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The h-vectors of PS ear-decomposable graphs

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We consider a family of simple graphs known as PS ear-decomposable graphs. These graphs are one-dimensional specializations of the more general class of PS ear-decomposable simplicial complexes, which were by Chari as a means of understanding matroid simplicial complexes. We outline a shifting algorithm for PS ear-decomposable graphs that allows us to explicitly show that the h-vector of a PS ear-decomposable graph is a pure $\mathbb{O}$-sequence.

1. Introduction

This paper concerns the combinatorial structure of a certain family of simple graphs known as *PS ear-decomposable* graphs. PS ear-decomposable graphs and, more generally, PS ear-decomposable simplicial complexes, were introduced by Chari [1997] and provide a unified framework for proving a number of combinatorial results about the combinatorial structure of matroid simplicial complexes.

Stanley [1977] conjectured that the h-vector of a matroid simplicial complex is a pure $\mathbb{O}$-sequence. Broadly speaking, the h-vector of a graph (or more generally a simplicial complex) encodes combinatorial information about its number of vertices and edges (respectively, the number of vertices, edges, and higher-dimensional faces in a simplicial complex), and a (pure) $\mathbb{O}$-sequence is the degree sequence of a (pure) family of monomials that is closed under divisibility. Thus Stanley's conjecture would impose extra structure on the number of vertices and edges that a graph in this family can have (or the number of vertices, edges, and higher-dimensional faces for the family of simplicial complexes).

Chari proved that all matroid simplicial complexes are PS ear-decomposable and used this extra structure to prove a number of results on h-vectors of matroid complexes. Thus it seems natural to conjecture that the h-vector of a PS ear-decomposable simplicial complex is a pure $\mathbb{O}$-sequence [Chari 1997, Conjecture 3],

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meaning that Stanley's conjecture would hold for this larger class of simplicial complexes.

In this paper, we focus our attention on the family of PS ear-decomposable graphs, which contains the family of all rank-2 matroids. The family of rank-2 matroids corresponds exactly to the family of complete multipartite graphs; but, as we will see, the family of PS ear-decomposable graphs is considerably larger. For any PS ear-decomposable graph Γ, we will define a canonical PS ear-decomposable graph $\mathcal{S}(\Gamma)$ with the same number of vertices and edges as Γ, called a *shifted* PS ear-decomposable graph. Having defined this shifted PS ear-decomposable graph, it will be easy to find a corresponding pure multicomplex whose F-vector is the h-vector of $\mathcal{S}(\Gamma)$. This approach of defining a shifting algorithm as a means of preserving combinatorial data while simplifying the algebraic or geometric structure of a simplicial complex is not new, and we refer to [Kalai 2002] and the references therein for further information. It is our hope that the shifting approach presented in this paper could be generalized to higher-dimensional PS ear-decomposable simplicial complexes as an alternative approach to solving Stanley's conjecture.

2. Background and definitions

We will be interested in studying two families of combinatorial objects in this paper. The first is the family of PS ear-decomposable graphs, and the second is the family of pure multicomplexes.

2.1. ***Graphs and PS ear-decompositions.*** In this paper we only consider finite, simple graphs, which we typically denote by Γ. The most natural combinatorial data that can be counted for a graph Γ are its number of vertices and edges, which we denote by $f_0(\Gamma)$ and $f_1(\Gamma)$ respectively. Here the subscripts indicate that a vertex is zero-dimensional and an edge is one-dimensional when we draw a graph. We are interested in studying certain integer linear transformations of these numbers, which are called the *h-numbers* of Γ. The h-numbers are defined by

$$h_0(\Gamma) = 1, \quad h_1(\Gamma) = f_0(\Gamma) - 2, \quad h_2(\Gamma) = f_1(\Gamma) - f_0(\Gamma) + 1.$$

Notice that $f_1(\Gamma) = h_0(\Gamma) + h_1(\Gamma) + h_2(\Gamma)$ and $f_0(\Gamma) = h_1(\Gamma) + 2$, so knowing the h-numbers of Γ is equivalent to knowing the number of vertices and edges in Γ. We encode the h-numbers of Γ in a vector called the *h-vector*, which is defined as $h(\Gamma) = (h_0(\Gamma), h_1(\Gamma), h_2(\Gamma))$.

Following [Chari 1997], we will study a certain family of simple graphs known as PS[1] ear-decomposable graphs, which are defined inductively as follows.

[1]Chari chose the name "PS ear-decomposable simplicial complexes" because products of simplices and their boundaries are fundamental to the construction.

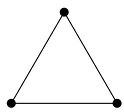
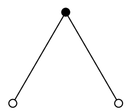
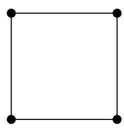
PS cycle	h-vector	PS ear	h-vector contribution
	(1, 1, 1)	Type 1 	(0, 1, 1)
	(1, 2, 1)	Type 2	(0, 0, 1)

Table 1. PS cycles and ears.

A *PS cycle* is a graph that is either a 3-cycle or a 4-cycle. A *PS ear* is a graph that is either a path of length two or a path of length one (a single edge). We call these *PS ears of Type 1* and *PS ears of Type 2* respectively. The *boundary* of a PS ear is defined as the set of vertices that are only incident to a single edge. It may seem counterintuitive to define an ear of Type 1 as a path of length two and an ear of Type 2 as a path of length one, but it will be more natural to consider ears of Type 1 first in our constructions later in the paper. Table 1 illustrates all possible PS cycles and PS ears. When illustrating PS ear-decompositions of graphs, we will adopt the practice of drawing the boundary vertices of a PS ear as unfilled circles and drawing all other vertices as filled circles.

Definition 2.1 [Chari 1997, Section 3.3]. A graph Γ is *PS ear-decomposable* if it can be decomposed as a union of the form $\Gamma = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_m$, such that

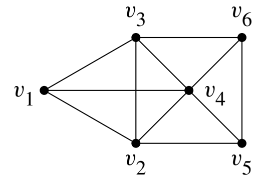
(1) Σ_0 is a PS cycle,

(2) Σ_j is a PS ear for all $0 < j \leq m$, and

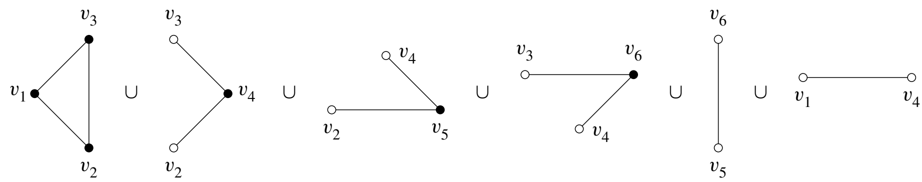
(3) the intersection $\Sigma_j \cap \bigcup_{i<j} \Sigma_i$ consists precisely of the boundary vertices of Σ_j for all $0 < j \leq m$.

One advantage to studying PS ear-decomposable graphs is that their h-vectors can also be computed inductively in terms of the ears of the decomposition. Specifically, adding an ear of Type 1 adds one vertex and two new edges to the graph, so it contributes (0, 1, 1) to the h-vector. Similarly, adding an ear of Type 2 adds one edge and zero vertices to the graph, so it contributes (0, 0, 1) to the h-vector.

Example 2.2. Consider the graph Γ on the top of page 746.



We exhibit the PS ear-decomposition of Γ.



Since Γ has 6 vertices and 11 edges, we can directly compute $h(\Gamma) = (1, 4, 6)$. We can also compute $h(\Gamma)$ in terms of the given PS ear-decomposition:

$$h(\Gamma) = (1, 1, 1) + (0, 1, 1) + (0, 1, 1) + (0, 1, 1) + (0, 0, 1) + (0, 0, 1) = (1, 4, 6).$$

We note that not all graphs are PS ear-decomposable (e.g., a tree or a graph containing an induced cycle of length at least five), and some graphs may admit several combinatorially distinct PS ear-decompositions. The family of graphs that are matroid simplicial complexes is precisely the family of complete multipartite graphs, while the family of PS ear-decomposable graphs is larger, as is exhibited in Example 2.2. Furthermore, any PS ear-decomposable graph is 2-connected, and a classical theorem of Whitney [1932, Theorem 19] states that any 2-connected graph admits an *ear-decomposition*. This definition extends that of a PS ear-decomposition by allowing one to begin with a cycle of arbitrary length (not just a 3-cycle or 4-cycle) and inductively attach paths of arbitrary length (not just paths of length one or two) along their boundary vertices. Thus the family of PS ear-decomposable graphs properly contains the family of all rank-2 matroids, and is properly contained within the family of all 2-connected graphs.

2.2. ***Multicomplexes.*** A collection of monomials $\mathcal{M}$ in the variables $x_0, x_1, \ldots, x_m$ is called a *multicomplex* if, whenever $\mu \in \mathcal{M}$ and ν divides μ, then $\nu \in \mathcal{M}$ as well. The name multicomplex comes from the fact that a simplicial complex is a family of sets that is closed under inclusion, so a multicomplex is a multiset analog of a simplicial complex. We refer to [Stanley 1996, Section II.2] for more information.

We say that a multicomplex $\mathcal{M}$ has *rank* d if d is the maximal degree of any monomial in $\mathcal{M}$. A multicomplex $\mathcal{M}$ is *pure of rank* d if each monomial in $\mathcal{M}$ divides into some monomial of degree d in $\mathcal{M}$.

For a given multicomplex $\mathcal{M}$ of rank d, we gather combinatorial data on $\mathcal{M}$ in the form of the *F-vector*, written $F(\mathcal{M}) = (F_0(\mathcal{M}), F_1(\mathcal{M}), \ldots, F_d(\mathcal{M}))$, where

degree	monomials					
2	x_0^2	x_1^2	x_2^2	x_3^2	x_0x_1	x_0x_2
1	x_0	x_1	x_2	x_3		
0	1					

Table 2. A pure multicomplex with F-vector $(1, 4, 6)$.

$F_j(\mathcal{M})$ counts the number of monomials of degree j in $\mathcal{M}$. An integer vector $\mathbf{F} = (F_0, F_1, \ldots, F_d)$ is a *(pure)* $\mathbb{O}$*-sequence* if there is a (pure) multicomplex $\mathcal{M}$ such that $\mathbf{F} = F(\mathcal{M})$.

Example 2.3. The vector $\mathbf{F} = (1, 3, 1)$ is an $\mathbb{O}$-sequence, but not a pure $\mathbb{O}$-sequence. The multicomplex $\mathcal{M} = \{1, x_0, x_1, x_2, x_0x_1\}$ has F-vector $F(\mathcal{M}) = (1, 3, 1)$, but $\mathbf{F}$ is not a pure $\mathbb{O}$-sequence since a pure multicomplex with one monomial of degree two supports at most two monomials of degree one.

Example 2.4. The vector $(1, 4, 6)$ is a pure $\mathbb{O}$-sequence. Table 2 exhibits a pure multicomplex whose F-vector is $(1, 4, 6)$.

3. h-vectors of PS ear-decomposable graphs

Stanley [1977] conjectured that the h-vector of any matroid simplicial complex is a pure $\mathbb{O}$-sequence. We will not define matroid simplicial complexes or their h-vectors here, but we refer to [Stanley 1996] for further details. Chari [1997] proved that any matroid simplicial complex is PS ear-decomposable, a definition that specializes to the given Definition 2.1 for graphs. Our main contribution in this paper is to show that Stanley's conjecture continues to hold for PS ear-decomposable graphs.

Theorem 3.1. *Let Γ be a PS ear-decomposable graph on $n+3$ vertices. Then there is a pure multicomplex $\mathcal{M}$ such that $h(\Gamma) = F(\mathcal{M})$. Moreover, there is a canonical PS ear-decomposable graph $\mathcal{S}(\Gamma)$ such that*

(1) $h(\Gamma) = h(\mathcal{S}(\Gamma))$,

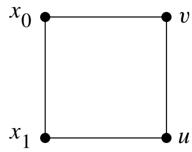
(2) *the vertices of $\mathcal{S}(\Gamma)$ are labeled as $\{u, v, x_0, x_1, \ldots, x_n\}$, and*

(3) *the multicomplex $\mathcal{M}$ arises naturally from the PS ear-decomposition of $\mathcal{S}(\Gamma)$ as a pure multicomplex on $\{x_0, x_1, \ldots, x_n\}$.*

Proof. We will prove Theorem 3.1 in two main steps. The first step is motivated by the observation that the h-vector of a PS ear-decomposable graph Γ depends only on the types of ears that are used in the PS ear-decomposition of Γ and is independent of the how these ears are attached. We begin by defining the graph $\mathcal{S}(\Gamma)$, which we call a *shifted PS ear-decomposable graph.*

Let Γ be a PS ear-decomposable graph on $n+3$ vertices with PS ear-decomposition $\Gamma = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_m$. For any $0 < j < m$, let $\Gamma_j := \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_j$. We define a new PS ear-decomposable graph $\mathcal{S}(\Gamma)$ satisfying conditions (1) and (2) of Theorem 3.1 by induction on the number of ears in the PS ear-decomposition of Γ.

If Σ_0 is a 3-cycle, we define $\mathcal{S}(\Gamma)_0$ to be a 3-cycle whose vertices are labeled u, v, and x_0. On the other hand, if Σ_0 is a 4-cycle, we define $\mathcal{S}(\Gamma)_0$ to be 4-cycle whose vertices are cyclically labeled u, v, x_0, and x_1 as follows.



For $0 < j \le m$, suppose we have inductively constructed a PS ear-decomposable graph $\mathcal{S}(\Gamma)_{j-1}$ that satisfies conditions (1) and (2) of Theorem 3.1. Suppose the vertices of $\mathcal{S}(\Gamma)_{j-1}$ are labeled as $\{u, v, x_0, x_1, \ldots, x_i\}$. If Σ_j is a PS ear of Type 1, we obtain $\mathcal{S}(\Gamma)_j$ from $\mathcal{S}(\Gamma)_{j-1}$ by adding a new vertex labeled x_{i+1} that is adjacent to vertices u and v. Otherwise, if Σ_j is a PS ear of Type 2, observe that there is a missing edge in $\mathcal{S}(\Gamma)_{j-1}$ because (i) $\mathcal{S}(\Gamma)_{j-1}$ has the same number of vertices and edges as Γ_{j-1} and (ii) Γ_j is obtained from Γ_{j-1} by adding a single edge. To form $\mathcal{S}(\Gamma)_j$, we add the lexicographically smallest missing edge to $\mathcal{S}(\Gamma)_{j-1}$ according to the alphabet order $u < v < x_0 < x_1 < \cdots < x_n$. Recall that an edge $\{a, b\}$ with $a < b$ precedes an edge $\{c, d\}$ with $c < d$ lexicographically if either $a < c$, or $a = c$ and $b < d$. By our construction it is clear that $h(\Gamma_j) = h(\mathcal{S}(\Gamma)_j)$.

In order to complete the proof of Theorem 3.1, we need to show that $h(\mathcal{S}(\Gamma))$ is a pure $\mathbb{O}$-sequence. Again, this will follow by induction on the number of ears in the PS ear-decomposition of Γ. For each $0 \le j \le m$, we will construct a pure multicomplex $\mathcal{M}_j$ such that $F(\mathcal{M}_j) = h(\mathcal{S}(\Gamma)_j)$.

We begin with the PS cycle Σ_0. If Σ_0 is a 3-cycle, then $h(\Sigma_0) = (1, 1, 1)$, which is the F-vector of the pure multicomplex $\mathcal{M}_0 = \{1, x_0, x_0^2\}$. On the other hand, if Σ_0 is a 4-cycle, then $h(\Sigma_0) = (1, 2, 1)$, which is the F-vector of the pure multicomplex $\mathcal{M}_0 = \{1, x_0, x_1, x_0 x_1\}$.

Inductively, for $0 < j \le m$, suppose we have constructed a pure multicomplex $\mathcal{M}_{j-1}$ on variables $\{x_0, \ldots, x_i\}$ such that $F(\mathcal{M}_{j-1}) = h(\mathcal{S}(\Gamma)_{j-1})$. We define a pure multicomplex $\mathcal{M}_j$ such that $F(\mathcal{M}_j) = h(\mathcal{S}(\Gamma)_j)$ as follows:

(1) If Σ_j is a PS ear of Type 1, define $\mathcal{M}_j := \mathcal{M}_{j-1} \cup \{x_{i+1}, x_{i+1}^2\}$. Clearly $F(\mathcal{M}_j) = F(\mathcal{M}_{j-1}) + (0, 1, 1)$, and hence $h(\mathcal{S}(\Gamma)_j) = F(\mathcal{M}_j)$. Moreover, it is clear that $\mathcal{M}_j$ is a pure multicomplex since $\mathcal{M}_{j-1}$ was a pure multicomplex, and we have added a new monomial of degree one and its square.

(2) If Σ_j is a PS ear of Type 2, define $\mathcal{M}_j := \mathcal{M}_{j-1} \cup \mathcal{X}$, where we define $\mathcal{X}$ according to the following rule.

(a) If the missing edge added to $\mathcal{S}(\Gamma)_{j-1}$ has the form $\{x_k, x_\ell\}$, then $\mathcal{X} := \{x_k x_\ell\}$. In this case, $\mathcal{M}_j$ is a multicomplex because the monomials of degree one that divide $x_k x_\ell$, which are x_k and x_ℓ, belong to $\mathcal{M}_{j-1}$ by construction; and $\mathcal{M}_j$ is pure because we have simply added another monomial of maximal degree.

(b) If the missing edge added to $\mathcal{S}(\Gamma)_{j-1}$ is $\{u, x_0\}$, then $\mathcal{X} := \{x_0^2\}$; if the missing edge is $\{v, x_1\}$, then $\mathcal{X} := \{x_1^2\}$. This only arises in the case that Σ_0 is a 4-cycle. The monomials x_0^2 and x_1^2 do not belong to $\mathcal{M}_0$ in this case, but their divisors, x_0 and x_1 respectively, do. Thus $\mathcal{M}_j$ is a multicomplex, and it is pure because we have only added a monomial of maximal degree to $\mathcal{M}_{j-1}$.

In either case, it is again clear that $F(\mathcal{M}_j) = F(\mathcal{M}_{j-1}) + (0, 0, 1)$ so $h(\mathcal{S}(\Gamma)_j) = F(\mathcal{M}_j)$.

This construction of the resulting pure multicomplex $\mathcal{M}$ is well-defined because we do not allow multiple edges in our graphs. In the case that Σ_0 is a 3-cycle, a monomial x_k^2 is introduced when the corresponding vertex labeled x_k is introduced, and this only happens when an ear of Type 1 is attached. Otherwise, all other monomials that are introduced have the form $x_k x_\ell$ with $k \neq \ell$, and correspond to an edge $\{x_k, x_\ell\}$ being introduced to the graph. The same argument applies when Σ_0 is a 4-cycle except that x_0^2 and x_1^2 are introduced to the multicomplex when the edges $\{v, x_0\}$ and $\{u, x_1\}$ are introduced. □

Here, we say that the graph $\mathcal{S}(\Gamma)$ is *shifted* for the following reason. At each step in the PS ear-decomposition, an ear is attached in such a way that its boundary vertices are the lexicographically smallest pair of vertices that support the required type of ear when we order the vertices $u < v < x_0 < \cdots < x_n$.

Example 3.2. Let Γ be the PS ear-decomposable graph presented in Example 2.2. The shifted PS ear-decomposable graph $\mathcal{S}(\Gamma)$ is shown in Figure 2. We exhibit the PS ear-decomposition outlined in Theorem 3.1, as well as the corresponding pure multicomplex encoded by $\mathcal{S}(\Gamma)$ in Figure 3.

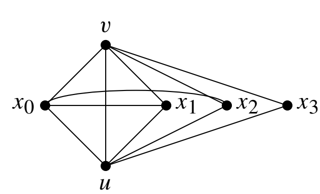


Figure 2. The shifted graph $\mathcal{S}(\Gamma)$.

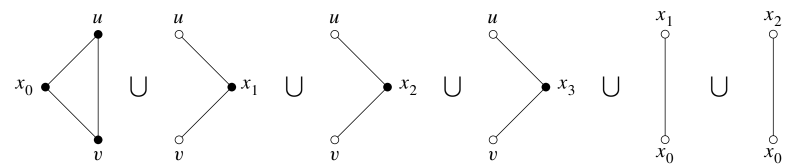
Ears	
Monomials	$\{1, x_0, x_0^2\} \cup \{x_1, x_1^2\} \cup \{x_2, x_2^2\} \cup \{x_3, x_3^2\} \cup \{x_0x_1\} \cup \{x_0x_2\}$

Figure 3. Decomposing the shifted graph $\mathscr{S}(\Gamma)$.

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