

# involve

a journal of mathematics

On commutators of matrices over unital rings

Michael Kaufman and Lillian Pasley





# On commutators of matrices over unital rings

Michael Kaufman and Lillian Pasley

(Communicated by Chi-Kwong Li)

Let  $R$  be a unital ring and let  $X \in M_n(R)$  be any upper triangular matrix of trace zero. Then there exist matrices  $A$  and  $B$  in  $M_n(R)$  such that  $X = [A, B]$ .

## 1. Introduction

Shoda [1936] proved that every matrix with trace zero over the complex numbers could be expressed as a commutator  $AB - BA$ . Albert and Muckenhoupt [1957] extended this result to matrices over any field. For matrices over commutative rings it is known that matrices of trace zero in general cannot be presented as commutators [Lissner 1961; Rosset and Rosset 2000]. Recently, Khurana and Lam [2012] showed every matrix with trace zero over any field can be expressed as a generalized commutator  $ABC - CBA$ . But the same result does not hold for matrices over commutative rings. Our work is motivated by the following question posed by Khurana and Lam: if  $n \geq 3$ , is every upper triangular matrix a generalized commutator over any ring  $S$  [Khurana and Lam 2012, Question 8.17]. In the case when  $n = 2$  this question has a negative answer as has been shown in [Khurana and Lam 2012, Theorem 8.11]. Using ideas due to Khurana and Lam we will give a simple proof of this case. We will also show that every  $n \times n$  upper triangular matrix of trace zero over any unital ring can be presented as a commutator.

## 2. Results

In this section, the trace of an  $n \times n$  matrix  $M = (x_{i,j})$  is denoted  $\text{tr}(M) = \sum_{k=1}^n x_{k,k}$ . Let  $R$  be any ring and  $S$  any commutative ring. We need some auxiliary results.

**Proposition 1** [Khurana and Lam 2012, Proposition 6.6]. *Let*

$$X = [A, B, C] = ABC - CBA,$$

where  $X, A, B, C \in M_n(S)$ . Then  $\text{tr}(BX) = 0$ .

*MSC2010:* primary 15A54; secondary 16S50.

*Keywords:* trace, matrix algebra, unital ring.

Supported in part by National Science Foundation, grant no. 1156798.

**Proposition 2** [Khurana and Lam 2012, Proposition 8.3]. *Let  $D \in R$  such that  $DC = CD \in Z(R)$  (the center of  $R$ ). If  $X = [A, B, C] \in R$ , then*

$$DX = [D, ABC] + [A, BCD] \quad \text{and} \quad XD = [D, CBA] + [A, BCD].$$

If  $X, D \in M_n(S)$ , then  $\text{tr}(XD) = \text{tr}(DX) = 0$ .

Khurana and Lam showed for  $n \geq 2$  there exist  $n \times n$  matrices that can not be expressed as generalized commutators. Now we use the preceding propositions to provide a different proof for the  $n = 2$  case.

**Theorem 3** [Khurana and Lam 2012, Theorem 8.11]. *There exists a  $2 \times 2$  upper triangular matrix that can not be expressed as a generalized commutator (i.e.,  $X \neq ABC - CBA$ ).*

*Proof.* Let  $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), A, B, C \in M_2(S)$ , where  $S = \mathbb{C}[x, y, z]$  and  $x, y$ , and  $z$  are indeterminates. Now suppose  $X \in M_2(S)$  is the upper triangular matrix  $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$  such that  $X = ABC - CBA$ .

We begin by observing that

$$BX = \begin{pmatrix} b_{11}x & b_{11}y + b_{12}z \\ b_{21}x & b_{21}y + b_{22}z \end{pmatrix}.$$

By Proposition 1,  $\text{tr}(BX) = b_{11}x + b_{21}y + b_{22}z = 0$ . This implies that polynomials  $b_{11}, b_{21}$ , and  $b_{22}$  cannot contain constant terms.

We consider the characteristic equation of  $A$ . From  $A^2 + \lambda A + \mu I = 0$  where

$$\lambda = -\text{tr}(A) = -a_{11} - a_{22} \quad \text{and} \quad \mu = \det(A) = a_{11}a_{22} - a_{12}a_{21},$$

we see that  $A(A + \lambda I) = -\mu I$ , and so  $A(A + \lambda I) \in Z(S)$ . Now we examine

$$(A + \lambda I)X = \begin{pmatrix} -a_{22}x & -a_{22}y + a_{12}z \\ a_{21}x & a_{21}y - a_{11}z \end{pmatrix}.$$

By Proposition 2,  $\text{tr}((A + \lambda I)X) = -a_{22}x + a_{21}y - a_{11}z = 0$ . This implies that polynomials  $a_{11}, a_{21}$ , and  $a_{22}$  cannot contain constant terms. Similarly, polynomials  $c_{11}, c_{21}$ , and  $c_{22}$  cannot contain constant terms. From  $X = ABC - CBA$  we obtain

$$x = a_{12}(b_{21}c_{11} + b_{22}c_{21}) + b_{12}(a_{11}c_{21} - a_{21}c_{11}) + c_{12}(-a_{11}b_{21} - a_{21}b_{22}). \quad (1)$$

Polynomials  $a_{11}, a_{21}, a_{22}, b_{11}, b_{21}, b_{22}, c_{11}, c_{21}$ , and  $c_{22}$  contain no constant terms, so the right-hand side of (1) cannot contain a linear term. Since the left-hand side of (1) is a polynomial of degree 1, namely  $x$ , we arrive at a contradiction.  $\square$

Since there exist upper triangular matrices in  $M_n(S)$  that cannot be expressed as generalized commutators, we consider what can be said about upper triangular matrices with respect to commutators.

**Theorem 4.** *Let  $R$  be a unital ring and let  $X \in M_n(R)$  be any upper triangular matrix of trace zero. Then there exist matrices  $A$  and  $B$  in  $M_n(R)$  such that  $X = [A, B]$ .*

This theorem is not true without the assumption that  $R$  is a unital ring. Let  $R$  be the ring of polynomials over  $\mathbb{C}$  with zero constant terms in variable  $x$ . Then

$$X = \begin{pmatrix} x & 0 & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x & 0 \\ 0 & 0 & \cdots & 0 & -(n-1)x \end{pmatrix}$$

is of trace zero. However, the entries of a nonzero commutator  $[A, B]$  in  $M_n(R)$  do not contain any linear terms.

*Proof of Theorem 4.* Let  $X \in M_n(R)$  be an upper triangular matrix of the form

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & x_{n-1,n-1} & x_{n-1,n} \\ 0 & \cdots & 0 & -\sum_{k=1}^{n-1} x_{k,k} \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We define the matrix  $B$  as follows: for  $1 \leq i - 1 \leq j \leq n$ , let

$$b_{ij} = \sum_{k=1}^{i-1} x_{k,j-i+k+1}.$$

All other terms of  $B$  are zero. Our goal is to show that  $X = [A, B]$ . Let  $[A, B] = (t_{i,j})$ . We want to prove  $t_{i,j} = x_{i,j}$  for  $i \geq j$ ,

$$t_{n,n} = -\sum_{k=1}^{n-1} x_{k,k},$$

and  $t_{i,j} = 0$  for  $i < j$ . We will split the proof into four cases.

Case 1. If  $i > j$ , then  $t_{i,j} = b_{i+1,j} - b_{i,j-1} = 0$ .

Case 2. If  $i = j = 1$ , then  $t_{i,j} = b_{21} = x_{11}$ .

Case 3. If  $i = j = n$ , then

$$t_{i,j} = 0 - b_{n,n-1} = 0 - \sum_{k=1}^{n-1} x_{k,k} = - \sum_{k=1}^{n-1} x_{k,k}.$$

Case 4. If  $i < j$  or  $i = j \in \{2, 3, \dots, n-1\}$ , then

$$t_{i,j} = b_{i+1,j} - b_{i,j-1} = \sum_{k=1}^i x_{k,j-i+k} - \sum_{k=1}^{i-1} x_{k,j-i+k} = x_{i,j}.$$

This completes the proof.  $\square$

This result may be used to give a proof of the well-known theorem due to Shoda [1936].

**Corollary 5.** *Let  $\mathbb{C}$  be the field of complex numbers and  $M_n(\mathbb{C})$  be the ring of  $n \times n$  matrices. Then every matrix of trace zero is a commutator.*

*Proof.* Let  $P$  be any matrix of trace zero and  $Q$  be Jordan normal form for  $P$ . So we have  $P = C^{-1}QC$  for some invertible  $C$ . Since  $P$  is upper triangular and of trace zero by Theorem 4 there exist  $A, B \in M_n(\mathbb{C})$  such that  $Q = [A, B]$ . Therefore,  $P = C^{-1}QC = C^{-1}[A, B]C = [C^{-1}AC, C^{-1}BC]$ .  $\square$

### Acknowledgments

We are grateful to our mentor Dr. Mikhail Chebotar for his guidance in writing this paper. We would also like to thank the referee for the careful reading of this paper.

### References

- [Albert and Muckenhoupt 1957] A. A. Albert and B. Muckenhoupt, “On matrices of trace zeros”, *Michigan Math. J.* **4** (1957), 1–3. MR 18,786b Zbl 0077.24304
- [Khurana and Lam 2012] D. Khurana and T. Y. Lam, “Generalized commutators in matrix rings”, *Linear Multilinear Algebra* **60**:7 (2012), 797–827. MR 2929647 Zbl 1255.15015
- [Lissner 1961] D. Lissner, “Matrices over polynomial rings”, *Trans. Amer. Math. Soc.* **98** (1961), 285–305. MR 23 #A171 Zbl 0111.01703
- [Rosset and Rosset 2000] M. Rosset and S. Rosset, “Elements of trace zero that are not commutators”, *Comm. Algebra* **28**:6 (2000), 3059–3072. MR 2001c:16055 Zbl 0954.16021
- [Shoda 1936] K. Shoda, “Einige Sätze über Matrizen”, *Jap. J. Math.* **13** (1936), 361–365. Zbl 0017.05101

Received: 2013-08-09

Revised: 2014-03-04

Accepted: 2014-03-08

mkaufma5@kent.edu

Kent State University, Kent, OH 44240, United States

lfpasley@ncsu.edu

North Carolina State University, Raleigh, NC 27695,  
United States

# involve

msp.org/involve

## EDITORS

### MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

### BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	La Trobe University, Australia P.Cerone@latrobe.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsteam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

## PRODUCTION


Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2014 is US \$120/year for the electronic version, and \$165/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2014 Mathematical Sciences Publishers

# involve

2014

vol. 7

no. 6

A median estimator for three-dimensional rotation data MELISSA BINGHAM AND ZACHARY FISCHER	713
Numerical results on existence and stability of steady state solutions for the reaction-diffusion and Klein–Gordon equations MILES ARON, PETER BOWERS, NICOLE BYER, ROBERT DECKER, ASLIHAN DEMIRKAYA AND JUN HWAN RYU	723
The $h$ -vectors of PS ear-decomposable graphs NIMA IMANI, LEE JOHNSON, MCKENZIE KEELING-GARCIA, STEVEN KLEE AND CASEY PINCKNEY	743
Zero-inflated Poisson (ZIP) distribution: parameter estimation and applications to model data from natural calamities SADIE BECKETT, JOSHUA JEE, THAPELO NCUBE, SOPHIA POMPILUS, QUINTEL WASHINGTON, ANSHUMAN SINGH AND NABENDU PAL	751
On commutators of matrices over unital rings MICHAEL KAUFMAN AND LILLIAN PASLEY	769
The nonexistence of cubic Legendre multiplier sequences TAMÁS FORGÁCS, JAMES HALEY, REBECCA MENKE AND CARLEE SIMON	773
Seating rearrangements on arbitrary graphs DARYL DEFORD	787
Fibonacci Nim and a full characterization of winning moves CODY ALLEN AND VADIM PONOMARENKO	807

