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On commutators of matrices over unital rings

Michael Kaufman and Lillian Pasley

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Let R be a unital ring and let $X \in M_n(R)$ be any upper triangular matrix of trace zero. Then there exist matrices A and B in $M_n(R)$ such that $X = [A, B]$.

1. Introduction

Shoda [1936] proved that every matrix with trace zero over the complex numbers could be expressed as a commutator $AB - BA$. Albert and Muckenhoupt [1957] extended this result to matrices over any field. For matrices over commutative rings it is known that matrices of trace zero in general cannot be presented as commutators [Lissner 1961; Rosset and Rosset 2000]. Recently, Khurana and Lam [2012] showed every matrix with trace zero over any field can be expressed as a generalized commutator $ABC - CBA$. But the same result does not hold for matrices over commutative rings. Our work is motivated by the following question posed by Khurana and Lam: if $n \geq 3$, is every upper triangular matrix a generalized commutator over any ring S [Khurana and Lam 2012, Question 8.17]. In the case when $n = 2$ this question has a negative answer as has been shown in [Khurana and Lam 2012, Theorem 8.11]. Using ideas due to Khurana and Lam we will give a simple proof of this case. We will also show that every $n \times n$ upper triangular matrix of trace zero over any unital ring can be presented as a commutator.

2. Results

In this section, the trace of an $n \times n$ matrix $M = (x_{i,j})$ is denoted $\operatorname{tr}(M) = \sum_{k=1}^{n} x_{k,k}$. Let R be any ring and S any commutative ring. We need some auxiliary results.

Proposition 1 [Khurana and Lam 2012, Proposition 6.6]. *Let*

$$X = [A, B, C] = ABC - CBA,$$

where $X, A, B, C \in M_n(S)$. *Then* $\operatorname{tr}(BX) = 0$.

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Proposition 2 [Khurana and Lam 2012, Proposition 8.3]. *Let $D \in R$ such that $DC = CD \in Z(R)$ (the center of R). If $X = [A, B, C] \in R$, then*

$$DX = [D, ABC] + [A, BCD] \quad \text{and} \quad XD = [D, CBA] + [A, BCD].$$

If $X, D \in M_n(S)$, then $\operatorname{tr}(XD) = \operatorname{tr}(DX) = 0$.

Khurana and Lam showed for $n \geq 2$ there exist $n \times n$ matrices that can not be expressed as generalized commutators. Now we use the preceding propositions to provide a different proof for the $n = 2$ case.

Theorem 3 [Khurana and Lam 2012, Theorem 8.11]. *There exists a 2×2 upper triangular matrix that can not be expressed as a generalized commutator (i.e., $X \neq ABC - CBA$).*

Proof. Let $A = (a_{ij}),\ B = (b_{ij}),\ C = (c_{ij})$, $A, B, C \in M_2(S)$, where $S = \mathbb{C}[x, y, z]$ and x, y, and z are indeterminates. Now suppose $X \in M_2(S)$ is the upper triangular matrix $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$ such that $X = ABC - CBA$.

We begin by observing that

$$BX = \begin{pmatrix} b_{11}x & b_{11}y + b_{12}z \\ b_{21}x & b_{21}y + b_{22}z \end{pmatrix}.$$

By Proposition 1, $\operatorname{tr}(BX) = b_{11}x + b_{21}y + b_{22}z = 0$. This implies that polynomials b_{11}, b_{21}, and b_{22} cannot contain constant terms.

We consider the characteristic equation of A. From $A^2 + \lambda A + \mu I = 0$ where

$$\lambda = -\operatorname{tr}(A) = -a_{11} - a_{22} \quad \text{and} \quad \mu = \det(A) = a_{11}a_{22} - a_{12}a_{21},$$

we see that $A(A + \lambda I) = -\mu I$, and so $A(A + \lambda I) \in Z(S)$. Now we examine

$$(A + \lambda I)X = \begin{pmatrix} -a_{22}x & -a_{22}y + a_{12}z \\ a_{21}x & a_{21}y - a_{11}z \end{pmatrix}.$$

By Proposition 2, $\operatorname{tr}((A + \lambda I)X) = -a_{22}x + a_{21}y - a_{11}z = 0$. This implies that polynomials a_{11}, a_{21}, and a_{22} cannot contain constant terms. Similarly, polynomials c_{11}, c_{21}, and c_{22} cannot contain constant terms. From $X = ABC - CBA$ we obtain

$$x = a_{12}(b_{21}c_{11} + b_{22}c_{21}) + b_{12}(a_{11}c_{21} - a_{21}c_{11}) + c_{12}(-a_{11}b_{21} - a_{21}b_{22}). \tag{1}$$

Polynomials a_{11}, a_{21}, a_{22}, b_{11}, b_{21}, b_{22}, c_{11}, c_{21}, and c_{22} contain no constant terms, so the right-hand side of (1) cannot contain a linear term. Since the left-hand side of (1) is a polynomial of degree 1, namely x, we arrive at a contradiction. □

Since there exist upper triangular matrices in $M_n(S)$ that cannot be expressed as generalized commutators, we consider what can be said about upper triangular matrices with respect to commutators.

Theorem 4. *Let R be a unital ring and let $X \in M_n(R)$ be any upper triangular matrix of trace zero. Then there exist matrices A and B in $M_n(R)$ such that $X = [A, B]$.*

This theorem is not true without the assumption that R is a unital ring. Let R be the ring of polynomials over $\mathbb{C}$ with zero constant terms in variable x. Then

$$X = \begin{pmatrix} x & 0 & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x & 0 \\ 0 & 0 & \cdots & 0 & -(n-1)x \end{pmatrix}$$

is of trace zero. However, the entries of a nonzero commutator $[A, B]$ in $M_n(R)$ do not contain any linear terms.

Proof of Theorem 4. Let $X \in M_n(R)$ be an upper triangular matrix of the form

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & x_{n-1,n-1} & x_{n-1,n} \\ 0 & \cdots & 0 & -\sum_{k=1}^{n-1} x_{k,k} \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We define the matrix B as follows: for $1 \le i-1 \le j \le n$, let

$$b_{ij} = \sum_{k=1}^{i-1} x_{k,j-i+k+1}.$$

All other terms of B are zero. Our goal is to show that $X = [A, B]$. Let $[A, B] = (t_{i,j})$. We want to prove $t_{i,j} = x_{i,j}$ for $i \ge j$,

$$t_{n,n} = -\sum_{k=1}^{n-1} x_{k,k},$$

and $t_{i,j} = 0$ for $i < j$. We will split the proof into four cases.

Case 1. If $i > j$, then $t_{i,j} = b_{i+1,j} - b_{i,j-1} = 0$.

Case 2. If $i = j = 1$, then $t_{i,j} = b_{21} = x_{11}$.

Case 3. If $i = j = n$, then

$$t_{i,j} = 0 - b_{n,n-1} = 0 - \sum_{k=1}^{n-1} x_{k,k} = -\sum_{k=1}^{n-1} x_{k,k}.$$

Case 4. If $i < j$ or $i = j \in \{2, 3, \cdots, n-1\}$, then

$$t_{i,j} = b_{i+1,j} - b_{i,j-1} = \sum_{k=1}^{i} x_{k,j-i+k} - \sum_{k=1}^{i-1} x_{k,j-i+k} = x_{i,j}.$$

This completes the proof. □

This result may be used to give a proof of the well-known theorem due to Shoda [1936].

Corollary 5. *Let $\mathbb{C}$ be the field of complex numbers and $M_n(\mathbb{C})$ be the ring of $n \times n$ matrices. Then every matrix of trace zero is a commutator.*

Proof. Let P be any matrix of trace zero and Q be Jordan normal form for P. So we have $P = C^{-1}QC$ for some invertible C. Since P is upper triangular and of trace zero by Theorem 4 there exist $A, B \in M_n(\mathbb{C})$ such that $Q = [A, B]$. Therefore, $P = C^{-1}QC = C^{-1}[A, B]C = [C^{-1}AC, C^{-1}BC]$. □

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