

involve

a journal of mathematics

A note on triangulations of sumsets

Károly J. Böröczky and Benjamin Hoffman



A note on triangulations of sumsets

Károly J. Böröczky and Benjamin Hoffman

(Communicated by Andrew Granville)

For finite subsets A and B of \mathbb{R}^2 , we write $A + B = \{a + b : a \in A, b \in B\}$. We write $\text{tr}(A)$ to denote the common number of triangles in any triangulation of the convex hull of A using the points of A as vertices. We consider the conjecture that $\text{tr}(A + B)^{\frac{1}{2}} \geq \text{tr}(A)^{\frac{1}{2}} + \text{tr}(B)^{\frac{1}{2}}$. If true, this conjecture would be a discrete two-dimensional analogue to the Brunn–Minkowski inequality. We prove the conjecture in three special cases.

1. Introduction

We write A, B to denote finite subsets of \mathbb{R}^d , and $|\cdot|$ to stand for their cardinality. For objects X_1, \dots, X_k in \mathbb{R}^d , $[X_1, \dots, X_k]$ denotes their convex hull. Our starting point is two classical results. One is due to Freiman from the 1960s; namely,

$$|A + B| \geq |A| + |B| - 1, \quad (1)$$

with equality if and only if A and B are arithmetic progressions of the same difference. The other result, the Brunn–Minkowski inequality, dates back to the 19th century. It says that if $X, Y \subset \mathbb{R}^d$ are compact sets, then

$$\lambda(X + Y)^{\frac{1}{d}} \geq \lambda(X)^{\frac{1}{d}} + \lambda(Y)^{\frac{1}{d}},$$

where λ stand for the Lebesgue measure, and equality holds if X and Y are convex homothetic sets. This theorem has been successfully applied to estimating the size of a sumset, for example by Ruzsa, Green, and Tao. In turn, various discrete analogues of the Brunn–Minkowski inequality have been established in papers by Bollobás and Leader, Gardner and Gronchi, Green and Tao and, most recently, by Gryniewicz and Serra in the planar case. All these papers use the method of compression, which changes a finite set into a set better suited for sumset estimates, but which cannot control the convex hull. See [Freiman 1973; 2002] for the earlier history, and [Ruzsa 2009] and [Tao and Vu 2006] for thorough surveys.

MSC2010: 11B75, 52C05.

Keywords: additive combinatorics, sumsets, Brunn–Minkowski inequality, triangulations.

Böröczky is supported by OTKA 109789.

Unfortunately the known analogues are not as simple in their form as the original Brunn–Minkowski inequality. A formula due to Gardner and Gronchi says that if A is not contained in any affine subspace of \mathbb{R}^d , then

$$|A + B| \geq (d!)^{-\frac{1}{d}} (|A| - d)^{\frac{1}{d}} + |B|^{\frac{1}{d}}.$$

In this paper, we discuss a more direct version of the Brunn–Minkowski inequality in the plane, which would improve Freiman’s inequality if both A and B are two-dimensional.

In the planar case ($d = 2$), a recent conjecture by Matolcsi and Ruzsa (personal communication, 2009) might point to the right version of the Brunn–Minkowski inequality. Let A be a finite noncollinear point set in \mathbb{R}^2 . We write $\text{tr } A$ to denote the common number of triangles in any triangulation of $[A]$ using the points of A as vertices. If b_A and i_A denote the number of points of A in $\partial[A]$ and $\text{int}[A]$, then the Euler formula yields

$$\text{tr } A = b_A + 2i_A - 2. \quad (2)$$

If Π is a polygon with vertices in \mathbb{Z}^2 , and $A = \mathbb{Z}^2 \cap \Pi$, then Pick’s theorem says that

$$\text{tr } A = 2\lambda(\Pi).$$

Now the Ruzsa–Matolcsi conjecture proposes that if A and B in the plane are not collinear, then

$$\text{tr}(A + B)^{\frac{1}{2}} \geq \text{tr}(A)^{\frac{1}{2}} + \text{tr}(B)^{\frac{1}{2}}. \quad (3)$$

We note that equality holds if for a polygon Π whose vertices are in \mathbb{Z}^2 and integers $k, m \geq 1$, we have $A = \mathbb{Z}^2 \cap k\Pi$ and $B = \mathbb{Z}^2 \cap m\Pi$.

In this paper, we verify (3) in some special cases. To present our main idea we note that if $\alpha, \beta > 0$, then

$$(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2), \quad (4)$$

with equality if and only if $\alpha = \beta$. Thus conjecture (3) follows from

$$\text{tr}(A + B) \geq 2[\text{tr } A + \text{tr } B]. \quad (5)$$

This inequality does not hold in general. For example, let Π be a polygon with vertices in \mathbb{Z}^2 , and let $A = \mathbb{Z}^2 \cap k\Pi$ and $B = \mathbb{Z}^2 \cap m\Pi$ for integers $k, m \geq 1$. If $k \neq m$, then we have equality in the Brunn–Minkowski theorem for $X = [A]$ and $Y = [B]$. Still, as we verify, (5) holds in several interesting cases.

The triangulation conjecture (3) can be written in the following form.

Conjecture 1 (main conjecture). *If A and B are finite noncollinear sets \mathbb{R}^2 , then*

$$\sqrt{2i_{A+B} + b_{A+B} - 2} \geq \sqrt{2i_A + b_A - 2} + \sqrt{2i_B + b_B - 2}.$$

In turn, (5) is equivalent with

$$2i_{A+B} + b_{A+B} \geq 4i_A + 4i_B + 2b_A + 2b_B - 6. \quad (6)$$

2. Remarks on the boundary

In the following, we need the notion of exterior normal. A vector u is an exterior normal at x_0 to $[A]$, where $x_0 \in A$, if

$$u \cdot x_0 = \max\{u \cdot x : x \in A\}.$$

It immediately follows that only points in the boundary of $[A]$ will have nonzero exterior normals. It also follows that if $a + b$ is a boundary point of $[A + B]$ for $a \in A$ and $b \in B$, then an exterior unit normal u at $a + b$ to $[A + B]$ is an exterior unit normal at a to $[A]$, and at b to $[B]$. We conclude the following:

Lemma 2. *If A and B are finite noncollinear sets in \mathbb{R}^2 , and $a \in A$ and $b \in B$, then $a + b$ lies on the boundary of $[A + B]$ with nonzero exterior unit normal vector u if and only if u is an exterior normal to $[A]$ at a and to $[B]$ at b .*

For a unit vector u , and finite set A , define the collinear set of points

$$A_u = \{x \in A : u \cdot x = \max_{y \in A}(u \cdot y)\}.$$

Lemma 3. *For any finite noncollinear sets A and B in \mathbb{R}^2 , we have*

$$b_{A+B} \geq b_A + b_B,$$

with equality if and only if the inequalities $|A_u| \geq 2$ and $|B_u| \geq 2$ for a unit vector u imply that A_u and B_u are arithmetic progressions of the same difference.

Proof. For a finite collinear set C , let $S(C) = |C| - 1$, namely, the number of segments the points of C divide the line into. Therefore if C and D are contained in parallel lines, then $S(C + D) \geq S(C) + S(D)$, with equality if and only if $|C| = 1$, $|D| = 1$, or C and D are arithmetic progressions of the same difference. Applying this observation to $C = A_u$ and $D = B_u$ for each unit vector which is an exterior normal to a side of $[A + B]$ yields the lemma. \square

3. Sums with unique representation for each point

In this section we consider the case where representation of points in $A + B$ is unique. We say that the representation is unique when for all $x \in A + B$, if $x = a_1 + b_1$ and $x = a_2 + b_2$, then $a_1 = a_2$ and $b_1 = b_2$.

Theorem 4. *If the representation of points in $A + B$ is unique, then [Conjecture 1](#) holds.*

Proof. From the previous section, we see that whether $x = a + b \in A + B$ lies on the boundary of $[A + B]$ depends only on the exterior normals of $a \in A$ and $b \in B$. So applying any transformation to A or B that preserves $|A + B|$, $\text{tr } A$, $\text{tr } B$, and the exterior normals of A and B will also preserve $\text{tr}(A + B)$. Note that scalar multiplication by ϵ , where $\epsilon A = \{\epsilon a : a \in A\}$, satisfies the latter three conditions immediately. Since the representation of points in $A + B$ is unique, picking ϵ so that the representation of points in $\epsilon A + B$ is also unique will satisfy the first condition.

We pick ϵ small enough so that, for fixed $b \in B$, letting $\epsilon A + b = \{a + b : a \in \epsilon A\}$, for any $x \in \epsilon A + B$, if $x \in [\epsilon A + b]$, then $x \in \epsilon A + b$. Geometrically, this amounts to shrinking A enough that $\epsilon A + B$ looks like a little copy of A placed at each point in B . It follows that the representation of points in $\epsilon A + B$ is unique, and hence $\text{tr}(\epsilon A + B) = \text{tr}(A + B)$.

Assume without loss of generality that $\text{tr } A = \text{tr}(\epsilon A) \geq \text{tr } B$. We begin to draw lines between points in $\epsilon A + B$ to form a partial triangulation, which can be extended to a triangulation of $\epsilon A + B$. For each $b \in B$, draw lines on $\epsilon A + b$ that form a triangulation of that set. Then, consider a triangulation T of B . For $b_1, b_2 \in B$ that are connected by a line in T , consider $\epsilon A + b_1$ and $\epsilon A + b_2$. Pick a point $b_1^* \in \epsilon A + b_1$ that has exterior normal $b_2 - b_1$ in $[\epsilon A + b_1]$. Pick a point $b_2^* \in \epsilon A + b_2$ that has exterior normal $b_1 - b_2$ in $[\epsilon A + b_2]$. Now, in $\epsilon A + B$, draw a line between b_1^* and b_2^* . Geometrically, we have mimicked a triangulation of A at each little copy of A , and a triangulation of B on a large scale, treating each little copy of A as a point in B . Letting $\text{ptr}(\epsilon A + B)$ denote the number of polygons enclosed in this partial triangulation, it follows that

$$\text{tr}(A + B) = \text{tr}(\epsilon A + B) \geq \text{ptr}(\epsilon A + B) = |B| \text{tr } A + \text{tr } B. \quad (7)$$

Conjecture 1 then follows from the inequality

$$\sqrt{|B| \text{tr } A + \text{tr } B} \geq \sqrt{\text{tr } A} + \sqrt{\text{tr } B}. \quad (8)$$

Since $|B| \geq 3$ and $\text{tr } A \geq \text{tr } B$, $(|B| - 2) \text{tr } A \geq \text{tr } B$ holds, which then implies (8). \square

4. The case $i_A = i_B = 1$

We see that **Lemma 3** yields that (6), and in turn **Conjecture 1**, would follow from

$$2i_{A+B} \geq 4i_A + 4i_B + b_A + b_B - 6, \quad (9)$$

which we have already noted does not always hold. However, in the remainder of this paper we show it holds for two special cases. The proof of the first case is simple:

Theorem 5. *When $i_A = i_B = 1$, **Conjecture 1** holds.*

Proof. From **Lemma 2**, it follows that if $a \in A_{\text{int}} = \{a \in A : a \in \text{int}[A]\}$, then $a + B \subset (A + B)_{\text{int}}$. So by (1), since i_A and i_B are nonempty, $i_{A+B} \geq i_A + |B| - 1$,

and similarly $i_{A+B} \geq i_B + |A| - 1$. Thus, since $|A| = i_A + b_A$ and $|B| = i_B + b_B$, we have

$$2i_{A+B} \geq 2i_A + 2i_B + b_A + b_B - 2. \tag{10}$$

In the case that $i_A = i_B = 1$, (9) follows. □

5. The case $|A| = b_A$ and $|B| = b_B$

We now turn to the case $|A| = b_A$ and $|B| = b_B$, or in other words, both A and B lie on the boundary of their convex hulls. In this case, (9) becomes

$$2i_{A+B} \geq b_A + b_B - 6. \tag{11}$$

The bad news is that (11) does not always hold. Let

$$\tilde{A} = \{(0, 0), (1, 0), (0, 1)\} = \{(x, y) \in \mathbb{N}^2 : x + y \leq 1\},$$

$$\tilde{B} = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)\} = \{(x, y) \in \mathbb{N}^2 : x + y \leq 2\}.$$

Therefore $|\tilde{A}| = b_{\tilde{A}} = 3$, $|\tilde{B}| = b_{\tilde{B}} = 6$, and $\tilde{A} + \tilde{B} = \{(x, y) \in \mathbb{N}^2 : x + y \leq 3\}$ yields $i_{\tilde{A}+\tilde{B}} = 1$. In particular, (11) fails to hold for \tilde{A} and \tilde{B} , but the good news is that [Conjecture 1](#) does hold for them.

We note that $\tilde{B} = \tilde{A} + \tilde{A}$. Actually, if A is any set of three noncollinear points, and $B = A + A$, then there exists a linear transformation φ such that A is a translate of $\varphi\tilde{A}$, and B is a translate of $\varphi\tilde{B}$. Therefore (11) does not hold for that A and B , as well. However, in the remainder of the paper, we prove the following theorem. From this result [Conjecture 1](#) holds for the case when $|A| = b_A$ and $|B| = b_B$.

Theorem 6. *If A and B are finite noncollinear sets in \mathbb{R}^2 such that $|A| = b_A$, $|B| = b_B$ and (11) fails to hold, then either $|A| = 3$, and B is a translate of $A + A$, or $|B| = 3$, and A is a translate of $B + B$.*

To prove [Theorem 6](#), we consider a unit vector v not parallel to any side of $[A]$ or $[B]$. We think of v as pointing vertically upwards. Let $l_{v,A}$ and $r_{v,A}$ be the leftmost and rightmost vertices of $[A]$, respectively. We note that $l_{v,A}$ and $r_{v,A}$ are unique, because v is not parallel to any side of $[A]$. Similarly, let $l_{v,B}$ and $r_{v,B}$ be the (unique) leftmost and rightmost vertices of $[B]$, respectively.

Remember that v points upwards. We observe that $l_{v,A}$ and $r_{v,A}$ divide the boundary of $[A]$ into one “upper” and one “lower” polygonal arc. Let $A_{v,\text{upp}}$ and $A_{v,\text{low}}$ denote the set of points of A in the upper and lower polygonal arcs, respectively, excluding $l_{v,A}$ and $r_{v,A}$. For $a \in A$, we have

$$a \in A_{v,\text{upp}} \text{ if and only if } u \cdot v > 0 \text{ for any unit exterior normal } u \text{ to } [A] \text{ at } a, \tag{12}$$

$$a \in A_{v,\text{low}} \text{ if and only if } u \cdot v < 0 \text{ for any unit exterior normal } u \text{ to } [A] \text{ at } a. \tag{13}$$

In addition, as $l_{v,A}$ and $r_{v,A}$ are excluded, we have

$$|A_{v,\text{upp}}| + |A_{v,\text{low}}| = b_A - 2. \quad (14)$$

Similarly, $l_{v,B}$ and $r_{v,B}$ divide the boundary of $[B]$ into an “upper” and a “lower” polygonal arc; let $B_{v,\text{upp}}$ and $B_{v,\text{low}}$ denote the set of points of B in the upper and lower polygonal arcs, respectively, excluding $l_{v,B}$ and $r_{v,B}$. For $b \in B$, we have

$$b \in B_{v,\text{upp}} \text{ if and only if } u \cdot v > 0 \text{ for any unit exterior normal } u \text{ to } [B] \text{ at } b, \quad (15)$$

$$b \in B_{v,\text{low}} \text{ if and only if } u \cdot v < 0 \text{ for any unit exterior normal } u \text{ to } [B] \text{ at } b, \quad (16)$$

$$|B_{v,\text{upp}}| + |B_{v,\text{low}}| = b_B - 2. \quad (17)$$

Lemma 7. *Let A and B be finite noncollinear sets in \mathbb{R}^2 , and let v be a unit vector not parallel to any side of $[A]$ or $[B]$. If $A_{v,\text{upp}}$, $A_{v,\text{low}}$, $B_{v,\text{upp}}$ and $B_{v,\text{low}}$ are all nonempty, then (11) holds.*

Proof. Lemma 2, (12) and (16) yield that $A_{v,\text{upp}} + B_{v,\text{low}} \subset \text{int}[A + B]$; therefore

$$i_{A+B} \geq |A_{v,\text{upp}} + B_{v,\text{low}}| \geq |A_{v,\text{upp}}| + |B_{v,\text{low}}| - 1.$$

In addition, Lemma 2, (13) and (15) yield that $A_{v,\text{low}} + B_{v,\text{upp}} \subset \text{int}[A + B]$; therefore

$$i_{A+B} \geq |A_{v,\text{low}} + B_{v,\text{upp}}| \geq |A_{v,\text{low}}| + |B_{v,\text{upp}}| - 1.$$

We deduce from (14) and (17) that

$$2i_{A+B} \geq |A_{v,\text{upp}}| + |B_{v,\text{low}}| + |A_{v,\text{low}}| + |B_{v,\text{upp}}| - 2 = b_A + b_B - 6. \quad \square$$

In other words, Lemma 7 says that if (11) does not hold, then at least one of the sets $A_{v,\text{upp}}$, $A_{v,\text{low}}$, $B_{v,\text{upp}}$ and $B_{v,\text{low}}$ is empty. We observe that replacing v by $-v$ simply exchanges $A_{v,\text{upp}}$ and $A_{v,\text{low}}$ on the one hand, and $B_{v,\text{upp}}$ and $B_{v,\text{low}}$ on the other hand. Therefore Proposition 9 will refine Lemma 7. Before that, we verify another auxiliary statement. Let $[p, q]$ denote the closed line segment with end points $p, q \in \mathbb{R}^2$.

Lemma 8. *Let A and B be finite noncollinear sets in \mathbb{R}^2 , and let v be a unit vector not parallel to any side of $[A]$ or $[B]$. If $A_{v,\text{low}} = \emptyset$, then $i_{A+B} \geq |B_{v,\text{upp}}| - 2$, where equality would imply that $B_{v,\text{low}} \subset [l_{v,B}, r_{v,B}]$, and the segments $[l_{v,A}, r_{v,A}]$ and $[l_{v,B}, r_{v,B}]$ are parallel.*

Proof. We drop the reference to v in the notation. After applying a linear transformation fixing v , we may assume that

$$w \cdot v = 0 \text{ for } w = r_A - l_A. \quad (18)$$

We may also assume that

$$l_A \cdot v = r_A \cdot v = 0. \quad (19)$$

If $r_B \cdot v > l_B \cdot v$, then we reflect both A and B through the line $\mathbb{R}v$. This keeps v , but interchanges the roles of l_A and r_A on the one hand, and the roles of l_B and r_B on the other hand. Therefore we may assume that

$$r_B \cdot v \leq l_B \cdot v. \quad (20)$$

Understanding exterior normals helps bound interior points in $[A + B]$. As A has some point above $[l_A, r_A]$ by $A_{\text{low}} = \emptyset$, (18) yields that

$$\text{either } u \cdot w > 0 \text{ or } u = -v \text{ for any exterior unit normal } u \text{ at } r_A \text{ to } [A], \quad (21)$$

$$\text{either } u \cdot w < 0 \text{ or } u = -v \text{ for any exterior unit normal } u \text{ at } l_A \text{ to } [A]. \quad (22)$$

We may assume that $B_{\text{upp}} \neq \emptyset$ (otherwise Lemma 8 trivially holds). We subdivide B_{upp} into the sets

$$B_{\text{upp}}^- = \{b \in B_{\text{upp}} : u \cdot w < 0 \text{ for any exterior unit normal } u \text{ at } b \text{ to } [B]\}, \quad (23)$$

$$B_{\text{upp}}^+ = \{b \in B_{\text{upp}} : u \cdot w > 0 \text{ for any exterior unit normal } u \text{ at } b \text{ to } [B]\}, \quad (24)$$

$$B_{\text{upp}}^0 = \{b \in B_{\text{upp}} : v \text{ is an exterior unit normal } u \text{ at } b \text{ to } [B]\}. \quad (25)$$

Since for any $b \in B$, the set of all exterior unit normals u at b to $[B]$ is an arc of the unit circle, the sets B_{upp}^- , B_{upp}^+ and B_{upp}^0 are pairwise disjoint, and their union is B_{upp} . In addition, we define

$$\tilde{B}_{\text{upp}}^- = \begin{cases} \{l_B\} \cup B_{\text{upp}}^- & \text{if there exists } b \in B \text{ with } b \cdot v < l_B \cdot v, \\ B_{\text{upp}}^- & \text{if } b \cdot v \geq l_B \cdot v \text{ for all } b \in B. \end{cases} \quad (26)$$

It follows that if $b \in \tilde{B}_{\text{upp}}^-$, then

$$\text{either } u \cdot w < 0 \text{ or } u = v \text{ for an exterior unit normal } u \text{ to } [B] \text{ at } b. \quad (27)$$

Turning to B_{upp}^0 , if $B_{\text{upp}}^0 \neq \emptyset$, then there exist $l_B^0, r_B^0 \in B_{\text{upp}}^0$ such that $r_B^0 - l_B^0 = sw$ for $s \geq 0$, and

$$B_{\text{upp}}^0 = B \cap [l_B^0, r_B^0], \quad (28)$$

$$v \cdot b_0 = \max\{v \cdot b : b \in B\} = H \text{ for all } b_0 \in B_{\text{upp}}^0. \quad (29)$$

To estimate i_{A+B} , we deduce from Lemma 2, and from (21) and (27) on the one hand, from (22) and (24) on the other hand, that

$$\begin{aligned} r_A + \tilde{B}_{\text{upp}}^- &\subset \text{int}[A + B] \text{ if } B_{\text{upp}}^- \neq \emptyset, \\ l_A + B_{\text{upp}}^+ &\subset \text{int}[A + B] \text{ if } B_{\text{upp}}^+ \neq \emptyset. \end{aligned} \quad (30)$$

We claim that if $\tilde{B}_{\text{upp}}^- \neq \emptyset$ and $B_{\text{upp}}^+ \neq \emptyset$, then

$$|(r_A + \tilde{B}_{\text{upp}}^-) \cap (l_A + B_{\text{upp}}^+)| \leq 1. \quad (31)$$

We observe that $r_A + x = l_A + y$ if and only if $y - x = w$, and hence $x \cdot v = y \cdot v$. However, if $x_1, x_2 \in \tilde{B}_{\text{upp}}^-$ and $y_1, y_2 \in B_{\text{upp}}^+$ with $x_1 \cdot v = y_1 \cdot v < x_2 \cdot v = y_2 \cdot v$, then $(y_2 - x_2) \cdot w < (y_1 - x_1) \cdot w$, which in turn yields (31). We conclude by (19), (29), (30) and (31) that

$$|\{z \in (A + B) \cap \text{int}[A + B] : z \cdot v < H\}| \geq |\tilde{B}_{\text{upp}}^-| + |B_{\text{upp}}^+| - 1. \quad (32)$$

We recall that there exists some $p \in A_{\text{upp}}$, and hence $p \cdot v > 0$ by $l_A \cdot v = 0$. Thus if $B_{\text{upp}}^0 \neq \emptyset$, and $z \in \{l_A, r_A\} + B_{\text{upp}}^0$ is different from $l_A + l_B^0$ and $r_A + r_B^0$, then these two points of $A + B$ lie left and right from z . Since $(l_A + l_B) \cdot v < z \cdot v$ and $(p + l_B^0) \cdot v > z \cdot v$, we have $z \in \text{int}[A + B]$. In particular, $|\{l_A, r_A\} + B_{\text{upp}}^0| \geq |B_{\text{upp}}^0| + 1$ yields that

$$|\{z \in (A + B) \cap \text{int}[A + B] : z \cdot v = H\}| \geq |B_{\text{upp}}^0| - 1. \quad (33)$$

Adding (32) and (33) implies $i_{A+B} \geq |B_{\text{upp}}| - 2$. If $i_{A+B} = |B_{\text{upp}}| - 2$, then $\tilde{B}_{\text{upp}}^- = B_{\text{upp}}^-$, and hence $r_B \cdot v = l_B \cdot v$ by (20) and (26), and $B_{\text{low}} \subset [l_B, r_B]$. In particular, (18) implies that $[l_{v,A}, r_{v,A}]$ and $[l_{v,B}, r_{v,B}]$ are parallel. \square

Proposition 9. *Let A and B be finite noncollinear sets in \mathbb{R}^2 , and let v be a unit vector not parallel to any side of $[A]$ or $[B]$. If (11) does not hold, then possibly after exchanging A and B , or v and $-v$, we have the following:*

- (i) $A_{v,\text{low}} = \emptyset$.
- (ii) $B_{v,\text{low}} \subset [l_{v,B}, r_{v,B}]$.
- (iii) $[l_{v,A}, r_{v,A}]$ and $[l_{v,B}, r_{v,B}]$ are parallel.
- (iv) Either $B_{v,\text{low}} = \emptyset$ and $b_B = b_A$, or $|B_{v,\text{upp}}| = |A_{v,\text{upp}}| + |B_{v,\text{low}}| + 1$ and $b_B > b_A$.

Proof. We drop the reference to v in the notation. To present the argument, we make some preparations. Again using that (11) does not hold, Lemma 7 yields that possibly after exchanging A and B , or v and $-v$, we may assume that

$$A_{\text{low}} = \emptyset.$$

Possibly after exchanging A and B again, we may assume that

$$\text{if } B_{\text{low}} = \emptyset, \text{ then } b_B \geq b_A. \quad (34)$$

Since (11) does not hold, we have

$$i_{A+B} < \frac{1}{2}(b_A + b_B) - 3. \quad (35)$$

First we show that

$$\begin{aligned} \text{either } |B_{\text{upp}}| &= \frac{b_A + b_B}{2} - 2, \quad B_{\text{low}} = \emptyset \text{ and } b_A = b_B, \\ \text{or } |B_{\text{upp}}| &> \frac{b_A + b_B}{2} - 2. \end{aligned} \tag{36}$$

If $B_{\text{low}} = \emptyset$, then $b_B \geq b_A$ by (34), and hence

$$|B_{\text{upp}}| = b_B - 2 \geq \frac{b_A + b_B}{2} - 2,$$

with equality only if $b_A = b_B$.

If $B_{\text{low}} \neq \emptyset$, then we use that $A_{\text{upp}} \neq \emptyset$ by $A_{\text{low}} = \emptyset$. Thus Lemma 2, (12) and (16) yield that $A_{\text{upp}} + B_{\text{low}}$ lies in the interior of $[A + B]$. Combining this fact with (17) leads to

$$\begin{aligned} i_{A+B} &\geq |A_{\text{upp}} + B_{\text{low}}| \geq |A_{\text{upp}}| + |B_{\text{low}}| - 1 \\ &= b_A - 2 + b_B - 2 - |B_{\text{upp}}| - 1 = b_A + b_B - |B_{\text{upp}}| - 5. \end{aligned} \tag{37}$$

Therefore

$$|B_{\text{upp}}| > \frac{b_A + b_B}{2} - 2$$

by (35), proving (36).

It follows from (35) and (36) that $i_{A+B} < |B_{\text{upp}}| - 1$; thus Lemma 8 implies that $i_{A+B} = |B_{\text{upp}}| - 2$, and in turn Proposition 9(ii) and (iii) hold. To prove (iv), we deduce from (35) that

$$b_A + b_B - 6 > 2i_{A+B} = 2|B_{\text{upp}}| - 4.$$

Therefore (36) yields that either $B_{\text{low}} = \emptyset$ and $b_A = b_B$, or

$$b_A + b_B - 4 < 2|B_{\text{upp}}| < b_A + b_B - 2.$$

In particular, $2|B_{\text{upp}}| = b_A + b_B - 3$ in the second case, which is in turn equivalent to $|B_{\text{upp}}| = |A_{\text{upp}}| + |B_{\text{low}}| + 1$ by $|A_{\text{upp}}| = b_A - 2$ and (17). In addition, $|B_{\text{upp}}| = |A_{\text{upp}}| + |B_{\text{low}}| + 1$ implies that $b_B > b_A$. \square

We have now developed enough machinery to prove Theorem 6, which we restate here:

Theorem 6. *If A and B are finite noncollinear sets in \mathbb{R}^2 such that $|A| = b_A$, $|B| = b_B$ and (11) fails to hold, then either $|A| = 3$, and B is a translate of $A + A$, or $|B| = 3$, and A is a translate of $B + B$.*

Proof. We follow Proposition 9, and choose A , B , and v as in that result. For each $x \in A$, we have that if x lies on a corner of $[A]$, there exist vectors $v_{x,l}$ and $v_{x,r}$ such that $x = l_{v_{x,l},A}$ and $x = r_{v_{x,r},A}$. Since $A_{v,\text{low}} = \emptyset$, in the first case

$r_{v_x, l, A} = r_{v, A}$, and in the second $l_{v_x, r, A} = l_{v, A}$. Consider one such $x \in A_{v, \text{upp}}$. By [Proposition 9](#), it follows that A is a subset of the triangle T_A formed by $l_{v, A}$, $r_{v, A}$, and x . And, by the same proposition, all the sides of $[B]$ must be parallel to sides in A , so B is a subset of some triangle $T_B = \phi T_A$, where ϕ is the composition of a transposition and scalar multiplication. Then the corners of $[B]$ are $l_{v, B}$, $r_{v, B}$, and some point $y \in B$. We define open line segments

$$\begin{aligned} s_1 &= (l_{v, A}, r_{v, A}), & s_2 &= (l_{v, A}, x), & s_3 &= (x, r_{v, A}), \\ t_1 &= (l_{v, B}, r_{v, B}), & t_2 &= (l_{v, B}, y), & t_3 &= (y, r_{v, B}). \end{aligned}$$

Let $A_i = s_i \cap A$ and $B_i = t_i \cap B$ for $i \in \{1, 2, 3\}$. Note that $A_1 = \emptyset$, and s_i is parallel to t_i , yet $A_i = \emptyset$ or $B_i = \emptyset$.

Assume for contradiction that $|A| > 3$. By [Proposition 9](#), $|B_{v, \text{upp}}| \geq 2$. Thus $B_i \neq \emptyset$ for one $i \in \{2, 3\}$. Assume without loss of generality that $B_3 \neq \emptyset$; then by [Proposition 9](#), $A_3 = \emptyset$ and so $A_2 \neq \emptyset$. Thus, letting $p \in A_2$, since B_1 and B_3 share no nonzero exterior normals with p , and since A_2 and $r_{v, B}$ share no nonzero exterior normals, $B_1 + p$, $A_2 + r_{v, B}$, and $B_3 + p$ are all in $(A + B)_{\text{int}}$. And since $T_B = \phi T_A$, these three sets are pairwise disjoint. So

$$i_{A+B} \geq |B_1 + p| + |A_2 + r_{v, B}| + |B_3 + p| = b_A + b_B - 6, \quad (38)$$

and thus [\(11\)](#) holds, contrary to our assumption. So $|A| = 3$.

By [Proposition 9](#), we have that if $B_{v, \text{low}} = \emptyset$, then $b_A = b_B = 3$. So, $2i_{A+B} \geq b_A + b_B - 6 = 0$, and again [\(11\)](#) holds. Thus, we have that $|B_{v, \text{low}}| \geq 1$, and so

$$|B_{v, \text{upp}}| = |B_{v, \text{low}}| + 2. \quad (39)$$

That is,

$$|B_2| + |B_3| = |B_1| + 1. \quad (40)$$

By the same argument, we get

$$|B_1| + |B_2| = |B_3| + 1, \quad (41)$$

$$|B_1| + |B_3| = |B_2| + 1. \quad (42)$$

It follows that $|B_1| = |B_2| = |B_3| = 1$ and so $b_B = 6$.

Now, $i_{A+B} > 0$, and if $i_{A+B} \geq 2$ then [\(11\)](#) holds, contradicting our assumption. Assuming then that $i_{A+B} = 1$, we let $b_i \in B_i$ for $i \in \{1, 2, 3\}$. Then we see that $x + b_1 = r_{v, A} + b_2 = l_{v, A} + b_3$. And since $T_B = \phi T_A$, B must just be a translated version of $A + A$. And, as was mentioned in the beginning of this section, [Conjecture 1](#) holds for A and B . \square

References

- [Freiman 1973] G. A. Freiman, *Foundations of a structural theory of set addition*, Translations of Math. Monographs **37**, American Mathematical Society, Providence, R. I., 1973. [MR 50 #12944](#) [Zbl 0271.10044](#)
- [Freiman 2002] G. A. Freiman, “Structure theory of set addition, II: Results and problems”, pp. 243–260 in *Paul Erdős and his mathematics, I* (Budapest, 1999), edited by M. S. Gábor Halász, László Lóvász and V. T. Sós, Bolyai Soc. Math. Stud. **11**, János Bolyai Math. Soc., Budapest, 2002. [MR 2004c:11190](#) [Zbl 1034.11056](#)
- [Ruzsa 2009] I. Z. Ruzsa, “Sumsets and structure”, pp. 87–210 in *Combinatorial number theory and additive group theory*, edited by A. Geroldinger and I. Z. Ruzsa, Birkhäuser, Basel, 2009. [MR 2010m:11013](#) [Zbl 1177.11005](#)
- [Tao and Vu 2006] T. Tao and V. Vu, *Additive combinatorics*, Cambridge Studies in Advanced Mathematics **105**, Cambridge University Press, 2006. [MR 2008a:11002](#)

Received: 2012-12-28

Revised: 2013-05-31

Accepted: 2013-09-22

carlos@renyi.hu

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda utca 13-15, Budapest 1053, Hungary

and

Central European University, 1051 Budapest, Nádor utca 9, Hungary

benjaminshoffman@gmail.com

Department of Mathematical Sciences, Lewis & Clark College, 615 Palatine Hill Road, Portland, OR 97219, United States

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	La Trobe University, Australia P.Cerone@latrobe.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tbriell@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsdam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA kgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew.andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sgupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nhritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION


Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2015 is US \$140/year for the electronic version, and \$190/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2015 Mathematical Sciences Publishers

involve

2015

vol. 8

no. 1

Efficient realization of nonzero spectra by polynomial matrices	1
NATHAN MCNEW AND NICHOLAS ORMES	
The number of convex topologies on a finite totally ordered set	25
TYLER CLARK AND TOM RICHMOND	
Nonultrametric triangles in diametral additive metric spaces	33
TIMOTHY FAVER, KATELYNN KOCHALSKI, MATHAV KISHORE MURUGAN, HEIDI VERHEGGEN, ELIZABETH WESSON AND ANTHONY WESTON	
An elementary approach to characterizing Sheffer A-type 0 orthogonal polynomial sequences	39
DANIEL J. GALIFFA AND TANYA N. RISTON	
Average reductions between random tree pairs	63
SEAN CLEARY, JOHN PASSARO AND YASSER TORUNO	
Growth functions of finitely generated algebras	71
ERIC FREDETTE, DAN KUBALA, ERIC NELSON, KELSEY WELLS AND HAROLD W. ELLINGSEN, JR.	
A note on triangulations of sumsets	75
KÁROLY J. BÖRÖCZKY AND BENJAMIN HOFFMAN	
An exploration of ideal-divisor graphs	87
MICHAEL AXTELL, JOE STICKLES, LANE BLOOME, ROB DONOVAN, PAUL MILNER, HAILEE PECK, ABIGAIL RICHARD AND TRISTAN WILLIAMS	
The failed zero forcing number of a graph	99
KATHERINE FETCIE, BONNIE JACOB AND DANIEL SAAVEDRA	
An Erdős–Ko–Rado theorem for subset partitions	119
ADAM DYCK AND KAREN MEAGHER	
Nonreal zero decreasing operators related to orthogonal polynomials	129
ANDRE BUNTON, NICOLE JACOBS, SAMANTHA JENKINS, CHARLES MCKENRY JR., ANDRZEJ PIOTROWSKI AND LOUIS SCOTT	
Path cover number, maximum nullity, and zero forcing number of oriented graphs and other simple digraphs	147
ADAM BERLINER, CORA BROWN, JOSHUA CARLSON, NATHANAEL COX, LESLIE HOGBEN, JASON HU, KATRINA JACOBS, KATHRYN MANTERNACH, TRAVIS PETERS, NATHAN WARNBERG AND MICHAEL YOUNG	
Braid computations for the crossing number of Klein links	169
MICHAEL BUSH, DANIELLE SHEPHERD, JOSEPH SMITH, SARAH SMITH-POLDERMAN, JENNIFER BOWEN AND JOHN RAMSAY	