A note on triangulations of sumsets

Károly J. Böröczky and Benjamin Hoffman

msp

A note on triangulations of sumsets

Károly J. Böröczky and Benjamin Hoffman

(Communicated by Andrew Granville)

For finite subsets A and B of $\mathbb{R}^2$, we write $A + B = \{a + b : a \in A, b \in B\}$. We write tr$(A)$ to denote the common number of triangles in any triangulation of the convex hull of A using the points of A as vertices. We consider the conjecture that tr$(A + B)^{\frac{1}{2}} \geq$ tr$(A)^{\frac{1}{2}} +$ tr$(B)^{\frac{1}{2}}$. If true, this conjecture would be a discrete two-dimensional analogue to the Brunn–Minkowski inequality. We prove the conjecture in three special cases.

1. Introduction

We write A, B to denote finite subsets of $\mathbb{R}^d$, and $|\cdot|$ to stand for their cardinality. For objects $X_1, \ldots, X_k$ in $\mathbb{R}^d$, $[X_1, \ldots, X_k]$ denotes their convex hull. Our starting point is two classical results. One is due to Freiman from the 1960s; namely,

$$|A + B| \geq |A| + |B| - 1, \tag{1}$$

with equality if and only if A and B are arithmetic progressions of the same difference. The other result, the Brunn–Minkowski inequality, dates back to the 19th century. It says that if $X, Y \subset \mathbb{R}^d$ are compact sets, then

$$\lambda(X + Y)^{\frac{1}{d}} \geq \lambda(X)^{\frac{1}{d}} + \lambda(Y)^{\frac{1}{d}},$$

where λ stand for the Lebesgue measure, and equality holds if X and Y are convex homothetic sets. This theorem has been successfully applied to estimating the size of a sumset, for example by Ruzsa, Green, and Tao. In turn, various discrete analogues of the Brunn–Minkowski inequality have been established in papers by Bollobás and Leader, Gardner and Gronchi, Green and Tao and, most recently, by Grynkiewicz and Serra in the planar case. All these papers use the method of compression, which changes a finite set into a set better suited for sumset estimates, but which cannot control the convex hull. See [Freiman 1973; 2002] for the earlier history, and [Ruzsa 2009] and [Tao and Vu 2006] for thorough surveys.

Unfortunately the known analogues are not as simple in their form as the original Brunn–Minkowski inequality. A formula due to Gardner and Gronchi says that if A is not contained in any affine subspace of $\mathbb{R}^d$, then

$$|A + B| \geq (d!)^{-\frac{1}{d}} (|A| - d)^{\frac{1}{d}} + |B|^{\frac{1}{d}}.$$

In this paper, we discuss a more direct version of the Brunn–Minkowski inequality in the plane, which would improve Freiman's inequality if both A and B are two-dimensional.

In the planar case ($d = 2$), a recent conjecture by Matolcsi and Ruzsa (personal communication, 2009) might point to the right version of the Brunn–Minkowski inequality. Let A be a finite noncollinear point set in $\mathbb{R}^2$. We write tr A to denote the common number of triangles in any triangulation of $[A]$ using the points of A as vertices. If b_A and i_A denote the number of points of A in $\partial[A]$ and $\text{int}[A]$, then the Euler formula yields

$$\text{tr}\, A = b_A + 2i_A - 2. \tag{2}$$

If Π is a polygon with vertices in $\mathbb{Z}^2$, and $A = \mathbb{Z}^2 \cap \Pi$, then Pick's theorem says that

$$\text{tr}\, A = 2\lambda(\Pi).$$

Now the Ruzsa–Matolcsi conjecture proposes that if A and B in the plane are not collinear, then

$$\text{tr}(A + B)^{\frac{1}{2}} \geq \text{tr}(A)^{\frac{1}{2}} + \text{tr}(B)^{\frac{1}{2}}. \tag{3}$$

We note that equality holds if for a polygon Π whose vertices are in $\mathbb{Z}^2$ and integers $k, m \geq 1$, we have $A = \mathbb{Z}^2 \cap k\Pi$ and $B = \mathbb{Z}^2 \cap m\Pi$.

In this paper, we verify (3) in some special cases. To present our main idea we note that if $\alpha, \beta > 0$, then

$$(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2), \tag{4}$$

with equality if and only if $\alpha = \beta$. Thus conjecture (3) follows from

$$\text{tr}(A + B) \geq 2[\text{tr}\, A + \text{tr}\, B]. \tag{5}$$

This inequality does not hold in general. For example, let Π be a polygon with vertices in $\mathbb{Z}^2$, and let $A = \mathbb{Z}^2 \cap k\Pi$ and $B = \mathbb{Z}^2 \cap m\Pi$ for integers $k, m \geq 1$. If $k \neq m$, then we have equality in the Brunn–Minkowski theorem for $X = [A]$ and $Y = [B]$. Still, as we verify, (5) holds in several interesting cases.

The triangulation conjecture (3) can be written in the following form.

Conjecture 1 (main conjecture). *If A and B are finite noncollinear sets $\mathbb{R}^2$, then*

$$\sqrt{2i_{A+B} + b_{A+B} - 2} \geq \sqrt{2i_A + b_A - 2} + \sqrt{2i_B + b_B - 2}.$$

In turn, (5) is equivalent with

$$2i_{A+B} + b_{A+B} \geq 4i_A + 4i_B + 2b_A + 2b_B - 6. \tag{6}$$

2. Remarks on the boundary

In the following, we need the notion of exterior normal. A vector u is an exterior normal at x_0 to $[A]$, where $x_0 \in A$, if

$$u \cdot x_0 = \max\{u \cdot x : x \in A\}.$$

It immediately follows that only points in the boundary of $[A]$ will have nonzero exterior normals. It also follows that if $a + b$ is a boundary point of $[A + B]$ for $a \in A$ and $b \in B$, then an exterior unit normal u at $a + b$ to $[A + B]$ is an exterior unit normal at a to $[A]$, and at b to $[B]$. We conclude the following:

Lemma 2. *If A and B are finite noncollinear sets in $\mathbb{R}^2$, and $a \in A$ and $b \in B$, then $a + b$ lies on the boundary of $[A + B]$ with nonzero exterior unit normal vector u if and only if u is an exterior normal to $[A]$ at a and to $[B]$ at b.*

For a unit vector u, and finite set A, define the collinear set of points

$$A_u = \{x \in A : u \cdot x = \max_{y \in A}(u \cdot y)\}.$$

Lemma 3. *For any finite noncollinear sets A and B in $\mathbb{R}^2$, we have*

$$b_{A+B} \geq b_A + b_B,$$

with equality if and only if the inequalities $|A_u| \geq 2$ and $|B_u| \geq 2$ for a unit vector u imply that A_u and B_u are arithmetic progressions of the same difference.

Proof. For a finite collinear set C, let $S(C) = |C| - 1$, namely, the number of segments the points of C divide the line into. Therefore if C and D are contained in parallel lines, then $S(C + D) \geq S(C) + S(D)$, with equality if and only if $|C| = 1$, $|D| = 1$, or C and D are arithmetic progressions of the same difference. Applying this observation to $C = A_u$ and $D = B_u$ for each unit vector which is an exterior normal to a side of $[A + B]$ yields the lemma. $\square$

3. Sums with unique representation for each point

In this section we consider the case where representation of points in $A + B$ is unique. We say that the representation is unique when for all $x \in A + B$, if $x = a_1 + b_1$ and $x = a_2 + b_2$, then $a_1 = a_2$ and $b_1 = b_2$.

Theorem 4. *If the representation of points in $A + B$ is unique, then* Conjecture 1 *holds.*

Proof. From the previous section, we see that whether $x = a + b \in A + B$ lies on the boundary of $[A + B]$ depends only on the exterior normals of $a \in A$ and $b \in B$. So applying any transformation to A or B that preserves $|A + B|$, tr A, tr B, and the exterior normals of A and B will also preserve tr$(A + B)$. Note that scalar multiplication by ϵ, where $\epsilon A = \{\epsilon a : a \in A\}$, satisfies the latter three conditions immediately. Since the representation of points in $A + B$ is unique, picking ϵ so that the representation of points in $\epsilon A + B$ is also unique will satisfy the first condition.

We pick ϵ small enough so that, for fixed $b \in B$, letting $\epsilon A + b = \{a + b : a \in \epsilon A\}$, for any $x \in \epsilon A + B$, if $x \in [\epsilon A + b]$, then $x \in \epsilon A + b$. Geometrically, this amounts to shrinking A enough that $\epsilon A + B$ looks like a little copy of A placed at each point in B. It follows that the representation of points in $\epsilon A + B$ is unique, and hence tr$(\epsilon A + B) = $ tr$(A + B)$.

Assume without loss of generality that tr $A = $ tr$(\epsilon A) \geq $ tr B. We begin to draw lines between points in $\epsilon A + B$ to form a partial triangulation, which can be extended to a triangulation of $\epsilon A + B$. For each $b \in B$, draw lines on $\epsilon A + b$ that form a triangulation of that set. Then, consider a triangulation T of B. For $b_1, b_2 \in B$ that are connected by a line in T, consider $\epsilon A + b_1$ and $\epsilon A + b_2$. Pick a point $b_1^* \in \epsilon A + b_1$ that has exterior normal $b_2 - b_1$ in $[\epsilon A + b_1]$. Pick a point $b_2^* \in \epsilon A + b_2$ that has exterior normal $b_1 - b_2$ in $[\epsilon A + b_2]$. Now, in $\epsilon A + B$, draw a line between b_1^* and b_2^*. Geometrically, we have mimicked a triangulation of A at each little copy of A, and a triangulation of B on a large scale, treating each little copy of A as a point in B. Letting ptr$(\epsilon A + B)$ denote the number of polygons enclosed in this partial triangulation, it follows that

$$\text{tr}(A + B) = \text{tr}(\epsilon A + B) \geq \text{ptr}(\epsilon A + B) = |B| \text{ tr } A + \text{tr } B. \qquad (7)$$

Conjecture 1 then follows from the inequality

$$\sqrt{|B| \text{ tr } A + \text{tr } B} \geq \sqrt{\text{tr } A} + \sqrt{\text{tr } B}. \qquad (8)$$

Since $|B| \geq 3$ and tr $A \geq $ tr B, $(|B| - 2)$ tr $A \geq $ tr B holds, which then implies (8). $\square$

4. The case $i_A = i_B = 1$

We see that Lemma 3 yields that (6), and in turn Conjecture 1, would follow from

$$2i_{A+B} \geq 4i_A + 4i_B + b_A + b_B - 6, \qquad (9)$$

which we have already noted does not always hold. However, in the remainder of this paper we show it holds for two special cases. The proof of the first case is simple:

Theorem 5. *When $i_A = i_B = 1$, Conjecture 1 holds.*

Proof. From Lemma 2, it follows that if $a \in A_{\text{int}} = \{a \in A : a \in \text{int}[A]\}$, then $a + B \subset (A + B)_{\text{int}}$. So by (1), since i_A and i_B are nonempty, $i_{A+B} \geq i_A + |B| - 1$,

and similarly $i_{A+B} \geq i_B + |A| - 1$. Thus, since $|A| = i_A + b_A$ and $|B| = i_B + b_B$, we have

$$2i_{A+B} \geq 2i_A + 2i_B + b_A + b_B - 2. \tag{10}$$

In the case that $i_A = i_B = 1$, (9) follows. $\qquad\square$

5. The case $|A| = b_A$ and $|B| = b_B$

We now turn to the case $|A| = b_A$ and $|B| = b_B$, or in other words, both A and B lie on the boundary of their convex hulls. In this case, (9) becomes

$$2i_{A+B} \geq b_A + b_B - 6. \tag{11}$$

The bad news is that (11) does not always hold. Let

$$\widetilde{A} = \{(0,0), (1,0), (0,1)\} = \{(x, y) \in \mathbb{N}^2 : x + y \leq 1\},$$
$$\widetilde{B} = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\} = \{(x, y) \in \mathbb{N}^2 : x + y \leq 2\}.$$

Therefore $|\widetilde{A}| = b_{\widetilde{A}} = 3$, $|\widetilde{B}| = b_{\widetilde{B}} = 6$, and $\widetilde{A} + \widetilde{B} = \{(x, y) \in \mathbb{N}^2 : x + y \leq 3\}$ yields $i_{\widetilde{A}+\widetilde{B}} = 1$. In particular, (11) fails to hold for $\widetilde{A}$ and $\widetilde{B}$, but the good news is that Conjecture 1 does hold for them.

We note that $\widetilde{B} = \widetilde{A} + \widetilde{A}$. Actually, if A is any set of three noncollinear points, and $B = A + A$, then there exists a linear transformation φ such that A is a translate of $\varphi \widetilde{A}$, and B is a translate of $\varphi \widetilde{B}$. Therefore (11) does not hold for that A and B, as well. However, in the remainder of the paper, we prove the following theorem. From this result Conjecture 1 holds for the case when $|A| = b_A$ and $|B| = b_B$.

Theorem 6. *If A and B are finite noncollinear sets in $\mathbb{R}^2$ such that $|A| = b_A$, $|B| = b_B$ and (11) fails to hold, then either $|A| = 3$, and B is a translate of $A + A$, or $|B| = 3$, and A is a translate of $B + B$.*

To prove Theorem 6, we consider a unit vector v not parallel to any side of $[A]$ or $[B]$. We think of v as pointing vertically upwards. Let $l_{v,A}$ and $r_{v,A}$ be the leftmost and rightmost vertices of $[A]$, respectively. We note that $l_{v,A}$ and $r_{v,A}$ are unique, because v is not parallel to any side of $[A]$. Similarly, let $l_{v,B}$ and $r_{v,B}$ be the (unique) leftmost and rightmost vertices of $[B]$, respectively.

Remember that v points upwards. We observe that $l_{v,A}$ and $r_{v,A}$ divide the boundary of $[A]$ into one "upper" and one "lower" polygonal arc. Let $A_{v,\text{upp}}$ and $A_{v,\text{low}}$ denote the set of points of A in the upper and lower polygonal arcs, respectively, excluding $l_{v,A}$ and $r_{v,A}$. For $a \in A$, we have

$a \in A_{v,\text{upp}}$ if and only if $u \cdot v > 0$ for any unit exterior normal u to $[A]$ at a, (12)

$a \in A_{v,\text{low}}$ if and only if $u \cdot v < 0$ for any unit exterior normal u to $[A]$ at a. (13)

In addition, as $l_{v,A}$ and $r_{v,A}$ are excluded, we have

$$|A_{v,\mathrm{upp}}| + |A_{v,\mathrm{low}}| = b_A - 2. \tag{14}$$

Similarly, $l_{v,B}$ and $r_{v,B}$ divide the boundary of $[B]$ into an "upper" and a "lower" polygonal arc; let $B_{v,\mathrm{upp}}$ and $B_{v,\mathrm{low}}$ denote the set of points of B in the upper and lower polygonal arcs, respectively, excluding $l_{v,B}$ and $r_{v,B}$. For $b \in B$, we have

$b \in B_{v,\mathrm{upp}}$ if and only if $u \cdot v > 0$ for any unit exterior normal u to $[B]$ at b, (15)

$b \in B_{v,\mathrm{low}}$ if and only if $u \cdot v < 0$ for any unit exterior normal u to $[B]$ at b, (16)

$$|B_{v,\mathrm{upp}}| + |B_{v,\mathrm{low}}| = b_B - 2. \tag{17}$$

Lemma 7. *Let A and B be finite noncollinear sets in $\mathbb{R}^2$, and let v be a unit vector not parallel to any side of $[A]$ or $[B]$. If $A_{v,\mathrm{upp}}$, $A_{v,\mathrm{low}}$, $B_{v,\mathrm{upp}}$ and $B_{v,\mathrm{low}}$ are all nonempty, then (11) holds.*

Proof. Lemma 2, (12) and (16) yield that $A_{v,\mathrm{upp}} + B_{v,\mathrm{low}} \subset \mathrm{int}[A + B]$; therefore

$$i_{A+B} \geq |A_{v,\mathrm{upp}} + B_{v,\mathrm{low}}| \geq |A_{v,\mathrm{upp}}| + |B_{v,\mathrm{low}}| - 1.$$

In addition, Lemma 2, (13) and (15) yield that $A_{v,\mathrm{low}} + B_{v,\mathrm{upp}} \subset \mathrm{int}[A+B]$; therefore

$$i_{A+B} \geq |A_{v,\mathrm{low}} + B_{v,\mathrm{upp}}| \geq |A_{v,\mathrm{low}}| + |B_{v,\mathrm{upp}}| - 1.$$

We deduce from (14) and (17) that

$$2i_{A+B} \geq |A_{v,\mathrm{upp}}| + |B_{v,\mathrm{low}}| + |A_{v,\mathrm{low}}| + |B_{v,\mathrm{upp}}| - 2 = b_A + b_B - 6. \qquad \square$$

In other words, Lemma 7 says that if (11) does not hold, then at least one of the sets $A_{v,\mathrm{upp}}$, $A_{v,\mathrm{low}}$, $B_{v,\mathrm{upp}}$ and $B_{v,\mathrm{low}}$ is empty. We observe that replacing v by $-v$ simply exchanges $A_{v,\mathrm{upp}}$ and $A_{v,\mathrm{low}}$ on the one hand, and $B_{v,\mathrm{upp}}$ and $B_{v,\mathrm{low}}$ on the other hand. Therefore Proposition 9 will refine Lemma 7. Before that, we verify another auxiliary statement. Let $[p, q]$ denote the closed line segment with end points $p, q \in \mathbb{R}^2$.

Lemma 8. *Let A and B be finite noncollinear sets in $\mathbb{R}^2$, and let v be a unit vector not parallel to any side of $[A]$ or $[B]$. If $A_{v,\mathrm{low}} = \varnothing$, then $i_{A+B} \geq |B_{v,\mathrm{upp}}| - 2$, where equality would imply that $B_{v,\mathrm{low}} \subset [l_{v,B}, r_{v,B}]$, and the segments $[l_{v,A}, r_{v,A}]$ and $[l_{v,B}, r_{v,B}]$ are parallel.*

Proof. We drop the reference to v in the notation. After applying a linear transformation fixing v, we may assume that

$$w \cdot v = 0 \quad \text{for } w = r_A - l_A. \tag{18}$$

We may also assume that

$$l_A \cdot v = r_A \cdot v = 0. \tag{19}$$

If $r_B \cdot v > l_B \cdot v$, then we reflect both A and B through the line $\mathbb{R}v$. This keeps v, but interchanges the roles of l_A and r_A on the one hand, and the roles of l_B and r_B on the other hand. Therefore we may assume that

$$r_B \cdot v \le l_B \cdot v. \tag{20}$$

Understanding exterior normals helps bound interior points in $[A + B]$. As A has some point above $[l_A, r_A]$ by $A_{\mathrm{low}} = \varnothing$, (18) yields that

either $u \cdot w > 0$ or $u = -v$ for any exterior unit normal u at r_A to $[A]$, (21)

either $u \cdot w < 0$ or $u = -v$ for any exterior unit normal u at l_A to $[A]$. (22)

We may assume that $B_{\mathrm{upp}} \ne \varnothing$ (otherwise Lemma 8 trivially holds). We subdivide B_{upp} into the sets

$$B_{\mathrm{upp}}^- = \{b \in B_{\mathrm{upp}} : u \cdot w < 0 \text{ for any exterior unit normal } u \text{ at } b \text{ to } [B]\}, \tag{23}$$

$$B_{\mathrm{upp}}^+ = \{b \in B_{\mathrm{upp}} : u \cdot w > 0 \text{ for any exterior unit normal } u \text{ at } b \text{ to } [B]\}, \tag{24}$$

$$B_{\mathrm{upp}}^0 = \{b \in B_{\mathrm{upp}} : v \text{ is an exterior unit normal } u \text{ at } b \text{ to } [B]\}. \tag{25}$$

Since for any $b \in B$, the set of all exterior unit normals u at b to $[B]$ is an arc of the unit circle, the sets B_{upp}^-, B_{upp}^+ and B_{upp}^0 are pairwise disjoint, and their union is B_{upp}. In addition, we define

$$\widetilde{B}_{\mathrm{upp}}^- = \begin{cases} \{l_B\} \cup B_{\mathrm{upp}}^- & \text{if there exists } b \in B \text{ with } b \cdot v < l_B \cdot v, \\ B_{\mathrm{upp}}^- & \text{if } b \cdot v \ge l_B \cdot v \text{ for all } b \in B. \end{cases} \tag{26}$$

It follows that if $b \in \widetilde{B}_{\mathrm{upp}}^-$, then

either $u \cdot w < 0$ or $u = v$ for an exterior unit normal u to $[B]$ at b. (27)

Turning to B_{upp}^0, if $B_{\mathrm{upp}}^0 \ne \varnothing$, then there exist $l_B^0, r_B^0 \in B_{\mathrm{upp}}^0$ such that $r_B^0 - l_B^0 = sw$ for $s \ge 0$, and

$$B_{\mathrm{upp}}^0 = B \cap [l_B^0, r_B^0], \tag{28}$$

$$v \cdot b_0 = \max\{v \cdot b : b \in B\} = H \text{ for all } b_0 \in B_{\mathrm{upp}}^0. \tag{29}$$

To estimate i_{A+B}, we deduce from Lemma 2, and from (21) and (27) on the one hand, from (22) and (24) on the other hand, that

$$\begin{aligned} r_A + \widetilde{B}_{\mathrm{upp}}^- &\subset \mathrm{int}[A + B] \text{ if } B_{\mathrm{upp}}^- \ne \varnothing, \\ l_A + B_{\mathrm{upp}}^+ &\subset \mathrm{int}[A + B] \text{ if } B_{\mathrm{upp}}^+ \ne \varnothing. \end{aligned} \tag{30}$$

We claim that if $\widetilde{B}_{\mathrm{upp}}^- \ne \varnothing$ and $B_{\mathrm{upp}}^+ \ne \varnothing$, then

$$\left| (r_A + \widetilde{B}_{\mathrm{upp}}^-) \cap (l_A + B_{\mathrm{upp}}^+) \right| \le 1. \tag{31}$$

We observe that $r_A + x = l_A + y$ if and only if $y - x = w$, and hence $x \cdot v = y \cdot v$. However, if $x_1, x_2 \in \widetilde{B}^-_{\text{upp}}$ and $y_1, y_2 \in B^+_{\text{upp}}$ with $x_1 \cdot v = y_1 \cdot v < x_2 \cdot v = y_2 \cdot v$, then $(y_2 - x_2) \cdot w < (y_1 - x_1) \cdot w$, which in turn yields (31). We conclude by (19), (29), (30) and (31) that

$$\left| \{z \in (A + B) \cap \text{int}[A + B] : z \cdot v < H\} \right| \geq |\widetilde{B}^-_{\text{upp}}| + |B^+_{\text{upp}}| - 1. \qquad (32)$$

We recall that there exists some $p \in A_{\text{upp}}$, and hence $p \cdot v > 0$ by $l_A \cdot v = 0$. Thus if $B^0_{\text{upp}} \neq \varnothing$, and $z \in \{l_A, r_A\} + B^0_{\text{upp}}$ is different from $l_A + l^0_B$ and $r_A + r^0_B$, then these two points of $A + B$ lie left and right from z. Since $(l_A + l_B) \cdot v < z \cdot v$ and $(p + l^0_B) \cdot v > z \cdot v$, we have $z \in \text{int}[A + B]$. In particular, $|\{l_A, r_A\} + B^0_{\text{upp}}| \geq |B^0_{\text{upp}}| + 1$ yields that

$$\left| \{z \in (A + B) \cap \text{int}[A + B] : z \cdot v = H\} \right| \geq |B^0_{\text{upp}}| - 1. \qquad (33)$$

Adding (32) and (33) implies $i_{A+B} \geq |B_{\text{upp}}| - 2$. If $i_{A+B} = |B_{\text{upp}}| - 2$, then $\widetilde{B}^-_{\text{upp}} = B^-_{\text{upp}}$, and hence $r_B \cdot v = l_B \cdot v$ by (20) and (26), and $B_{\text{low}} \subset [l_B, r_B]$. In particular, (18) implies that $[l_{v,A}, r_{v,A}]$ and $[l_{v,B}, r_{v,B}]$ are parallel. $\qquad \square$

Proposition 9. *Let A and B be finite noncollinear sets in $\mathbb{R}^2$, and let v be a unit vector not parallel to any side of $[A]$ or $[B]$. If (11) does not hold, then possibly after exchanging A and B, or v and $-v$, we have the following*:

(i) $A_{v,\text{low}} = \varnothing$.

(ii) $B_{v,\text{low}} \subset [l_{v,B}, r_{v,B}]$.

(iii) $[l_{v,A}, r_{v,A}]$ *and* $[l_{v,B}, r_{v,B}]$ *are parallel.*

(iv) *Either* $B_{v,\text{low}} = \varnothing$ *and* $b_B = b_A$, *or* $|B_{v,\text{upp}}| = |A_{v,\text{upp}}| + |B_{v,\text{low}}| + 1$ *and* $b_B > b_A$.

Proof. We drop the reference to v in the notation. To present the argument, we make some preparations. Again using that (11) does not hold, Lemma 7 yields that possibly after exchanging A and B, or v and $-v$, we may assume that

$$A_{\text{low}} = \varnothing.$$

Possibly after exchanging A and B again, we may assume that

$$\text{if } B_{\text{low}} = \varnothing, \text{ then } b_B \geq b_A. \qquad (34)$$

Since (11) does not hold, we have

$$i_{A+B} < \tfrac{1}{2}(b_A + b_B) - 3. \qquad (35)$$

First we show that

$$\text{either } |B_{\text{upp}}| = \frac{b_A + b_B}{2} - 2, \ B_{\text{low}} = \varnothing \text{ and } b_A = b_B,$$

$$\text{or } |B_{\text{upp}}| > \frac{b_A + b_B}{2} - 2. \tag{36}$$

If $B_{\text{low}} = \varnothing$, then $b_B \geq b_A$ by (34), and hence

$$|B_{\text{upp}}| = b_B - 2 \geq \frac{b_A + b_B}{2} - 2,$$

with equality only if $b_A = b_B$.

If $B_{\text{low}} \neq \varnothing$, then we use that $A_{\text{upp}} \neq \varnothing$ by $A_{\text{low}} = \varnothing$. Thus Lemma 2, (12) and (16) yield that $A_{\text{upp}} + B_{\text{low}}$ lies in the interior of $[A + B]$. Combining this fact with (17) leads to

$$i_{A+B} \geq |A_{\text{upp}} + B_{\text{low}}| \geq |A_{\text{upp}}| + |B_{\text{low}}| - 1$$
$$= b_A - 2 + b_B - 2 - |B_{\text{upp}}| - 1 = b_A + b_B - |B_{\text{upp}}| - 5. \tag{37}$$

Therefore

$$|B_{\text{upp}}| > \frac{b_A + b_B}{2} - 2$$

by (35), proving (36).

It follows from (35) and (36) that $i_{A+B} < |B_{\text{upp}}| - 1$; thus Lemma 8 implies that $i_{A+B} = |B_{\text{upp}}| - 2$, and in turn Proposition 9(ii) and (iii) hold. To prove (iv), we deduce from (35) that

$$b_A + b_B - 6 > 2i_{A+B} = 2|B_{\text{upp}}| - 4.$$

Therefore (36) yields that either $B_{\text{low}} = \varnothing$ and $b_A = b_B$, or

$$b_A + b_B - 4 < 2|B_{\text{upp}}| < b_A + b_B - 2.$$

In particular, $2|B_{\text{upp}}| = b_A + b_B - 3$ in the second case, which is in turn equivalent to $|B_{\text{upp}}| = |A_{\text{upp}}| + |B_{\text{low}}| + 1$ by $|A_{\text{upp}}| = b_A - 2$ and (17). In addition, $|B_{\text{upp}}| = |A_{\text{upp}}| + |B_{\text{low}}| + 1$ implies that $b_B > b_A$. □

We have now developed enough machinery to prove Theorem 6, which we restate here:

Theorem 6. *If A and B are finite noncollinear sets in $\mathbb{R}^2$ such that $|A| = b_A$, $|B| = b_B$ and (11) fails to hold, then either $|A| = 3$, and B is a translate of $A + A$, or $|B| = 3$, and A is a translate of $B + B$.*

Proof. We follow Proposition 9, and choose A, B, and v as in that result. For each $x \in A$, we have that if x lies on a corner of $[A]$, there exist vectors $v_{x,l}$ and $v_{x,r}$ such that $x = l_{v_{x,l},A}$ and $x = r_{v_{x,r},A}$. Since $A_{v,\text{low}} = \varnothing$, in the first case

$r_{v_{x,l},A} = r_{v,A}$, and in the second $l_{v_{x,r},A} = l_{v,A}$. Consider one such $x \in A_{v,\text{upp}}$. By Proposition 9, it follows that A is a subset of the triangle T_A formed by $l_{v,A}$, $r_{v,A}$, and x. And, by the same proposition, all the sides of $[B]$ must be parallel to sides in A, so B is a subset of some triangle $T_B = \phi T_A$, where ϕ is the composition of a transposition and scalar multiplication. Then the corners of $[B]$ are $l_{v,B}$, $r_{v,B}$, and some point $y \in B$. We define open line segments

$$s_1 = (l_{v,A}, r_{v,A}), \quad s_2 = (l_{v,A}, x), \quad s_3 = (x, r_{v,A}),$$
$$t_1 = (l_{v,B}, r_{v,B}), \quad t_2 = (l_{v,B}, y), \quad t_3 = (y, r_{v,B}).$$

Let $A_i = s_i \cap A$ and $B_i = t_i \cap B$ for $i \in \{1, 2, 3\}$. Note that $A_1 = \varnothing$, and s_i is parallel to t_i, yet $A_i = \varnothing$ or $B_i = \varnothing$.

Assume for contradiction that $|A| > 3$. By Proposition 9, $|B_{v,\text{upp}}| \geq 2$. Thus $B_i \neq \varnothing$ for one $i \in \{2, 3\}$. Assume without loss of generality that $B_3 \neq \varnothing$; then by Proposition 9, $A_3 = \varnothing$ and so $A_2 \neq \varnothing$. Thus, letting $p \in A_2$, since B_1 and B_3 share no nonzero exterior normals with p, and since A_2 and $r_{v,B}$ share no nonzero exterior normals, $B_1 + p$, $A_2 + r_{v,B}$, and $B_3 + p$ are all in $(A + B)_{\text{int}}$. And since $T_B = \phi T_A$, these three sets are pairwise disjoint. So

$$i_{A+B} \geq |B_1 + p| + |A_2 + r_{v,B}| + |B_3 + p| = b_A + b_B - 6, \tag{38}$$

and thus (11) holds, contrary to our assumption. So $|A| = 3$.

By Proposition 9, we have that if $B_{v,\text{low}} = \varnothing$, then $b_A = b_B = 3$. So, $2i_{A+B} \geq b_A + b_B - 6 = 0$, and again (11) holds. Thus, we have that $|B_{v,\text{low}}| \geq 1$, and so

$$|B_{v,\text{upp}}| = |B_{v,\text{low}}| + 2. \tag{39}$$

That is,

$$|B_2| + |B_3| = |B_1| + 1. \tag{40}$$

By the same argument, we get

$$|B_1| + |B_2| = |B_3| + 1, \tag{41}$$
$$|B_1| + |B_3| = |B_2| + 1. \tag{42}$$

It follows that $|B_1| = |B_2| = |B_3| = 1$ and so $b_B = 6$.

Now, $i_{A+B} > 0$, and if $i_{A+B} \geq 2$ then (11) holds, contradicting our assumption. Assuming then that $i_{A+B} = 1$, we let $b_i \in B_i$ for $i \in \{1, 2, 3\}$. Then we see that $x + b_1 = r_{v,A} + b_2 = l_{v,A} + b_3$. And since $T_B = \phi T_A$, B must just be a translated version of $A + A$. And, as was mentioned in the beginning of this section, Conjecture 1 holds for A and B. $\qquad\square$

References

[Freiman 1973] G. A. Freiman, *Foundations of a structural theory of set addition*, Translations of Math. Monographs **37**, American Mathematical Society, Providence, R. I., 1973. MR 50 #12944 Zbl 0271.10044

[Freiman 2002] G. A. Freiman, "Structure theory of set addition, II: Results and problems", pp. 243–260 in *Paul Erdős and his mathematics, I* (Budapest, 1999), edited by M. S. Gábor Halász, László Lóvász and V. T. Sós, Bolyai Soc. Math. Stud. **11**, János Bolyai Math. Soc., Budapest, 2002. MR 2004c:11190 Zbl 1034.11056

[Ruzsa 2009] I. Z. Ruzsa, "Sumsets and structure", pp. 87–210 in *Combinatorial number theory and additive group theory*, edited by A. Geroldinger and I. Z. Ruzsa, Birkhäuser, Basel, 2009. MR 2010m:11013 Zbl 1177.11005

[Tao and Vu 2006] T. Tao and V. Vu, *Additive combinatorics*, Cambridge Studies in Advanced Mathematics **105**, Cambridge University Press, 2006. MR 2008a:11002

carlos@renyi.hu   *Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda utca 13-15, Budapest 1053, Hungary*

and

Central European University, 1051 Budapest, Nádor utca 9, Hungary

benjaminsshoffman@gmail.com

                    Department of Mathematical Sciences, Lewis & Clark College, 615 Palatine Hill Road, Portland, OR 97219, United States

involve

msp.org/involve

involve