  nor  $\overline{B_{\epsilon_2}(w_2)}$ . Continuing in this fashion, we can find an infinite sequence  $\{w_i\}_i$  in  $W$  with corresponding radii  $\{\epsilon_i\}_i$ . Let  $W_m = B_{\epsilon_m}(w_m) \cap W$ . Then  $\{W_i\}_i$  is a family of infinite disjoint open subsets of  $C$  contained in  $W$ .

In either case, we find a family  $\{W_i\}_i$  of infinite pairwise disjoint open subsets of  $C$  contained in  $W$ . Let  $2^{\mathbb{Z}}$  be the power set of  $\mathbb{Z}$  and define a map  $g : 2^{\mathbb{Z}} \rightarrow \Upsilon_t$  by

$$g(S) = \bigcup_{i \in S} W_i.$$

First, we note that for any  $S \subset \mathbb{Z}$ , we have  $g(S) \in \Upsilon$  by Lemma 4.2, using the fact that each  $W_i$  is an open subset of  $W \in \Upsilon$ , so  $\bigcup_{i \in S} W_i$  is also an open subset of  $W$ . Next, we claim that  $g$  is injective. But this is clear from the fact that the sets in  $\{W_i\}_i$  are disjoint: if  $S \neq S'$ , then without loss of generality there is some  $m \in S$  such that  $m \notin S'$ , so  $W_m \subset g(S)$  whereas  $W_m \cap g(S') = \emptyset$ , and thus  $g(S) \neq g(S')$ . It follows directly that  $|\Upsilon| \geq |2^{\mathbb{Z}}| = |\mathbb{R}|$ . We conclude that  $\#([A, B])_t \geq |\mathbb{R}|$ , proving the theorem.  $\square$

### Acknowledgments

We thank Steve Schlicker for his help with the revision of this paper. This research was partially supported by National Science Foundation grant DMS-0451254.

## References

- [Blackburn et al. 2009] C. C. Blackburn, K. Lund, S. Schlicker, P. Sigmon, and A. Zupan, “A missing prime configuration in the Hausdorff metric geometry”, *J. Geom.* **92**:1–2 (2009), 28–59. MR 2010d:51020 Zbl 1171.54008
- [Bogdewicz 2000] A. Bogdewicz, “Some metric properties of hyperspaces”, *Demonstratio Math.* **33**:1 (2000), 135–149. MR 1759874 Zbl 0948.54015
- [Braun et al. 2005] D. Braun, J. Mayberry, A. Powers, and S. Schlicker, “A singular introduction to the Hausdorff metric geometry”, *Pi Mu Epsilon Journal* **12**:3 (2005), 129–138.
- [Edgar 1990] G. A. Edgar, *Measure, topology, and fractal geometry*, Springer, New York, 1990. MR 92a:54001 Zbl 0727.28003

Received: 2010-09-22    Revised: 2014-04-23    Accepted: 2014-05-11

zupan@math.utexas.edu

*Department of Mathematics, University of Texas at Austin,  
1 University Station C1200, Austin, TX 78712, United States*



# involve

msp.org/involve

## EDITORS

### MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

### BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	La Trobe University, Australia P.Cerone@latrobe.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsteam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnrit
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

## PRODUCTION

Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2014 is US \$120/year for the electronic version, and \$165/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2014 Mathematical Sciences Publishers

# involve

2014

vol. 7

no. 5

Infinite cardinalities in the Hausdorff metric geometry ALEXANDER ZUPAN	585
Computing positive semidefinite minimum rank for small graphs STEVEN OSBORNE AND NATHAN WARNBERG	595
The complement of Fermat curves in the plane SETH DUTTER, MELISSA HAIRE AND ARIEL SETNIKER	611
Quadratic forms representing all primes JUSTIN DEBENEDETTO	619
Counting matrices over a finite field with all eigenvalues in the field LISA KAYLOR AND DAVID OFFNER	627
A not-so-simple Lie bracket expansion JULIE BEIER AND MCCABE OLSEN	647
On the omega values of generators of embedding dimension-three numerical monoids generated by an interval SCOTT T. CHAPMAN, WALTER PUCKETT AND KATY SHOUR	657
Matrix coefficients of depth-zero supercuspidal representations of $GL(2)$ ANDREW KNIGHTLY AND CARL RAGSDALE	669
The sock matching problem SARAH GILLIAND, CHARLES JOHNSON, SAM RUSH AND DEBORAH WOOD	691
Superlinear convergence via mixed generalized quasilinearization method and generalized monotone method VINCHENCIA ANDERSON, COURTNEY BETTIS, SHALA BROWN, JACQKIS DAVIS, NAEEM TULL-WALKER, VINODH CHELLAMUTHU AND AGHALAYA S. VATSALA	699



1944-4176(2014)7:5;1-4