

involve

a journal of mathematics

A not-so-simple Lie bracket expansion

Julie Beier and McCabe Olsen



A not-so-simple Lie bracket expansion

Julie Beier and McCabe Olsen

(Communicated by Robert W. Robinson)

Lie algebras and quantum groups are not usually studied by an undergraduate. However, in the study of these structures, there are interesting questions that are easily accessible to an upper-level undergraduate. Here we look at the expansion of a nested set of brackets that appears in relations presented in a paper of Lum on toroidal algebras. We illuminate certain terms that must be in the expansion, providing a partial answer for the closed form.

1. Introduction

Lie algebras and quantum groups are not topics that you are apt to hear undergraduates math majors discussing in their spare time. However, there are a surprising number of nontrivial questions in this area that are undergraduate appropriate. In this paper, we will give a brief overview of the broad mathematical setting, and then discuss an accessible problem that involves expanding a nested set of brackets.

Lie algebras, their universal enveloping algebras and quantum groups are a fundamental part of representation theory that have many applications within mathematics and mathematical physics. Lie algebras and Lie groups were originally discovered by Sophus Lie in the late nineteenth century [Borel 2001]. Given a Lie algebra, we associate a unique associative algebra called the universal enveloping algebra. In 1985, Jimbo and Drinfeld discovered q -analogues of these universal enveloping algebras called “quantum groups”, which have been a recent area of study (see [Lusztig 1993]).

In order to find the quantum analogue of a Lie algebra it is often desirable to understand the defining relationships of the Lie algebra inside of its universal enveloping algebra. The motivation for this project came from a paper by Lum in which he gives a nice presentation of a toroidal Lie algebra that could be useful in understanding this Lie algebra’s quantum group [Lum 1998]. All of these relations utilize a nested set of brackets called $t(k)$. For simplicity, we have modified $t(k)$ by a scalar. In this paper we seek to understand the expansion of this object.

MSC2010: 17B67.

Keywords: Lie algebra, toroidal algebra.

2. The Lie bracket $t(k)$

Recall that a *Lie algebra* is defined as a vector space L over a field F that is equipped with a bilinear map $L \times L \rightarrow L$, known as a *Lie bracket*, satisfying certain conditions. The Lie bracket $(x, y) \rightarrow [x, y]$ for all $x, y \in L$ must satisfy the *alternating property*, namely

$$[x, x] = 0$$

and the *Jacobi identity*, an analog of associativity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,$$

for all $x, y, z \in L$.

Example 2.1. The set $gl(n, \mathbb{R})$ of n -by- n matrices with real entries, together with the operation defined by $[A, B] := AB - BA$, is a Lie algebra. To see this, consider matrices $A, B, C \in gl(n, \mathbb{R})$. It is easy to show that the bracket is bilinear. Since $[A, A] = AA - AA = 0$ alternation is satisfied. To verify that the Jacobi identity holds, note that

$$\begin{aligned} [A, [B, C]] + [B, [C, A]] + [C, [A, B]] &= [A, BC - CB] + [B, CA - AC] + [C, AB - BA] \\ &= A(BC - CB) - (BC - CB)A + B(CA - AC) \\ &\quad - (CA - AC)B + C(AB - BA) - (AB - BA)C \\ &= ABC - ACB - BCA + CBA + BCA - BAC \\ &\quad - CAB + ACB + CAB - CBA - ABC + BAC \\ &= 0. \end{aligned}$$

The nested set of brackets which we seek to understand is denoted $t(k)$. They are defined recursively as

$$t(1) := [x, y] = xy - yx, \quad t(k) := \underbrace{[\dots [[x, y], x], y] \dots]}_{k \text{ } xy\text{-pairs}}$$

Example 2.2. The case of $k = 3$ is given as follows:

$$\begin{aligned} t(3) &= [[[[x, y], x], y], x], y] \\ &= [[[xy - yx, x], y], x], y] \\ &= [2xyxy - 2yxxy + y^2x^2 - x^2y^2, x], y] \\ &= 4xyxyxy - 4yxxyyx + 2y^2xyx^2 - 2x^2yxy^2 + 2yx^2yxy - 2yxxy^2y \\ &\quad + y^2x^3y - yx^3y^2 + yx^2y^2x - xy^2x^2y + yxy^2x^2 - x^2y^2xy + x^3y^3 - y^3x^3. \end{aligned}$$

We choose to view the output of $t(k)$ as *words*. Unlike the combinatorial definition of words, we include the coefficient. For example, $t(1)$ consists of two words, namely xy and $-yx$. We know some additional properties of words because of how the bracket functions. No word can begin and end with an x nor can a word begin and end with y^2 .

We define the *antiword* to be the associated word with reverse ordering of x 's and y 's, opposite sign, and same coefficient. In the example of $t(1)$, the antiword of xy is $-yx$. Similarly, $-yx$ has antiword xy . For a more interesting example, consider Example 2.2. Notice that each word is written next to its antiword. We have $4xyxyxy$ followed by $-4yxxyxy$ which is a word-antiword pair, $2y^2xyx^2$ followed by $-2x^2yx^2y$, and so on. This observation works in general and cuts the problem in half.

Theorem 2.3. *Every word in $t(k)$ has an antiword in $t(k)$.*

Proof. For the base case $k = 1$, we have $t(1) := [x, y] = xy - yx$, so the statement is clearly valid. Now assume for some integer $k \geq 1$ every word appears in $t(k)$ together with its antiword. We want to show that each word in the $t(k + 1)$ has an antiword in $t(k + 1)$. So consider an arbitrary word ω and its antiword $\bar{\omega}$ in the k -th iteration. By bracket expansion, we have

$$\begin{aligned} [[\omega + \bar{\omega}, x], y] &= [\omega x + \bar{\omega}x - x\omega - x\bar{\omega}, y] \\ &= \omega xy + \bar{\omega}xy - x\omega y - x\bar{\omega}y - (y\omega x + y\bar{\omega}x - yx\omega - yx\bar{\omega}) \\ &= \omega xy + \bar{\omega}xy - x\omega y - x\bar{\omega}y - y\omega x - y\bar{\omega}x + yx\omega + yx\bar{\omega}. \end{aligned}$$

Since ω and $\bar{\omega}$ are a word-antiword pair, the following are word-antiword pairs: ωxy and $yx\bar{\omega}$, $-x\omega y$ and $-y\bar{\omega}x$, $-y\omega x$ and $-x\bar{\omega}y$, and $yx\omega$ and $\bar{\omega}xy$. Each of these words will have the same coefficient as ω and $\bar{\omega}$. If two of these words in $t(k + 1)$ are the same, the words in $t(k)$ that generated them have corresponding antiwords in $t(k)$. Bracketing these will necessarily give the same antiword in $t(k + 1)$ causing coefficients to be preserved. Thus, while a whole pair may cancel, no word can independently disappear. □

From this result, we know that it is not possible to have any symmetric words in the output of an arbitrary $t(k)$.

3. Word patterns

Given our goal to determine the content of $t(k)$ in the universal enveloping algebra, we first look to locate patterns universal to all $t(k)$. Here we prove the existence of several such patterns of words. First we consider two fundamental lemmas.

Lemma 3.1. *If the word $x^k y^k$ exists in $t(k)$, it must be generated by the word $x^{k-1} y^{k-1}$ in $t(k-1)$. Similarly, if the word $y^k x^k$ exists in $t(k)$, it must be generated by the word $y^{k-1} x^{k-1}$ in $t(k-1)$.*

Proof. Assume the word $x^k y^k$ exists in $t(k)$. By the definition of the bracket, in order to arrive at this word, we must multiply some word in $t(k-1)$ by both an x and a y . Working backwards, we remove a y and an x in all possible ways to obtain possible root words for $x^k y^k$. Our only option is to remove a y from the end and an x from the beginning. Therefore our only root word is $x^{k-1} y^{k-1}$. Showing that $y^k x^k$ is only generated by the root word $y^{k-1} x^{k-1}$ is analogous. \square

The lemma below follows in an identical fashion.

Lemma 3.2. *If the word $(xy)^k$ exists in $t(k)$, it must be generated by $(xy)^{k-1}$ or $(yx)^{k-1}$ in $t(k-1)$. Similarly, if the word $(yx)^k$ exists in $t(k)$ it must be generated by $(xy)^{k-1}$ or $(yx)^{k-1}$ in $t(k-1)$.*

The proceeding propositions use these lemmas to show some universal patterns appearing in $t(k)$ for all k .

Proposition 3.3. *The words $(-1)^{k+1} x^k y^k$ and $(-1)^k y^k x^k$ appear in $t(k)$.*

Proof. These words appear in the case of $k = 1$ since

$$[x, y] = xy - yx = (-1)^2 xy + (-1)yx.$$

Assume that for some integer $k \geq 1$, we have the words $(-1)^{k+1} x^k y^k + (-1)^k x^k y^k$. We now show that the words $(-1)^{k+2} x^{k+1} y^{k+1}$ and $(-1)^{k+1} x^{k+1} y^{k+1}$ appear in $t(k+1)$. By the definition of the bracket, we have

$$\begin{aligned} & [[(-1)^{k+1} x^k y^k, x], y] \\ &= [(-1)^{k+1} x^{k+1} y^k - (-1)^{k+1} x^k y^k x, y] \\ &= [(-1)^{k+1} x^{k+1} y^k + (-1)^{k+2} x^k y^k x, y] \\ &= (-1)^{k+1} y x^{k+1} y^k + (-1)^{k+2} y x^k y^k x - ((-1)^{k+1} x^{k+1} y^{k+1} + (-1)^{k+2} x^k y^k x y) \\ &= (-1)^{k+1} y x^{k+1} y^k + (-1)^{k+2} y x^k y^k x + (-1)^{k+2} x^{k+1} y^{k+1} + (-1)^{k+3} x^k y^k x y. \end{aligned}$$

The word $(-1)^{k+2} x^{k+1} y^{k+1}$ appears as desired. It is an identical process to prove the existence of $(-1)^{k+1} y^{k+1} x^{k+1}$. Furthermore, we know from Lemma 3.1 that $x^{k+1} y^{k+1}$ and $y^{k+1} x^{k+1}$ cannot be generated by any other root words. Therefore the coefficient is as given. \square

Using this same technique we find two more words that appear in $t(k)$.

Proposition 3.4. *The words $2^{k-1} (xy)^k$ and $-2^{k-1} (yx)^k$ appear in $t(k)$.*

4. More general recurring words

We now look to find broader patterns of words which necessarily appear in $t(k)$. Similar to before, we need a foundational lemma.

Lemma 4.1. *For $k \geq 1$ and $2 \leq j \leq k$, the word $x^j(yx)^{k-j}y^j$, if it exists in $t(k)$, can only be generated by the word $x^{j-1}(yx)^{((k-1)-(j-1))}y^{j-1}$ in $t(k-1)$ and the word $y^j(xy)^{k-j}x^j$, if it exists in $t(k)$, can only be generated by the word $y^{j-1}(xy)^{((k-1)-(j-1))}x^{j-1}$ in $t(k-1)$.*

In order to prove this lemma, we use similar techniques to that of the previous lemmas. We begin by assuming that the words appear in the k -th iteration of the bracket and we work backwards to determine possible root words. This relatively simple procedure is all that is needed to show that the lemma holds. Using this lemma, we now expand the notions of Proposition 3.3 and Proposition 3.4.

Theorem 4.2. *For $k \geq 1$ and $1 \leq j \leq k$, the word $-\tau_k^j x^j (yx)^{k-j} y^j + \tau_k^j y^j (xy)^{k-j} x^j$ appears in $t(k)$, where we have set*

$$\tau_k^j := (-1)^j 2^{k-j}.$$

To prove this more encompassing theorem, we use double induction. We know that this theorem holds for the base case $k = 1$ and $j = 1$

$$[x, y] = xy - yx = (-1)^2 2^0 x(yx)^0 y + (-1)^1 2^0 y(xy)^0 x$$

and by Proposition 3.4, we know the statement holds for arbitrary k and $j = 1$. Subsequently, we use this as a starting point for the second induction. Simply use a bracket argument similar to the one in Proposition 3.3. This argument yields all of the desired words except in the case of $j = k$. However, Proposition 3.3 already accounts for this case. Therefore, the statement is satisfied.

Returning to our running example of $k = 3$, notice that Theorem 4.2 asserts the existence of the following words: $4xyxyxy$, $-4yxxyxy$, $2y^2xyx^2$, $-2x^2yxy^2$, x^3y^3 , and $-y^3x^3$. In Example 2.2, we see that all of these do indeed appear in $t(3)$.

This collection of words accounts for a share of the words in $t(k)$. Unfortunately, it does not even account for all of the words in the case of $k = 3$. However, repeated bracketing of words in Theorem 4.2 will result in more words that are always present. We leave showing the following corollary by bracket as an exercise.

Corollary 4.3. *For $k \geq 1$ and $1 \leq j \leq k$, the following sum appears in $t(k+1)$:*

$$\begin{aligned} &-\tau_k^j x^j (yx)^{k-j} y^j x y + \tau_k^j y^j (xy)^{k-j} x^{j+1} y + \tau_k^j x^{j+1} (yx)^{k-j} y^{j+1} \\ &-\tau_k^j x y^j (xy)^{k-j} x^j y + \tau_k^j y x^j (yx)^{k-j} y^j x - \tau_k^j y^{j+1} (xy)^{k-j} x^{j+1} \\ &-\tau_k^j y x^{j+1} (yx)^{k-j} y^j + \tau_k^j y x y^j (xy)^{k-j} x^j. \end{aligned}$$

Indeed, the words from Corollary 4.3 actually include all of the words in Theorem 4.2 as shown below.

Proposition 4.4. *All words in $t(k + 1)$ of the form $-\tau_k^{j+1}x^{j+1}(yx)^{k-(j+1)}y^{j+1}$ and $\tau_k^{j+1}y^{j+1}(xy)^{k-(j+1)}x^{j+1}$ can be expressed by a form given in Corollary 4.3.*

Proof. Consider the word in $t(k + 1)$ generated by Theorem 4.2 given by

$$(-1)^{(j+1)+1}2^{(k+1)-(j+1)}x^{j+1}(yx)^{(k+1)-(j+1)}y^{j+1} = \tau_k^j x^{j+1}(yx)^{k-j}y^{j+1}.$$

This word is also a word of the form given in Corollary 4.3. Furthermore, consider the other word in $t(k + 1)$ generated by Theorem 4.2:

$$(-1)^{j+1}2^{(k+1)-(j+1)}y^{j+1}(xy)^{(k+1)-(j+1)}x^{j+1} = -\tau_k^j y^{j+1}(xy)^{k-j}x^{j+1}$$

which is indeed a word of the desired form. These two general words account for all words of the form given by Theorem 4.2 in $t(k + 1)$ except for the case of $j = 1$.

First consider $(-1)^{j+1}2^{(k+1)-j}x^j(yx)^{(k+1)-j}y^j$ generated by Theorem 4.2 evaluated at $j = 1$. This yields

$$(-1)^2 2^k x(yx)^k y = 2^k (xy)^{k+1} = 2^{k-1}x(yx)^{k-1}yxy + 2^{k-1}xy(xy)^{k-1}xy$$

which are two words in Corollary 4.3 evaluated at $j = 1$, namely $-\tau_k^j x^j(yx)^{k-j}y^j xy$ and $-\tau_k^j xy^j(xy)^{k-j}x^j y$. The proof that $(-1)^j 2^{(k+1)-j}y^j(xy)^{(k+1)-j}x^j$ can be expressed in a desired form when $j = 1$ is identical. □

Using Proposition 4.4, we account for all of the words in $t(1)$, $t(2)$, and $t(3)$. We leave showing that Corollary 4.3 produces all of $t(3)$ as an exercise. Moreover, we believe that we can identify an even larger pattern of words.

As seen in previous cases, we first identify how the particular words can be generated.

Proposition 4.5. *If the words in the left column of the table below exist in $t(k)$, they must be generated by the corresponding root word listed on the right.*

<i>Generated word in $t(k)$</i>	<i>Root word in $t(k - 1)$</i>
$y^m x^j (yx)^{k-(j+m)} y^j x^m$	$y^{m-1} x^j (yx)^{(k-1)-(j+(m-1))} y^j x^{m-1}$
$x^m y^j (xy)^{k-(j+m)} x^j y^m$	$x^{m-1} y^j (xy)^{k-(j+(m-1))} x^j y^{m-1}$
$x^j (yx)^{k-(j+m)} y^j (xy)^m$	$x^j (yx)^{k-(j+(m-1))} y^j (xy)^{m-1}$
$(yx)^m y^j (yx)^{k-(j+m)} x^j$	$(yx)^{m-1} y^j (yx)^{k-(j+(m-1))} x^j$
$(yx)^m x^j (yx)^{k-(j+m)} y^j$	$(yx)^{m-1} x^j (yx)^{k-(j+(m-1))} y^j$
$y^j (xy)^{k-(j+m)} x^j (xy)^m$	$y^j (xy)^{k-(j+(m-1))} x^j (xy)^{m-1}$

Despite the larger number of words in question, the proof of each follows in the same manner as our previous proofs for necessary root words, except in the more

complicated case of $m = 1$. In this instance, there are more ways to remove one x and y . However, it can be shown that some of these violate the properties of words and thus do not exist in $t(k + 1)$.

Building from all of our previous work we present our largest list of necessary words.

Theorem 4.6. *Let $k \geq 1$.*

- *If $k \geq 2, j = 1, \text{ and } m = 1, \text{ then}$*

$$\tau_k^{j+m} (yx)^m x^j (yx)^{k-(j+m)} y^j - \tau_k^{j+m} y^j (xy)^{k-(j+m)} x^j (xy)^m$$

appears in $t(k)$.

- *If $1 \leq j \leq k \text{ and } m = 0, \text{ then}$*

$$-\tau_k^{j+m} y^m x^j (yx)^{k-(j+m)} y^j x^m + \tau_k^{j+m} x^m y^j (xy)^{k-(j+m)} x^j y^m$$

appears in $t(k)$.

- *If $m \geq 1, j \geq 2, \text{ and } j + m \leq k, \text{ then}$*

$$\begin{aligned} &-\tau_k^{j+m} y^m x^j (yx)^{k-(j+m)} y^j x^m + \tau_k^{j+m} x^m y^j (xy)^{k-(j+m)} x^j y^m \\ &+ (-1)^m (-\tau_k^{j+m} x^j (yx)^{k-(j+m)} y^j (xy)^m + \tau_k^{j+m} (yx)^m y^j (yx)^{k-(j+m)} x^j \\ &\quad - \tau_k^{j+m} (yx)^m x^j (yx)^{k-(j+m)} y^j + \tau_k^{j+m} y^j (xy)^{k-(j+m)} x^j (xy)^m) \end{aligned}$$

appears in $t(k)$.

Proof. Proof of the $j = 1, m = 1$ case follows directly from Corollary 4.3 by evaluating $-\tau_k^j yx^{j+1} (yx)^{k-j} y^j$ at $j = 1$.

Now consider the case of $m = 0$. We have

$$(-1)^{j+0+1} 2^{k-(j+0)} y^0 x^j (yx)^{k-(j+0)} y^j x^0 = -\tau_k^j x^j (yx)^{k-j} y^j.$$

This word was shown to exist in $t(k)$ by Theorem 4.2. For the same reason, we know that $\tau_k^{j+m} x^m y^j (xy)^{k-(j+m)} x^j y^m$ exists in $t(k)$ when $m = 0$.

Now we show $-\tau_k^{j+m} y^m x^j (yx)^{k-(j+m)} y^j x^m$ and $\tau_k^{j+m} x^m y^j (xy)^{k-(j+m)} x^j y^m$ appear in $t(k)$ if $j \geq 2$ and $3 \leq m + j \leq k$. We just argued the case of $m = 0$ for arbitrary $1 \leq j \leq k$ for all $t(k)$. So, we perform induction on m . In Corollary 4.3, we bracket $-\tau_k^j x^j (yx)^{k-j} y^j + (-1)^j 2^{k-j} y^j (xy)^{k-j} x^j$ in $t(k)$ with $k \geq 2$ to generate the term $(-1)^{j+2} 2^{k-j} yx^j (yx)^{k-j} y^j x - \tau_k^j xy^j (xy)^{k-j} x^j y$ which is the desired term for $m = 1$ in $t(k + 1)$.

Now, assume that $k \geq 3$ and that for some $m \geq 1$ with $m + j \leq k$, the words $-\tau_k^{j+m} y^m x^j (yx)^{k-(j+m)} y^j x^m + \tau_k^{j+m} x^m y^j (xy)^{k-(j+m)} x^j y^m$ appear in $t(k)$. We want to show that

$$(-1)^{j+(m+1)+1} 2^{k-(j+(m+1))} y^{m+1} x^j (yx)^{k-(j+(m+1))} y^j x^{m+1}$$

$$+(-1)^{j+(m+1)}2^{(k+1)-(j+(m+1))}x^{m+1}y^j(xy)^{(k+1)-(j+(m+1))}x^jy^{m+1}$$

appears in $t(k+1)$. Using our bracket, we have

$$\begin{aligned} & [[-\tau_k^{j+m}y^m x^j (yx)^{k-(j+m)}y^j x^m + \tau_k^{j+m}x^m y^j (xy)^{k-(j+m)}x^j y^m, x], y] \\ &= [-\tau_k^{j+m}y^m x^j (yx)^{k-(j+m)}y^j x^{m+1} + \tau_k^{j+m}x^m y^j (xy)^{k-(j+m)}x^j y^m x \\ &\quad + \tau_k^{j+m}x y^m x^j (yx)^{k-(j+m)}y^j x^m - \tau_k^{j+m}x^{m+1}y^j (xy)^{k-(j+m)}x^j y^m, y] \\ &= -\tau_k^{j+m}y^m x^j (yx)^{k-(j+m)}y^j x^{m+1}y + \tau_k^{j+m}x^m y^j (xy)^{k-(j+m)}x^j y^m x y \\ &\quad + \tau_k^{j+m}x y^m x^j (yx)^{k-(j+m)}y^j x^m y - \tau_k^{j+m}x^{m+1}y^j (xy)^{k-(j+m)}x^j y^{m+1} \\ &\quad + \tau_k^{j+m}y^{m+1}x^j (yx)^{k-(j+m)}y^j x^{m+1} + \tau_k^{j+m}y x^m y^j (xy)^{k-(j+m)}x^j y^m x \\ &\quad - \tau_k^{j+m}y x y^m x^j (yx)^{k-(j+m)}y^j x^m + \tau_k^{j+m}y x^{m+1}y^j (xy)^{k-(j+m)}x^j y^m. \end{aligned}$$

We have the resulting words (line 3 word 1 and line 2 word 2)

$$\begin{aligned} & \tau_k^{j+m}y^{m+1}x^j (yx)^{k-(j+m)}y^j x^{m+1} \\ &= (-1)^{j+(m+1)+1}2^{(k+1)-(j+(m+1))}y^{m+1}x^j (yx)^{(k+1)-(j+(m+1))}y^j x^{m+1} \end{aligned}$$

and

$$\begin{aligned} & -\tau_k^{j+m}x^{m+1}y^j (xy)^{k-(j+m)}x^j y^{m+1} \\ &= (-1)^{j+(m+1)}2^{(k+1)-(j+(m+1))}x^{m+1}y^j (xy)^{(k+1)-(j+(m+1))}x^j y^{m+1}. \end{aligned}$$

By Proposition 4.5, we know that these words cannot be generated by any other root word. The other four remaining desired words can be shown through an analogous process. \square

5. Moving forward

We could consider continuing our current course of action by looking for new patterns beginning in the $k = 4, 5, 6$ cases to try to detect another significant margin of words. One difficulty with this avenue is that an entirely new class of words appears every few cases. A second difficulty is that these become time consuming for the computer to compute. Maple 15 was unable to compute these brackets at $t(8)$ after a full day of computation for $t(7)$. It appears that every time this version of Maple encounters a noncommuting term like xyx it computes $x * y * x$. However, Sage (sagemath.org) treats xyx as an element and can compute the values much faster. Despite this, at $t(11)$ it starts to take minutes for the computation to occur, and it is expected that even using SAGE the computational time would be too high before reaching $t(20)$.

The reader may be wondering why we have taken this particular approach to the problem. The answer is quite simple. We have been unable to find a nice

k	word count	k	word count	k	word count
4	46	7	1648	10	61512
5	152	8	5506	11	206028
6	500	9	18380	12	691126

Table 1. Number of words in $t(k)$.

combinatorial method to simplify the problem. The number of terms in $t(k)$ grows rapidly; see Table 1. Our initial use of dominoes, strips, and tableaux illustrated interesting connections but did not yield useful results. Then we used the Online Encyclopedia of Integer Sequences (oeis.org) to try and find connections to other less obvious options. However, despite searching a number of related sequences, we were unable to locate any connections. It would be ideal if one could find such a connection in order to continue this problem.

This problem is thus still open, as is the question of expanding the full relations given in Lum's paper. We encourage readers to improve on our method and find connections to solve these problems. After this is done, it will be possible to give a nice presentation of the toroidal quantum group.

References

- [Borel 2001] A. Borel, *Essays in the history of Lie groups and algebraic groups*, History of Mathematics **21**, American Math. Soc. and London Math. Soc., 2001. MR 2002g:01010 Zbl 1087.01011
- [Lum 1998] K. H. Lum, "A presentation of toroidal algebras as homomorphic images of G.I.M. algebras", *Comm. Algebra* **26**:12 (1998), 4051–4063. MR 99j:17033
- [Lusztig 1993] G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics **110**, Birkhäuser, Boston, 1993. MR 94m:17016 Zbl 0788.17010

Received: 2013-05-14 Revised: 2014-02-03 Accepted: 2014-03-03

beierju@earlham.edu

Department of Mathematics, Earlham College, 801 National Road West, Drawer 138, Richmond, IN 47374, United States

mccabe.olsen@gmail.com

Mercer University, Department of Mathematics, 1400 Coleman Avenue, Macon, GA 31207, United States

involve

msp.org/involve

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	La Trobe University, Australia P.Cerone@latrobe.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsteam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnrit
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION


Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2014 is US \$120/year for the electronic version, and \$165/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2014 Mathematical Sciences Publishers

involve

2014

vol. 7

no. 5

Infinite cardinalities in the Hausdorff metric geometry ALEXANDER ZUPAN	585
Computing positive semidefinite minimum rank for small graphs STEVEN OSBORNE AND NATHAN WARNBERG	595
The complement of Fermat curves in the plane SETH DUTTER, MELISSA HAIRE AND ARIEL SETNIKER	611
Quadratic forms representing all primes JUSTIN DEBENEDETTO	619
Counting matrices over a finite field with all eigenvalues in the field LISA KAYLOR AND DAVID OFFNER	627
A not-so-simple Lie bracket expansion JULIE BEIER AND MCCABE OLSEN	647
On the omega values of generators of embedding dimension-three numerical monoids generated by an interval SCOTT T. CHAPMAN, WALTER PUCKETT AND KATY SHOUR	657
Matrix coefficients of depth-zero supercuspidal representations of $GL(2)$ ANDREW KNIGHTLY AND CARL RAGSDALE	669
The sock matching problem SARAH GILLIAND, CHARLES JOHNSON, SAM RUSH AND DEBORAH WOOD	691
Superlinear convergence via mixed generalized quasilinearization method and generalized monotone method VINCHENCIA ANDERSON, COURTNEY BETTIS, SHALA BROWN, JACQKIS DAVIS, NAEEM TULL-WALKER, VINODH CHELLAMUTHU AND AGHALAYA S. VATSALA	699



1944-4176(2014)7:5;1-4