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On the omega values of generators of embedding dimension-three numerical monoids generated by an interval

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We offer a formula to compute the omega values of the generators of the numerical monoid $S = \langle k, k+1, k+2 \rangle$ where k is a positive integer greater than 2.

1. Introduction and the main result

The notion of a prime element is a central focus in the study of algebra and number theory. Several recent papers [Anderson and Chapman 2010; 2012; Anderson et al. 2011] have considered the following generalization of the notion of prime elements in the context of numerical monoids. This definition, which we state for a general commutative cancellative monoid, originally appeared in [Geroldinger and Hassler 2008].

Definition 1.1. Let M be a commutative, cancellative, atomic monoid with set of units $M^\times$ and set of irreducibles (or atoms) $\mathcal{A}(M)$. For $x \in M \setminus M^\times$, we define $\omega_M(x) = n$ if n is the smallest positive integer with the property that whenever $x \mid a_1 \cdots a_t$, where each $a_i \in \mathcal{A}(M)$, there is a $T \subseteq \{1, 2, \ldots, t\}$ with $|T| \leq n$ such that $x \mid \prod_{k \in T} a_k$. If no such n exists, then $\omega_M(x) = \infty$. For $x \in M^\times$, we define $\omega_M(x) = 0$.

As in [Anderson et al. 2011], when our context is clear, we will shorten $\omega_M(x)$ to $\omega(x)$. It follows easily from the definition that an element $x \in M \setminus M^\times$ is prime if and only if $\omega(x) = 1$. Hence, in some sense the omega function measures how far an element is from being prime. Some basic properties of this function can be found not only in the papers mentioned above, but also in [Geroldinger and Halter-Koch 2006]. Anderson and Chapman [2010; 2012] study the behavior of the omega function in the setting of the multiplicative monoid of a commutative ring.

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Anderson, Chapman, Kaplan and Torkornoo [Anderson et al. 2011, Section 3] offer a finite time algorithm for computing $\omega(x)$ when x is an element in a numerical monoid S. Recall that a numerical monoid is an additive submonoid of the nonnegative integers (which we denote by $\mathbb{N}_0$). Using elementary number theory, it is easy to show that such a submonoid is finitely generated and possesses a unique minimal (in terms of cardinality) generating set. If $n_1, n_2, \ldots, n_t$ is the minimal generating set for a numerical monoid S, then we write

$$S = \langle n_1, \ldots, n_t \rangle = \big\{x_1 n_1 + \cdots + x_t n_t \mid x_i \in \mathbb{N}_0 \text{ for each } i\big\}.$$

The value t is known as the *embedding dimension of* S. The elements $n_1, \ldots, n_t$ are the irreducibles of S, and as noted in Definition 1.1, we will write $\mathscr{A}(S) = \{n_1, \ldots, n_t\}$. When considering the complete class of numerical monoids, elementary isomorphism arguments allow us to reduce to the case where $\gcd(n_1, \ldots, n_t) = 1$. Such a numerical monoid is called *primitive*. [Rosales and García-Sánchez 2009] is a good general reference on numerical monoids and semigroups. [Bowles et al. 2006; Chapman et al. 2006; 2009; Omidali 2012] examine factorization properties of numerical monoids which are related in various ways to the omega function.

A version of the algorithm in [Anderson et al. 2011] mentioned above has been programmed and can be found in the numerical semigroups package available for Gap (gap-system.org/Manuals/pkg/numericalsgps/doc/manual.pdf). Using data generated by this program, much of the work in [Anderson et al. 2011] is dedicated to showing that closed forms for particular values of $\omega(x)$ are highly nontrivial to determine. In [Anderson et al. 2011, Propositions 3.1 and 3.2], the authors determine formulas for this when $S = \langle n, n+1, \ldots, 2n-1 \rangle$ and $S = \langle n, n+1, \ldots, 2n-2 \rangle$ (where $n \geq 2$), and in [Anderson et al. 2011, Theorem 4.4] they handle the case where $S = \langle n_1, n_2 \rangle$. The paper also takes interest in computing the values $\omega(n_1)$, $\omega(n_2)$, and $\omega(n_3)$ when $S = \langle n_1, n_2, n_3 \rangle$ is of embedding dimension 3. In particular, they offer a chart [Anderson et al. 2011, p. 101] to illustrate how these omega values can differ. We include a modified form in Table 1.

There are 5 possibilities that Table 1 omits. With the programs then available, Anderson et al. [2011] were unable to find examples of these missing orderings. With some improved programming techniques, the present authors were able to compute $\omega(n_1)$, $\omega(n_2)$ and $\omega(n_3)$ for all embedding dimension-three numerical monoids with generators less than or equal to 100. This yielded two of the remaining five cases.

(i) $S = \langle 6, 7, 9 \rangle$ yields $\omega(6) = 3$, $\omega(7) = 5$, and $\omega(9) = 3$. Hence, $\omega(6) < \omega(7)$, $\omega(9) < \omega(7)$, and $\omega(6) = \omega(9)$.

(ii) $S = \langle 7, 8, 20 \rangle$ yields $\omega(7) = 6$, $\omega(8) = 4$, and $\omega(20) = 5$. Hence, $\omega(7) > \omega(8)$, $\omega(8) < \omega(20)$, and $\omega(7) > \omega(20)$.

$\langle n_1, n_2, n_3\rangle$	$\omega(n_1)$	$\omega(n_2)$	$\omega(n_3)$	Ordering of the omega values
$\langle 6, 8, 13\rangle$	3	4	7	$\omega(6)<\omega(8)<\omega(13)$
$\langle 5, 7, 11\rangle$	3	5	5	$\omega(5)<\omega(7)=\omega(11)$
$\langle 4, 5, 6\rangle$	2	4	3	$\omega(4)<\omega(5),\ \omega(5)>\omega(6), \omega(4)<\omega(6)$
$\langle 6, 9, 11\rangle$	3	3	7	$\omega(6)=\omega(9)<\omega(11)$
$\langle 7, 11, 17\rangle$	5	5	5	$\omega(7)=\omega(11)=\omega(17)$
$\langle 6, 7, 11\rangle$	4	3	5	$\omega(6)>\omega(7),\ \omega(7)<\omega(11), \omega(6)<\omega(11)$
$\langle 7, 8, 12\rangle$	5	4	4	$\omega(7)>\omega(8)=\omega(12)$
$\langle 7, 8, 13\rangle$	5	4	5	$\omega(7)>\omega(8),\ \omega(8)<\omega(13), \omega(7)=\omega(13)$

Table 1. Differing values of omega (modified from [Anderson et al. 2011]).

We strongly suspect the final three orderings are not possible. Hence, we state this as a potential problem.

Problem. Let $S = \langle n_1, n_2, n_3\rangle$ be an embedding dimension-3 numerical monoid. Show that the sequence $\omega(n_1)$, $\omega(n_2)$, and $\omega(n_3)$ does not satisfy any of the following three orderings:

- $\omega(n_1) > \omega(n_2) > \omega(n_3)$.
- $\omega(n_1) = \omega(n_2) > \omega(n_3)$.
- $\omega(n_1) < \omega(n_2),\ \omega(n_2) > \omega(n_3),\ \omega(n_3) < \omega(n_1)$.

In the course of attempting to solve this problem, numerous classes of embedding dimension-3 numerical monoids were studied. We encountered one with especially nice omega values on the generators. The remainder of this paper will consist of a proof of the following theorem.

Theorem 1.2. *Let k be a positive integer.*

(a) *If $S_1 = \langle 2k+1, 2k+2, 2k+3\rangle$, then*

$$\omega(2k+1) = k+1 \quad \textit{and} \quad \omega(2k+2) = \omega(2k+3) = k+2.$$

(b) *If $k \geq 2$ and $S_2 = \langle 2k, 2k+1, 2k+2\rangle$, then*

$$\omega(2k) = k,\ \omega(2k+1) = k+2 \quad \textit{and} \quad \omega(2k+2) = k+1.$$

The proof will require two results from the literature. The first allows one to reduce the definition of $\omega(x)$ from that of checking arbitrary products to checking only products of irreducibles.

Theorem 1.3 [Anderson and Chapman 2010, Theorem 2.1]. *Let M be a commutative cancellative monoid and suppose that $x \in M \backslash M^\times$. Then the following statements are equivalent*:

(a) $\omega(x) = m \in \mathbb{N}$.

(b) *m is the least positive integer such that if $x \mid x_1 \cdots x_n$ with each $x_i \in M$ irreducible, then $x \mid x_{i_1} \cdots x_{i_t}$ for some $\{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$ with $t \le m$.*

(c) *If $x \mid x_1 \cdots x_n$ with each $x_i \in M$ irreducible and $n \ge m$, then $x \mid x_{i_1} \cdots x_{i_m}$ for some $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, and there are irreducible $x_1, \ldots, x_m \in M$ such that $x \mid x_1 \cdots x_m$, but x divides no proper subproduct of the x_i.*

For an element $x \in S$, the product $x_1 \cdots x_m$ alluded to in part (c) above will be called a *bullet* for x.

The second necessary result is an amazing characterization of the membership problem for a numerical monoid generated by an interval of integers.

Theorem 1.4 [García-Sánchez and Rosales 1999, Corollary 2]. *An element $n \in \mathbb{N}$ belongs to $S = \langle a, a+1, \ldots, a+x \rangle$ if and only if*

$$n \pmod a \le \left\lfloor \frac{n}{a} \right\rfloor x,$$

where $\lfloor \cdot \rfloor$ represents the greatest integer function and residues are assumed to be least.

To prove Theorem 1.2, we will verify the 6 claimed values of the omega function. To do this, we will pivot on Theorem 1.3(c) and produce a bullet for each of the six elements. The condition in Theorem 1.4 will be vital in these arguments. In the two monoids we consider, the condition will reduce to

$$n \pmod{2k+1} \le \left\lfloor \frac{n}{2k+1} \right\rfloor 2$$

for $S_1 = \langle 2k+1, 2k+2, 2k+3 \rangle$ and

$$n \pmod{2k} \le \left\lfloor \frac{n}{2k} \right\rfloor 2$$

for $S_2 = \langle 2k, 2k+1, 2k+2 \rangle$. To finish the proof, we will then verify the first part of Theorem 1.3(c); namely if the bullet is of length j, then divisibility by a sum of length greater than or equal to j yields divisibility by a subsum of length j or less.

2. Proof of Theorem 1.2 for S_1

Lemma 2.1. *In S_1 we have the following divisibility relationships*:

(a) $(2k+1) \,\Big|\, \sum_{i=1}^{k+1} (2k+3)$;

(b) $(2k+2) \,\Big|\, \sum_{i=1}^{k+2} (2k+1)$;

(c) $(2k+3) \,\Big|\, \sum_{i=1}^{k+2} (2k+1)$.

Proof. (a) Now, $\sum_{i=1}^{k+1}(2k+3) = (k+1)(2k+3) = 2k^2+5k+3$. To prove the claim, we must show that $(2k^2+5k+3)-(2k+1) \in S_1$. Now (a) follows since $(2k^2+5k+3)-(2k+1) = 2k^2+3k+2 = k(2k+1)+(2k+2)$.

For the proof of (b) and (c), note that $\sum_{i=1}^{k+2}(2k+1) = (k+2)(2k+1) = 2k^2+5k+2$. For (b), we must show that $(2k^2+5k+2)-(2k+2) \in S_1$. Since

$$(2k^2+5k+2)-(2k+2) = 2k^2+3k = k(2k+3),$$

part (b) follows. For (c), we must show that $(2k^2+5k+2)-(2k+3) \in S_1$. Since $(2k^2+5k+2)-(2k+3) = 2k^2+3k-1 = 2k+2+(k-1)(2k+3)$, part (c) follows and the proof of the lemma is complete. □

In the next three lemmas, we show that the sums produced in Lemma 2.1 are actually bullets for $2k+1$, $2k+2$ and $2k+3$.

Lemma 2.2. *In S_1, $2k+1$ does not divide any proper subsum of* $\sum_{i=1}^{k+1}(2k+3)$.

Proof. To prove the claim, we must show that $2k+1$ does not divide $j(2k+3)$ for $1 \le j \le k$. This is equivalent to showing that

$$j(2k+3)-(2k+1) \notin S_1,$$

for each $1 \le j \le k$. Using Theorem 1.4, we must show that

$$j(2k+3)-(2k+1) \pmod{2k+1} > \left\lfloor \frac{j(2k+3)-(2k+1)}{2k+1} \right\rfloor \cdot 2, \tag{1}$$

for each $1 \le j \le k$. Now, (1) reduces to

$$2j > \left\lfloor \frac{j(2k+3)-(2k+1)}{2k+1} \right\rfloor 2, \tag{2}$$

and hence

$$j > \left\lfloor j\left(\frac{2k+3}{2k+1}\right) - 1 \right\rfloor. \tag{3}$$

Equation (3) can be rewritten as

$$j > \left\lfloor j\left(\frac{2k+3}{2k+1}\right) - 1 \right\rfloor = \left\lfloor j - 1 + j\left(\frac{2}{2k+1}\right) \right\rfloor = j - 1 + \left\lfloor \frac{2j}{2k+1} \right\rfloor.$$

Since $j \le k$, we have

$$\frac{2j}{2k+1} \le \frac{2k}{2k+1} < \frac{2k+1}{2k+1} = 1,$$

so $\lfloor 2j/(2k+1) \rfloor = 0$ and Equation (3) is true, which completes the proof. □

Lemma 2.3. *In S_1, $2k+2$ does not divide any proper subsum of* $\sum_{i=1}^{k+2}(2k+1)$.

Proof. To prove the claim, we must show that $2k+2$ does not divide $j(2k+1)$ for $1 \le j \le k+1$. This is equivalent to showing that

$$j(2k+1)-(2k+2) \notin S_1,$$

for each $1 \le j \le k+1$. Using Theorem 1.4 again, we must show that

$$j(2k+1)-(2k+2) \pmod{2k+1} > \left\lfloor \frac{j(2k+1)-(2k+2)}{2k+1} \right\rfloor 2, \tag{4}$$

for each $1 \le j \le k$. Now,

$$j(2k+1)-(2k+2) \equiv -1 \equiv 2k \pmod{2k+1},$$

and thus (4) reduces to

$$k > \left\lfloor j - \frac{2k+2}{2k+1} \right\rfloor. \tag{5}$$

Note that

$$\left\lfloor j - \frac{2k+2}{2k+1} \right\rfloor = \left\lfloor j-1-\frac{1}{2k+1} \right\rfloor = j-1+\left\lfloor -\frac{1}{2k+1} \right\rfloor = j-2.$$

Since $j \le k$, Equation (5) holds which completes the proof. $\square$

Lemma 2.4. *In S_1, $2k+3$ does not divide any proper subsum of* $\sum_{i=1}^{k+2}(2k+1)$.

Proof. To prove the claim, we must show that $2k+3$ does not divide $j(2k+1)$ for $1 \le j \le k+1$. This is equivalent to showing that

$$j(2k+1)-(2k+3) \notin S_1,$$

for each $1 \le j \le k+1$. Using Theorem 1.4 again, we must show that

$$j(2k+1)-(2k+3) \pmod{2k+1} > \left\lfloor \frac{j(2k+1)-(2k+3)}{2k+1} \right\rfloor 2, \tag{6}$$

for each $1 \le j \le k$. Now,

$$j(2k+1)-(2k+3) \equiv -2 \equiv (2k-1) \pmod{2k+1},$$

and thus (6) reduces to

$$2k-1 > \left\lfloor j - \frac{2k+3}{2k+1} \right\rfloor 2.$$

Notice that $1 < (2k+3)/(2k+1) < 2$, and so $\lfloor j-(2k+3)/(2k+1) \rfloor = j-2$. Hence,

$$2k-1 > 2(j-2) = 2j-4,$$

and thus

$$k+\tfrac{3}{2} > j.$$

The last statement is true since $1 \le j \le k+1$, which completes the proof of the lemma. □

To complete the argument for S_1, we must verify that the first condition in Theorem 1.3(c) holds.

Proposition 2.5. (a) *If* $(2k+1) \mid \alpha_1+\cdots+\alpha_t$ *where each* α_i *is irreducible in* S_1 *and* $t \ge k+1$, *then there is a proper subsum* $\alpha_{i_1}+\cdots+\alpha_{i_r}$ *of* $\alpha_1+\cdots+\alpha_t$ *with* $r \le k+1$ *such that* $(2k+1) \mid \alpha_{i_1}+\cdots+\alpha_{i_r}$.

(b) *If* $(2k+2) \mid \alpha_1+\cdots+\alpha_t$ *where each* α_i *is irreducible in* S_1 *and* $t \ge k+2$, *then there is a proper subsum* $\alpha_{i_1}+\cdots+\alpha_{i_r}$ *of* $\alpha_1+\cdots+\alpha_t$ *with* $r \le k+2$ *such that* $(2k+2) \mid \alpha_{i_1}+\cdots+\alpha_{i_r}$.

(c) *If* $(2k+3) \mid \alpha_1+\cdots+\alpha_t$ *where each* α_i *is irreducible in* S_1 *and* $t \ge k+2$, *then there is a proper subsum* $\alpha_{i_1}+\cdots+\alpha_{i_r}$ *of* $\alpha_1+\cdots+\alpha_t$ *with* $r \le k+2$ *such that* $(2k+3) \mid \alpha_{i_1}+\cdots+\alpha_{i_r}$.

Proof. (a) We can clearly reduce to the case where all the α_i are of the form $2k+2$ or $2k+3$. We also note that since $(2k+2)+(2k+2) = 4k+4 = (2k+1)+(2k+3)$, it follows that $(2k+1) \mid (2k+2)+(2k+2)$. Hence, if the sum $\alpha_1+\cdots+\alpha_t$ contains two or more irreducibles of the form $2k+2$, then we are done. Assume that this is not the case. If there are no irreducibles of the form $2k+2$, then the result follows by Lemma 2.1(a). If there is exactly one copy of $2k+2$, then consider $k(2k+3)+(2k+2) = 2k^2+5k+2$. It follows that

$$(2k^2+5k+2)-(2k+1) = 2k^2+3k+1 = (k+1)(2k+1).$$

Hence, $(2k+1) \mid k(2k+3)+(2k+2)$, which completes the proof.

(b) It is only necessary to look at the case where all the α_i are of the form $2k+1$ or $2k+3$. We first note that since $(2k+1)+(2k+3) = 4k+4 = 2(2k+2)$, it follows $(2k+2) \mid (2k+1)+(2k+3)$, and if the sum $\alpha_1+\cdots+\alpha_t$ contains at least one of each irreducible $2k+1$ and $2k+3$, then we are done. If the sum contains no copies of $2k+3$, then the result holds by Lemma 2.1(b). If the sum contains no copies of $2k+1$, then the equality

$$(k+1)(2k+3)-(2k+2) = 2k^2+3k+1 = (k+1)(2k+1)$$

completes the proof.

(c) It is only necessary to look at the case where the α_i are of the form $2k+1$ or $2k+2$. Now, $(2k+2)+(2k+2) = (2k+3)+(2k+1)$ and thus, if the sum $\alpha_1+\cdots+\alpha_t$ contains at least 2 irreducibles of the form $2k+2$, then we are done. If there are no irreducibles of the form $2k+2$, then this result follows by Lemma 2.1(c). If there is exactly one irreducible of the form $2k+2$, then consider $(k+1)(2k+1)+(2k+2) = 2k^2+5k+3$. Now,

$$(2k^2+5k+3)-(2k+3) = 2k^2+3k = k(2k+3),$$

and thus $(2k+3) \mid (k+1)(2k+1)+(2k+2)$, which completes the proof. □

3. Proof of Theorem 1.2 for S_2

Lemma 3.1. *In S_2, we have the following divisibility relationships*:

(a) $2k \,\Big|\, \sum_{i=1}^{k}(2k+2)$;

(b) $(2k+1) \,\Big|\, \sum_{i=1}^{k+2} 2k$;

(c) $(2k+2) \,\Big|\, \sum_{i=1}^{k+1} 2k$.

Proof. (a) $\sum_{i=1}^{k}(2k+2) = 2k^2+2k$. Now, $(2k^2+2k)-(2k) = 2k^2 = 2k(k)$. Thus, $2k \mid k(2k+2)$ and the result follows.

(b) $\sum_{i=1}^{k+2}(2k) = (k+2)(2k) = 2k^2+4k$. Now, $(2k^2+4k)-(2k+1) = 2k^2+2k-1 = (k-1)(2k+2)+(2k+1) \in S_2$. Thus, $(2k+1) \mid (k+2)(2k)$ and the result follows.

(c) $\sum_{i=1}^{k+1}(2k) = (k+1)(2k) = 2k^2+2k$. Now, $(2k^2+2k)-(2k+2) = (k-1)(2k+2) \in S_1$. Thus, $(2k+2) \mid 2k(k+1)$ and the result follows. □

Lemma 3.2. *In S_2, $2k$ does not divide any proper subsum of* $\sum_{i=1}^{k}(2k+2)$.

Proof. To prove this claim, we must show that $2k$ does not divide $j(2k+2)$ for $1 \le j \le k-1$. This is equivalent to showing that

$$j(2k+2)-2k \notin S_2,$$

for each $1 \le j \le k-1$. Using Theorem 1.4, we must show that

$$j(2k+2)-2k \pmod{2k} > 2\left\lfloor \frac{j(2k+2)-2k}{2k} \right\rfloor.$$

As in the arguments in Section 2, this reduces to

$$j > \left\lfloor \frac{jk+j-k}{k} \right\rfloor,$$

which is equivalent to

$$j > \left\lfloor j+\frac{j}{k}-1 \right\rfloor.$$

Since $j \le k-1$, we have $j/k < 1$. So,

$$j-1 = \left\lfloor j + \frac{j}{k} - 1 \right\rfloor \quad \text{and} \quad j > j-1 = \left\lfloor j + \frac{j}{k} - 1 \right\rfloor.$$

Thus, no subsum is in S_2. □

Lemma 3.3. *In* S_2, $2k+1$ *does not divide any proper subsum of* $\sum_{i=1}^{k+2}(2k)$.

Proof. To prove this claim, we must show that $2k+1$ does not divide $j(2k)$ for $1 \le j \le k+1$. This is equivalent to showing that

$$j(2k) - (2k+1) \notin S_2,$$

for each $1 \le j \le k-1$. Using Theorem 1.4, we must show that

$$j(2k) - (2k+1) \pmod{2k} > 2\left\lfloor \frac{j2k - (2k+1)}{2k} \right\rfloor.$$

This is equivalent to

$$(2k-1) > 2\left\lfloor \frac{j2k - (2k+1)}{2k} \right\rfloor.$$

Since $1 < (2k+1)/2k < 2$, we know that

$$j-2 = \lfloor j-2 \rfloor = \left\lfloor j - \frac{2k+1}{2k} \right\rfloor = \left\lfloor \frac{j2k - (2k+1)}{2k} \right\rfloor.$$

By the limits on j, it follows that $k - \frac{1}{2} > j-2$. Combining the last two inequalities and multiplying by 2 yields the desired result. □

Lemma 3.4. *In* S_2, $2k+2$ *does not divide any proper subsum of* $\sum_{i=1}^{k+1} 2k$.

Proof. To prove this claim, we must show that $2k+2$ does not divide $j(2k)$ for $1 \le j \le k$. This is equivalent to showing that

$$j(2k) - (2k+2) \notin S_2,$$

for each $1 \le j \le k$. Using Theorem 1.4, we must show that

$$j(2k) - (2k+2) \pmod{2k} > 2\left\lfloor \frac{j2k - (2k+2)}{2k} \right\rfloor.$$

This is equivalent to

$$k-1 > \left\lfloor \frac{jk - k - 1}{k} \right\rfloor = \left\lfloor j - 1 - \frac{1}{k} \right\rfloor = j-2,$$

from which the result follows. □

Proposition 3.5. (a) *If* $2k \mid \alpha_1 + \cdots + \alpha_t$ *where each* α_i *is irreducible in* S_1 *and* $t \geq k$, *then there is a proper subsum* $\alpha_{i_1} + \cdots + \alpha_{i_r}$ *of* $\alpha_1 + \cdots + \alpha_t$ *with* $r \leq k$ *such that* $2k \mid \alpha_{i_1} + \cdots + \alpha_{i_r}$.

(b) *If* $(2k+1) \mid \alpha_1 + \cdots + \alpha_t$ *where each* α_i *is irreducible in* S_1 *and* $t \geq k+2$, *then there is a proper subsum* $\alpha_{i_1} + \cdots + \alpha_{i_r}$ *of* $\alpha_1 + \cdots + \alpha_t$ *with* $r \leq k+2$ *such that* $(2k+1) \mid \alpha_{i_1} + \cdots + \alpha_{i_r}$.

(c) *If* $(2k+2) \mid \alpha_1 + \cdots + \alpha_t$ *where each* α_i *is irreducible in* S_1 *and* $t \geq k+1$, *then there is a proper subsum* $\alpha_{i_1} + \cdots + \alpha_{i_r}$ *of* $\alpha_1 + \cdots + \alpha_t$ *with* $r \leq k+1$ *such that* $(2k+1) \mid \alpha_{i_1} + \cdots + \alpha_{i_r}$.

Proof. (a) We can clearly reduce to the case where the α_i are of the form $2k+1$ and $2k+2$. Also note that we can assume that $t > k$, as the result clearly holds if $t = k$. Since

$$(2k+1) + (2k+1) = 4k+2 = (2k+2) + (2k),$$

the result holds if at least two of the α_i are of the form $2k+1$. If at least k of the α_i are of the form $2k+2$, then the result holds by Lemma 3.1(a). If not, then we have at least two of the form $2k+1$, which completes the proof of (a).

(b) We can clearly reduce to the case where the α_i are of the form $2k$ and $2k+2$. We proceed as in (a) and assume that $t > k+2$. Note that

$$(2k) + (2k+2) = (2k+1) + (2k+1).$$

Hence if at least one of the α_i is of each type, then we are done. If all the α_i are of the form $2k$, then we are done by Lemma 3.1(b). To complete the argument, note that $(k+1)(2k+2) = 2k^2 + 4k + 2$ and

$$2k^2 + 4k + 2 - (2k+1) = 2k^2 + 2k + 1 = k(2k) + (2k+1) \in S_2.$$

(c) We can clearly reduce to the case where the α_i are of the form $2k$ and $2k+1$. Assume as in (a) and (b) that $t > k+1$. As before,

$$(2k+1) + (2k+1) = 4k+2 = (2k+2) + (2k),$$

and if at least two of the α_i are of the form $2k+1$, then we are done. Otherwise, we have at least $2k+1$ copies of $2k$, and the result follows by Lemma 3.1(c). □

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