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a journal of mathematics

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msp

2014 vol. 7, no. 5

INVOLVE 7:5 (2014)
dx.doi.org/10.2140/involve.2014.7.699

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(Communicated by Johnny Henderson)

The method of upper and lower solutions guarantees the interval of existence of nonlinear differential equations with initial conditions. To compute the solution on this interval, we need coupled lower and upper solutions on the interval of existence. We provide both theoretical as well as numerical methods to compute coupled lower and upper solutions by using a superlinear convergence method. Further, we develop monotone sequences which converge uniformly and monotonically, and with superlinear convergence, to the unique solution of the nonlinear problem on this interval. We accelerate the superlinear convergence by means of the Gauss–Seidel method. Numerical examples are developed for the logistic equation. Our method is applicable to more general nonlinear differential equations, including Riccati type differential equations.

1. Introduction

Qualitative study such as existence, uniqueness of nonlinear differential equations with initial and boundary conditions play an important role in modeling science and engineering problems. Explicit solutions of such nonlinear problems are rarely possible [Adams et al. 2012; Cronin 1994; Holt and Pickering 1985; Lakshmikantham et al. 1989; Jin et al. 2004]. Approximate methods such as Picard's method provide only local existence. The generalized monotone method combined with coupled lower and upper solutions provides a method to compute coupled

MSC2010: primary 34A12; secondary 34A34.
Keywords: coupled lower and upper solutions, superlinear convergence.
This work was done under the auspices of the 2013 "Smooth Transition for Advancement to Graduate Education" (STAGE) for Underrepresented Minorities in Mathematical Sciences, an undergraduate summer research and professional development experience organized by the University of Louisiana at Lafayette. STAGE is a pilot project that is supported by National Science Foundation grant DMS-1043223.

minimal and maximal solutions [Bhaskar and McRae 2002; Sokol and Vatsala 2001; Stutson and Vatsala 2011; West and Vatsala 2004]. Noel, Sheila, Zenia, Dayonna, Jasmine, Vatsala, and Sowmya [Noel et al. 2012] have developed both theoretical and numerical approaches to compute coupled minimal and maximal solutions using the idea of generalized monotone method. See [Muniswamy and Vatsala 2013; Noel et al. 2012] for details. However, the order of convergence of the sequences generated is linear. We note that the generalized monotone method is useful when the nonlinear function is the sum of increasing and decreasing functions.

In this paper, we develop a method when the nonlinear function is the sum of a convex function and a decreasing function. The sequences constructed yield quadratic convergence when the decreasing function is not present and yields linear convergence when the convex term is not present. We develop a methodology to compute coupled lower and upper solutions whose convergence rate is superlinear on any desired interval. Using the computed coupled upper and lower solutions, we can develop sequences which converge uniformly and monotonically to the unique solution of the nonlinear problem. The rate of convergence of the sequences is superlinear. In addition, the superlinear convergence can be accelerated by using the Gauss–Seidel approach. We have presented some numerical examples of the population model of single species, namely the logistic equation. Our method is applicable to more general nonlinear problems such as Ricatti type differential equation.

2. Preliminary results

In this section, we recall known definitions and results which we need to develop our main results. For that purpose, consider the first-order differential equation of the form

$$u' = f(t,u) + g(t,u), \quad u(0) = u_0 \quad \text{on } [0,T] = J, \tag{2-1}$$

where f, g lie in $C(J \times \mathbb{R}, \mathbb{R})$, the space of continuous functions from $J \times \mathbb{R}$ to $\mathbb{R}$.

Definition 2.1. The functions $v_0, w_0 \in C^1(J, \mathbb{R})$ are called *natural lower and upper solutions* of (2-1) if

$$\begin{aligned} v_0' &\le f(t, v_0) + g(t, v_0), &\quad v_0(0) &\le u_0, \\ w_0' &\ge f(t, w_0) + g(t, w_0), &\quad w_0(0) &\ge u_0. \end{aligned}$$

Definition 2.2. The functions $v_0, w_0 \in C^1(J, \mathbb{R})$ are called *coupled lower and upper solutions* of (2-1) of type I if

$$\begin{aligned} v_0' &\le f(t, v_0) + g(t, w_0), &\quad v_0(0) &\le u_0, \\ w_0' &\ge f(t, w_0) + g(t, v_0), &\quad w_0(0) &\ge u_0. \end{aligned}$$

Next we recall a comparison theorem which will be useful in establishing the uniqueness of the solution of (2-1). For that purpose we assume

$$f(t,u)+g(t,u)=F(t,u).$$

Theorem 2.3. *Let $v, w \in C^1(J,\mathbb{R})$ be lower and upper solutions of (2-1) respectively. Suppose that $F(t,x)-F(t,y) \le L(x-y)$ whenever $x \ge y$, and $L>0$ is a constant, then $v(0) \le w(0)$ implies that $v(t) \le w(t)$, $t \in J$.*

Corollary 2.4. *Let $p(t) \in C(J,\mathbb{R})$ be a function such that $p'(t) \le L(t)p(t)$, where $L(t) \in C(J,\mathbb{R})$. Then $p(0) \le 0$ implies $p(t) \le 0$.*

Remark. If in Corollary 2.4 all the inequalities are reversed, the conclusion holds with reversed inequality.

We define the following sector Ω for convenience. That is,

$$\Omega = \big\{(t,u) \mid v(t) \le u(t) \le w(t),\, t \in J\big\}.$$

Theorem 2.5. *Suppose $v, w \in C^1(J,\mathbb{R})$ are natural upper and lower solutions of (2-1) such that $v(t) \le w(t)$ on J and $F \in C(\Omega,\mathbb{R})$. Then there exists a solution $u(t)$ of (2-1) such that $v(t) \le u(t) \le w(t)$ on J, provided $v(0) \le u(0) \le w(0)$.*

Proof. See [Ladde et al. 1985] for details. □

Remark. If $g(t,u)$ in (2-1) is nonincreasing in u, then the existence of coupled lower and upper solutions of (2-1) on J implies that they are also natural lower and upper solutions. From Theorem 2.5 it follows that there exists a solution of (2-1) such that $v(t) \le u(t) \le w(t)$ on J, provided $v(0) \le u(0) \le w(0)$.

The next result is to prove the existence of coupled lower and upper solutions by the generalized monotone method.

Theorem 2.6. *Let $v_0, w_0 \in C^1(J,\mathbb{R})$ be coupled upper and lower solutions of type I such that $v_0(t) \le w_0(t)$ on J, and assume that f, g are elements of $C(J \times \mathbb{R},\mathbb{R})$ such that $f(t,u)$ is nondecreasing in u and $g(t,u)$ is nonincreasing in u on J.*

There exist monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ on J such that

$$v_n(t) \to v(t) \quad \textit{and} \quad w_n(t) \to w(t)$$

uniformly and monotonically, and (v, w) are coupled minimal and maximal solutions, respectively, to (2-1). That is, (v, w) satisfy on J the equations

$$v' = f(t,v)+g(t,w), \qquad v(0)=u_0, \tag{2-2}$$

$$w' = f(t,w)+g(t,v), \qquad w(0)=u_0. \tag{2-3}$$

Here the iterative scheme is given on J by

$$v'_{n+1} = f(t, v_n) + g(t, w_n), \quad v_{n+1}(0) = u_0, \tag{2-4}$$
$$w'_{n+1} = f(t, w_n) + g(t, v_n), \quad w_{n+1}(0) = u_0. \tag{2-5}$$

Proof. See [Noel et al. 2012; Sokol and Vatsala 2001; West and Vatsala 2004] for details of the proof. □

We now give an existence result based on the generalized monotone method using natural lower and upper solutions.

Theorem 2.7. *Let $v_0, w_0 \in C^1(J, \mathbb{R})$ be natural lower and upper solutions with $v_0 \le w_0$ on J, and assume that f, g are elements of $C(J \times \mathbb{R}, \mathbb{R})$ such that $f(t, u)$ is nondecreasing in u and $g(t, u)$ is nonincreasing in u on J.*

There exist monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ on J such that

$$v_n(t) \to v(t) \quad \text{and} \quad w_n(t) \to w(t)$$

uniformly and monotonically, and (v, w) are coupled minimal and maximal solutions, respectively, to (2-1). That is, (v, w) satisfy

$$v' = f(t, v) + g(t, w), \quad v(0) = u_0,$$
$$w' = f(t, w) + g(t, v), \quad w(0) = u_0,$$

on J, provided also that $v_0 \le v_1$ and $w_1 \le w_0$ on J.

Proof. See [Noel et al. 2012; West and Vatsala 2004] for details of the proof. □

Noel et al. [2012] have developed computational methods to compute coupled lower and upper solutions to any desired interval by applying Theorem 2.7, by redefining the sequences on the interval J. However, the rate of convergence of these sequences is linear.

Before we recall the next result, which provides quadratic convergence, we need the Gronwall lemma which will be used to compute the rate of convergence.

Lemma 2.8 (Gronwall lemma). *Let $v \in C^1(J, \mathbb{R}^N)$ and $v' \le Av + \sigma$, where $A = (a_{ij})$ is an $N \times N$ constant matrix satisfying $a_{ij} \ge 0$, $i \ne j$, and $\sigma \in C(J, \mathbb{R}^N)$. Then we have*

$$v' \le v(0)e^{At} + \int_0^t e^{A(t-s)}\sigma(s)\, ds, \quad t \in J. \tag{2-6}$$

Theorem 2.9. *Assume that*

(i) *$v_0, w_0 \in C^1(J, \mathbb{R})$, $v_0(t) \le w_0(t)$ on J, with $v_0(t)$ and $w_0(t)$ coupled lower and upper solutions of type I for (2-1), such that $v_0(t) \le w_0(t)$ on J;*

(ii) *$f, g \in C(\Omega, \mathbb{R})$, f_u, g_u, f_{uu}, g_{uu} exist, are continuous and satisfy $f_{uu}(t, u) \ge 0$, $g_{uu}(t, u) \le 0$ for $(t, u) \in \Omega = \{t \in J \mid v_0(t) \le u \le w_0(t)\}$;*

(iii) *$g_u(t, u) \le 0$ on Ω.*

Then there exist monotone sequences $\{v_n(t)\}$, $\{w_n(t)\}$ *that converge uniformly to the unique solution of* (2-1). *The convergence is quadratic.*

Proof. See [Lakshmikantham and Vatsala 1998] for details of the proof. □

In Theorem 2.9 the iterations are as follows:

$$\begin{cases} v_n' = f(t, v_{n-1}) + f_u(t, v_{n-1})(v_n - v_{n-1}) \\ \qquad\qquad + g(t, w_{n-1}) + g_u(t, v_{n-1})(w_n - w_{n-1}), \\ v_n(0) = u_0, \end{cases} \tag{2-7}$$

$$\begin{cases} w_n' = f(t, w_{n-1}) + f_u(t, v_{n-1})(w_n - w_{n-1}) \\ \qquad\qquad + g(t, v_{n-1}) + g_u(t, v_{n-1})(v_n - v_{n-1}), \\ w_n(0) = u_0. \end{cases} \tag{2-8}$$

In this theorem the sequences $\{v_n\}$ and $\{w_n\}$ are solutions of the two linear systems of coupled equations with variable coefficients, which are not easy to compute. In the next result under a slightly weaker assumption, we obtain superlinear convergence.

Theorem 2.10. *Assume that*

(i) $v_0, w_0 \in C^1(J, \mathbb{R})$, $v_0(t) \le w_0(t)$ *on* J, *with* $v_0(t)$ *and* $w_0(t)$ *coupled lower and upper solutions of type I for* (2-1), *such that* $v_0(t) \le w_0(t)$ *on* J;

(ii) $f, g \in C(\Omega, \mathbb{R})$, f_u, g_u, f_{uu} *exist, are continuous and satisfy* $f_{uu}(t, u) \ge 0$, *for* $(t, u) \in \Omega = \{t \in J \mid v_0(t) \le u \le w_0(t)\}$;

(iii) $g_u(t, u) \le 0$ *on* Ω.

Then there exist monotone sequences $\{v_n(t)\}$, $\{w_n(t)\}$ *that converge uniformly to the unique solution of* (2-1), *and the convergence is superlinear.*

Proof. See [Muniswamy and Vatsala 2013] for details of proof. □

3. Main results

In this section we will provide a method to compute coupled lower and upper solutions of (2-1) to any desired interval when we have natural lower and upper solutions. Natural lower and upper solutions are relatively easy to compute. For example, equilibrium solutions are natural lower and upper solutions for all time. This means the solution of the nonlinear problem exists for all time by upper and lower solution method. In order to develop this method, we modify Theorem 2.10 using natural lower and upper solutions.

Theorem 3.1. *Let*

(i) $v_0, w_0 \in C^1(J, R)$, $v_0(t) \le w_0(t)$ *on* J, *with* $v_0(t)$ *and* $w_0(t)$ *natural lower and upper solutions of* (2-1), *such that* $v_0(t) \le w_0(t)$ *on* J;

(ii) *$f, g \in C(\Omega, \mathbb{R})$, f_u, g_u, f_{uu} exist, are continuous and satisfy $f_{uu}(t,u) \geq 0$ for $(t,u) \in \Omega = \{t \in J \mid v_0(t) \leq u \leq w_0(t)\}$;*

(iii) *$g_u(t,u) \leq 0$ on Ω;*

(iv) *$v_0 \leq v_1$ and $w_1 \leq w_0$ on J.*

Then there exist monotone sequences $\{v_n(t)\}$, $\{w_n(t)\}$ that converge uniformly to the unique solution of (2-1). The convergence is superlinear.

Here and in Theorem 2.10 the iterations are computed as follows:

$$v_n' = f(t, v_{n-1}) + f_u(t, w_{n-1})(v_n - v_{n-1}) + g(t, w_{n-1}), \qquad v_n(0) = u_0, \tag{3-1}$$

$$w_n' = f(t, w_{n-1}) + f_u(t, v_{n-1})(w_n - w_{n-1}) + g(t, v_{n-1}), \qquad w_n(0) = u_0. \tag{3-2}$$

Proof. The proof follows on the same lines as in Theorem 2.10. Here we briefly prove the superlinear convergence part. In order to prove superlinear convergence, we let $p_n(t) = u(t) - v_n(t)$ and $q_n(t) = w_n(t) - u(t)$ on J, where u, v_n, and w_n are solutions of (2-1), (3-1), and (3-2) respectively. It is easy to see that $p_n(0) = 0 = q_n(0)$. Using Gronwall lemma and the estimate on $|f_{uu}(+,\cdot)|$ and $|g_u(+,\cdot)|$ on J, we can prove that

$$\max_J\big(|p_n + q_n|\big) \leq L_1 \max_J\big(|p_{n-1} + q_{n-1}|^2\big) + L_2 \max_J\big(|p_{n-1} + q_{n-1}|\big),$$

where L_1 and L_2 depends on bounds of $|f_{uu}(+,\cdot)|$ and $|g_u(+,\cdot)|$ on J. If $g \equiv 0$, then $L_2 \equiv 0$, we have quadratic convergence. If $f \equiv 0$, then $L_1 \equiv 0$, which means we have linear convergence. □

Consider the following example

$$u' = u - u^2, \quad u(0) = \tfrac{1}{2}, \quad t \in [0, T], \quad T \geq 1.$$

It is easy to observe that $v_0(t) = 0$ and $w_0(t) = 1$ are natural lower and upper solutions. Starting with $v_0 = 0$ and $w_0 = 1$, which are natural lower and upper solutions, and using the iterations as in Theorem 3.1, we get

$$v_1 = 1 - \tfrac{1}{2}e^t \quad \text{and} \quad w_1 = \tfrac{1}{2}e^t.$$

We can see that

$$\begin{aligned} v_0 &\leq v_1, & 0 &\leq 1 - \tfrac{1}{2}e^t && \text{on } [0, 0.69], \\ w_1 &\leq w_0, & \tfrac{1}{2}e^t &\leq 1 && \text{on } [0, 0.69]. \end{aligned}$$

This means $v_0 \leq v_1$ and $w_1 \leq w_0$ on $[0, 0.69]$. Here $0.69 < T$. This is the motivation for our next main result: developing a method to compute coupled lower and upper solutions to any desired interval.

Theorem 3.2. *Let all the hypothesis of Theorem 3.1 hold.*

Then there exist monotone sequences $\{v_n(t)\}$ *and* $\{w_n(t)\}$ *on* J *such that*

$$v_n(t) \to v(t) \quad \text{and} \quad w_n(t) \to w(t)$$

uniformly and monotonically and (v, w) *are coupled minimal and maximal solutions, respectively to* (2-1).

The sequences $\{v_n\}$ *and* $\{w_n\}$ *are computed using the following iterative scheme*:

$$v_n' = f(t, v_{n-1}) + f_u(t, v_{n-1})(v_n - v_{n-1}) + g(t, w_{n-1}), \qquad v_n(0) = u_0,$$
$$w_n' = f(t, w_{n-1}) + f_u(t, v_{n-1})(w_n - w_{n-1}) + g(t, v_{n-1}), \quad w_n(0) = u_0.$$

Proof. We compute the iterations using v_0 and w_0 in the following form:

$$v_1' = f(t, v_0) + f_u(t, v_0)(v_1 - v_0) + g(t, w_0), \qquad v_1(0) = u_0,$$
$$w_1' = f(t, w_0) + f_u(t, v_0)(w_1 - w_0) + g(t, v_0), \quad w_1(0) = u_0.$$

After computing v_1 and w_1, if $v_0 \le v_1$ and $w_1 \le w_0$ on $[0, T]$, then there is nothing to prove. If not, then $v_1(t_1) = v_0(t_1)$ and $w_1(\bar{t}_1) = w_0(\bar{t}_1)$. It is obvious that t_1 and $\bar{t}_1$ are less than T. We relabel $v_1(t)$ and $w_1(t)$ as

$$v_1(t) = v_1(t) \text{ on } [0, t_1], \qquad w_1(t) = w_1(t) \text{ on } [0, \bar{t}_1],$$
$$v_1(t) = v_0(t) \text{ on } [t_1, T], \qquad w_1(t) = w_0(t) \text{ on } [\bar{t}_1, T].$$

It is easy to see that $v_0 \le v_1$ and $w_1 \le w_0$ on $[0, T]$. Continuing this process, we can compute $v_n(t)$ and $w_n(t)$ as

$$v_n' = f(t, v_{n-1}) + f_u(t, v_{n-1})(v_n - v_{n-1}) + g(t, w_{n-1}), \qquad v_n(0) = u_0, \tag{3-3}$$
$$w_n' = f(t, w_{n-1}) + f_u(t, v_{n-1})(w_n - w_{n-1}) + g(t, v_{n-1}), \quad w_n(0) = u_0, \tag{3-4}$$

on $[0, t_n]$ and $[0, \bar{t}_n]$, respectively. Again relabeling $v_n(t)$ and $w_n(t)$ on $[0, T]$ as before, we can prove

$$v_0 \le v_1 \le \cdots \le v_n \le u \le w_n \le \cdots \le w_1 \le w_0$$

on $[0, T]$. Note that this is the redefined sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ on $[0, T]$. We can show that the redefined sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are equicontinuous and uniformly bounded on J. We will show that the sequence $\{v_n(t)\}$ is uniformly bounded. Since $v_0 \le v_n \le w_0$, and v_0, w_0 are continuous functions on a closed bounded set, it follows that $0 \le |v_n(t) - v_0(t)| \le |w_0(t) - v_0(t)| \le K_1$ on $[0, T]$. From this, and using the triangle inequality, we can show that

$$|v_n| = |v_n - v_0 + v_0| \le |v_n - v_0| + |v_0| \le K_1 + K_2 = K,$$

on $[0, T]$ where K is independent of n and t. This proves that $\{v_n(t)\}$ is uniformly bounded. Similarly, we can prove that $\{w_n(t)\}$ is uniformly bounded. The equicontinuity of these sequences follows from the integral representation of v_n and w_n. This is achieved using the fact that the functions $f(t,u)$, $g(t,u)$ are continuous on Ω, and the uniform boundedness of $v_n(t)$ and $w_n(t)$. Hence, by the Arzelà–Ascoli theorem, a subsequence converges uniformly and monotonically. Since the sequences are monotone, the entire sequence converges uniformly and monotonically to v and w respectively. Further, we can prove the rate of convergence for these sequences are superlinear. □

We note that the elements of the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are also coupled lower and upper solutions of (2-1) on $[0, T]$. We demonstrate this here.

Since $v_{n-1} \le v_n$ on $[0, T]$ and $w_n \le w_{n-1}$ on $[0, T]$, we get

$$f(t, v_{n-1}) + f_u(t, v_{n-1})(v_n - v_{n-1}) \le f(t, v_n) \quad \text{on } [0, T]$$

and

$$g(t, w_{n-1}) \le g(t, w_n),$$

using the assumptions on f and g from the hypothesis. This proves

$$v_n' \le f(t, v_n) + g(t, w_n),\ v_n(0) = u_0 \quad \text{on } [0, T].$$

Using the nature of $f(t,u)$ and $g(t,u)$, we can prove that

$$f(t, w_{n-1}) + f_u(t, v_{n-1})(w_n - w_{n-1}) \ge f(t, w_n) \quad \text{and} \quad g(t, w_{n-1}) \ge g(t, v_n).$$

Now from the iterates w_n, we can show that

$$w_n' \ge f(t, w_n) + g(t, v_n),\ w_n(0) = u_0 \quad \text{on } [0, T].$$

This proves that v_n and w_n are coupled lower and upper solutions of type I on the interval $[0, T]$.

Remark. Note that we can accelerate the rate of convergence of the sequences in Theorem 3.2 by using the Gauss–Seidel method.

We will apply the Gauss–Seidel method to Theorem 2.10.

Theorem 3.3. *We assume that*

(i) $v_0, w_0 \in C^1(J, R)$, $v_0(t) \le w_0(t)$ *on* J, *with* $v_0(t)$ *and* $w_0(t)$ *coupled lower and upper solutions of type I for* (2-1), *such that* $v_0(t) \le w_0(t)$ *on* J;

(ii) $f, g \in C(\Omega, \mathbb{R})$, f_u, g_u, f_{uu} *exist, are continuous and satisfy* $f_{uu}(t,u) \ge 0$ *for* $(t,u) \in \Omega = \{t \in J \mid v_0(t) \le u \le w_0(t)\}$;

(iii) $g_u(t,u) \le 0$ *on* Ω.

Then there exist monotone sequences $\{v_n^*(t)\}$, $\{w_n^*(t)\}$ *that converge uniformly to the unique solution of* (2-1), *and the convergence is faster than superlinear.*

The iterative scheme is given by:

$$\left(v_n^*\right)' = f(t, v_{n-1}^*) + f_u(t, v_{n-1}^*)(v_n^* - v_{n-1}) + g(t, w_{n-1}^*), \qquad v_n^*(0) = u_0, \quad (3\text{-}5)$$

$$\left(w_n^*\right)' = f(t, w_{n-1}^*) + f_u(t, v_n^*)(w_n^* - w_{n-1}^*) + g(t, v_n^*), \qquad w_n^*(0) = u_0, \quad (3\text{-}6)$$

starting with $v_0^* = v_1$ *on J.*

Remark. Here $v_1(t)$ is computed using Theorem 2.10.

Proof. Initially, compute $v_1(t)$ using

$$v_1' = f(t, v_0) + f_u(t, v_0)(v_1 - v_0) + g(t, w_0), \qquad v_1(0) = u_0.$$

Relabel $v_1(t)$ as $v_0^*(t)$. Now compute $w_1(t)$ using $w_0(t)$ and $v_0^*(t)$. That is, $w_1(t)$ is the solution of

$$w_1' = f(t, w_0) + f_u(t, v_0^*)(w_1 - w_0) + g(t, v_0^*), \qquad w_1(0) = u_0.$$

Relabel $w_1(t)$ as $w_0^*(t)$ and continue the process.

It is obvious that $v_0(t) \le v_1(t) = v_0^*(t)$ and $w_1(t) \le w_0(t)$ on J. Therefore $g(v_0) \ge g(v_0^*)$ and $f_u(t, v_0) \le f_u(t, v_0^*)$ from the hypothesis. Let $p(t) = w_1(t) - w_0^*(t)$. Then $p(0) = 0$. Also,

$$\begin{aligned} p'(t) &= (w_1)'(t) - (w_0^*)'(t) \\ &= f_u(t, v_0)(w_1 - w_0) + g(t, v_0) - f_u(t, v_0^*)(w_0^* - w_0) - g(t, v_0^*) \\ &\ge f_u(t, v_0)(w_1 - w_0) - f_u(t, v_0^*)(w_0^* - w_0) \\ &\ge f_u(t, v_0^*)(w_1 - w_0) - f_u(t, v_0^*)(w_0^* - w_0) \\ &= f_u(t, v_0^*)p(t). \end{aligned}$$

It follows that $p'(t) \ge f_u(t, v_0^*)p(t)$. Using Corollary 2.4, we know $p(t) \ge 0$, that is, $w_1 \ge w_0^*$ on J. Continuing the process, we will be able to show that the sequences $\{v_n^*\}$ and $\{w_n^*\}$ converges faster than the sequences $\{v_n\}$ and $\{w_n\}$ computed using Theorem 2.10. □

4. Numerical results

Here we develop numerical results as an application of the theoretical main results in Section 3. All the numerical simulations are done using Euler's method, implemented in Matlab.

To begin with, consider the simple logistic equations

$$u' = u - u^2, \qquad u_0(0) = \tfrac{1}{2}, \qquad (4\text{-}1)$$

$$u' = 2u - 3u^2, \quad u_0(0) = \tfrac{1}{2}. \qquad (4\text{-}2)$$

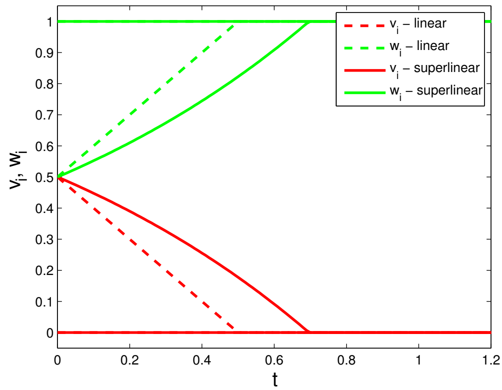

Figure 1. Comparison between linear and superlinear convergence.

It is easy to observe that $v_0(t) = 0$ and $w_0(t) = 1$ are the equilibrium solutions. In addition, they are also natural lower and upper solutions. Using the existence of solution by upper and lower solution method, the solution of (4-1) exists for all time. In Figure 1, we have computed coupled lower and upper solution using our superlinear convergence method as well as the linear convergence method as in [Noel et al. 2012]. In Figure 1, $v_0 \le v_1$ and $w_1 \le w_0$ on $[0, 0.5]$ by the generalized monotone method, whereas $v_0 \le v_1$ and $w_1 \le w_0$ on $[0, 0.7]$ by the superlinear convergence method.

Using Theorem 3.2, we computed v_i and w_i for (4-1), for $i = 1, 2, 3$. In Figure 2, we can see that in three iterations, that is, v_3 and w_3 are coupled lower and upper solutions of (4-1) on $[0, 1]$.

Using v_3 and w_3 from Figure 2 as v_0 and w_0 in Theorem 2.10, we compute the unique solution of (4-1), in Figure 3. This is achieved in four iterations. In Figure 4, we use superlinear convergence and the Gauss–Seidel method to compute coupled lower and upper solutions of (4-1) on $[0, 1]$. We achieved this in two iterations.

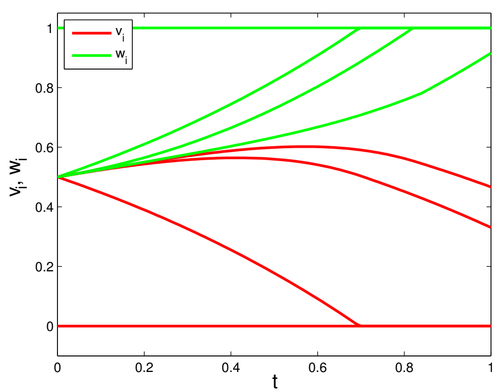

Figure 2. Coupled lower and upper solutions of (4-1) using Theorem 3.2.

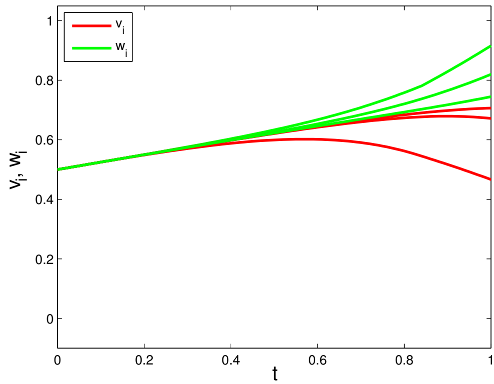

Figure 3. Four iterations of (4-1) using Theorem 2.10.

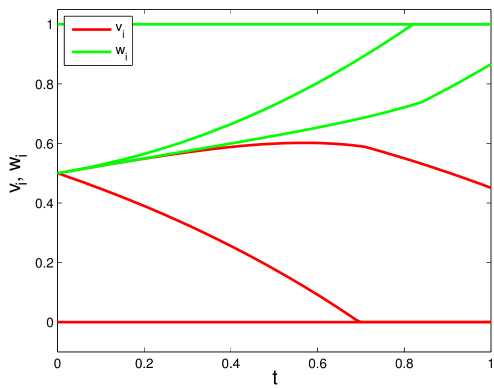

Figure 4. Coupled lower and upper solutions of (4-1) using Theorem 3.3.

Using the coupled lower and upper solution of Figure 4, we have computed the unique solution of (4-1) on [0, 1] using Theorem 3.3; see Figure 5. This combines

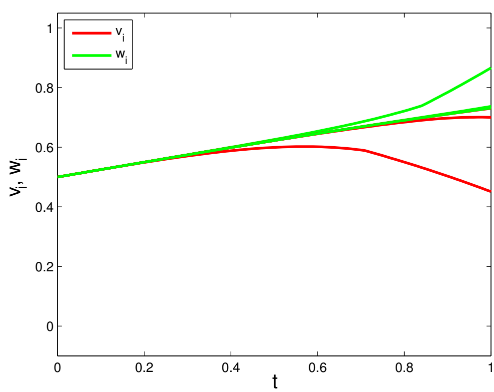

Figure 5. Three iterations of (4-1) using Theorem 3.3.

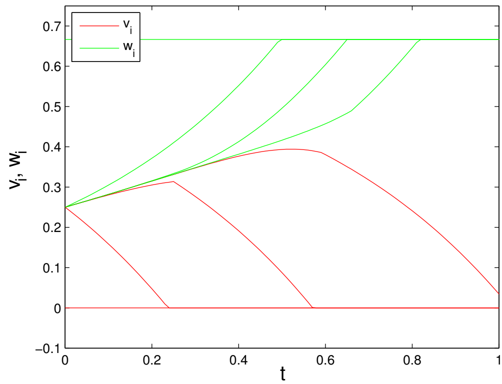

Figure 6. Three iterations of (4-2) using Theorem 3.2.

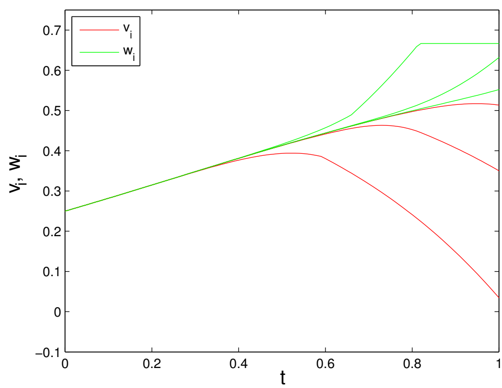

Figure 7. Three iterations of (4-2) using Theorem 2.10.

superlinear convergence and the Gauss–Seidel method. We achieved this in three iterations.

We have used the superlinear convergence method to compute coupled lower and upper solutions of (4-2) on [0, 1]. We achieved this in Figure 6 in three iterations.

Using the coupled lower and upper solutions of Figure 6, we have computed the unique solution of (4-2) on [0, 1] using Theorem 2.10. In Figure 7, this is achieved in three iterations.

5. Conclusion

We have developed a method to compute coupled upper and lower solutions for a nonlinear differential equation with initial conditions to any desired interval or to the interval of existence. We note that the natural lower and upper solutions guarantee the interval of existence of the solution. However, to compute the solutions by the generalized monotone method or the generalized quasilinearization method, we need coupled lower and upper solutions of type I on that interval. The method we have developed requires the construction of sequences or iterates that are solutions of the linear equation. The rate of convergence of these sequences is superlinear. Further, the rate of convergence can be accelerated using the Gauss–Seidel acceleration method. Linear convergence methods are developed in [Noel et al. 2012]. Although we have applied our theoretical method to the logistic equation in our numerical results, our method is applicable to a variety of nonlinear problems, including Ricatti type differential equations. We plan to extend our method to two or more systems of differential equations. We anticipate being able to apply it to two species biological models (the Lotka–Volterra equation, for example), which can be cooperative, competitive or predator–prey models.

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Received: 2013-09-15 Revised: 2013-11-22 Accepted: 2013-11-24

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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