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An Erdős–Ko–Rado theorem for subset partitions

Adam Dyck and Karen Meagher

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A kl-subset partition, or (k, l)-*subpartition*, is a kl-subset of an n-set that is partitioned into l distinct blocks, each of size k. Two (k, l)-subpartitions are said to t-*intersect* if they have at least t blocks in common. In this paper, we prove an Erdős–Ko–Rado theorem for intersecting families of (k, l)-subpartitions. We show that for $n \geq kl$, $l \geq 2$ and $k \geq 3$, the number of (k, l)-subpartitions in the largest 1-intersecting family is at most $\binom{n-k}{k}\binom{n-2k}{k}\cdots\binom{n-(l-1)k}{k}/(l-1)!$, and that this bound is only attained by the family of (k, l)-subpartitions with a common fixed block, known as the *canonical intersecting family of* (k, l)-*subpartitions*. Further, provided that n is sufficiently large relative to k, l and t, the largest t-intersecting family is the family of (k, l)-subpartitions that contain a common set of t fixed blocks.

1. Introduction

We prove here an Erdős–Ko–Rado theorem for intersecting families of subset partitions. The EKR theorem gives the size and structure of the largest family of intersecting sets, all of the same size, from a base set. This theorem has an interesting history: Erdős [1987] wrote that the work was done in 1938, but due to lack of interest in combinatorics at the time, it wasn't until 1961 that the paper was published. Once the result did appear in the literature, it sparked a great deal of interest in extremal set theory.

To start, we must consider some relevant notation and background information. For any positive integer n, denote $[n] := \{1, \ldots, n\}$. A k-*set* is a subset of size k from $[n]$. Two k-sets A and B are said to *intersect* if $|A \cap B| \geq 1$, and for $1 \leq t \leq k$, they are said to be t-*intersecting* if $|A \cap B| \geq t$. A *canonical* t-*intersecting family of* k-*sets* is one that contains all k-sets with t fixed elements.

EKR theorem. [Erdős et al. 1961] *Let* $n \geq k \geq t \geq 1$, *and let* $\mathcal{F}$ *be a* t-*intersecting family of* k-*sets from* $[n]$. *If* n *is sufficiently large compared to* k *and* t, *then*

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$|\mathcal{F}| \leq \binom{n-t}{k-t}$; *further, equality holds if and only if* $\mathcal{F}$ *is a canonical t-intersecting family of k-sets.*

The exact bound on n is known to be $n \geq (t+1)(k-t+1)$ (an elegant proof of this that uses algebraic graph theory is given by Wilson [1984]). If n is smaller than this bound, then there are t-intersecting families that are larger than the canonical t-intersecting family. A complete characterization of the families of maximum size for all values of n is given by Ahlswede and Khachatrian [1997].

From here, many EKR-type theorems have been developed by incorporating other combinatorial objects. Frankl and Wilson [1986] have considered this theorem for vector spaces over a finite field, Rands [1982] for blocks in a design, Cameron and Ku [2003] for permutations, Ku and Leader [2006] for partial permutations, Brunk and Huczynska [2010] for injections, and Ku and Renshaw [2008] for set partitions and cycle-intersecting permutations. All of these cases consider combinatorial objects that are made up of what we shall call *atoms*, and two objects intersect if they contain a common atom and t-intersect if they contain t common atoms. To say that "an EKR-type theorem holds" means that the largest set of intersecting (or t-intersecting) objects is the set of all objects that contain a common atom (or a common t-set of atoms).

In this paper, we shall prove that an EKR-type theorem holds for an object which we call a *subset partition*. We begin by outlining the appropriate notation.

A *uniform l-partition* of $[n]$ is a division of $[n]$ into l distinct, nonempty subsets, known as *blocks*, where each block has the same size and the union of these blocks is $[n]$. Further, a *uniform kl-subset partition* P is a uniform l-partition of a subset of kl elements from $[n]$. We shall also call P a (k,l)-*subpartition*. If P is a (k,l)-subpartition of $[n]$, then $P = \{P_1, \ldots, P_l\}$ and $|P_i| = k$ for $i \in \{1, \ldots, l\}$, with $|\bigcup_{i=1}^{l} P_i| = kl$. Let $U_{l,k}^n$ denote the set of all (k,l)-subpartitions from $[n]$, and define

$$U(n,l,k) := |U_{l,k}^n| = \frac{1}{l!}\binom{n}{k}\binom{n-k}{k}\cdots\binom{n-(l-1)k}{k} = \frac{1}{l!}\prod_{i=0}^{l-1}\binom{n-ik}{k}.$$

Two (k,l)-subpartitions $P = \{P_1, \ldots, P_l\}$ and $Q = \{Q_1, \ldots, Q_l\}$ are said to be *intersecting* if $P_i = Q_j$ for some $i, j \in \{1, \ldots, l\}$. Further, for $1 \leq t \leq l$, P and Q are said to be *t-intersecting* if there is an ordering of the blocks such that $P_i = Q_i$ for $i = 1, \ldots, t$.

A *canonical t-intersecting family of* (k,l)-*subpartitions* is a family that contains every (k,l)-subpartition with a fixed set of t blocks. Such a family has size

$$U(n-tk, l-t, k) = \frac{1}{(l-t)!}\prod_{i=t}^{l-1}\binom{n-ik}{k}. \tag{*}$$

In particular, a *canonical intersecting family of* (k,l)*-subpartitions* has size

$$U(n-k, l-1, k) = \frac{1}{(l-1)!} \prod_{i=1}^{l-1} \binom{n-ik}{k}. \tag{**}$$

Finally, note that

$$U(n, l, k) = \frac{1}{l} \binom{n}{k} U(n-k, l-1, k), \tag{†}$$

and $U(n, 0, 0) = 1$ for $n \geq 0$.

We shall not consider the cases when $k = 1$, as this reduces to the original EKR theorem when $l = 1$, where intersection is trivial, or when $t = l$, where intersection is also trivial.

Theorem 1. *Let* n, k, l *be positive integers with* $n \geq kl$, $l \geq 2$, *and* $k \geq 3$. *If* $\mathcal{P}$ *is an intersecting family of* (k,l)*-subpartitions, then*

$$|\mathcal{P}| \leq \frac{1}{(l-1)!} \prod_{i=1}^{l-1} \binom{n-ik}{k}.$$

Moreover, this bound can only be attained by a canonical intersecting family of (k,l)*-subpartitions.*

Theorem 2. *Let* n, k, l, t *be positive integers with* $n \geq n_0(k, l, t)$ *and* $1 \leq t \leq l-1$. *If* $\mathcal{P}$ *is a* t*-intersecting family of* (k,l)*-subpartitions, then*

$$|\mathcal{P}| \leq \frac{1}{(l-t)!} \prod_{i=t}^{l-1} \binom{n-ik}{k}.$$

Moreover, this bound can only be attained by a canonical t*-intersecting family of* (k,l)*-subpartitions.*

Meagher and Moura [2005] introduced Erdős–Ko–Rado theorems for t-intersecting partitions, which fall under the case $n = kl$. Additionally, for the case $k = 2$ with $n > kl$, a (k,l)-subpartition is a partial matching; in their recent paper, Kamat and Misra [2013] presented the corresponding EKR theorems for these objects. They incorporate a very nice Katona-style proof, but interestingly, it does not appear that the Katona method would work very well for (k,l)-subpartitions (it seems that this proof would require an additional lower bound on n). The goal of this work is to complete the work done in both [Meagher and Moura 2005] and [Kamat and Misra 2013] by showing that an EKR-type theorem holds for subpartitions. In this paper, we specifically do not consider the case where $k = 2$ (as this is done in Kamat and Misra's work). In [Meagher and Moura 2005], the only difficult case is $k = 2$; it is possible that our counting method will work for the partial matchings if some of the tricks used in [loc. cit.] are applied.

2. Three technical lemmas

We shall require results similar to Lemma 3 in [Meagher and Moura 2005] — the proofs of which use similar counting arguments. The first of these, Lemma 3, is just the $t = 1$ case of the third, Lemma 5. We present proofs for both of these lemmas since the proof of Lemma 3 is straight-forward and presenting it first makes the proof of Lemma 5 clearer.

As we shall see, it is worthwhile to consider the size of a canonical t-intersecting family of (k, l)-subpartitions and find when this is an upper bound for the size of any t-intersecting family of (k, l)-subpartitions.

Define a *dominating set* for a family of (k, l)-subpartitions to be a set of blocks, each of size k, that intersects with every (k, l)-subpartition in the family. For the intersecting families being investigated here, each (k, l)-subpartition in the family is also a dominating set. In [Meagher and Moura 2005], dominating sets are called *blocking sets*. We use the term dominating set here because if the blocks in the (k, l)-subpartitions (the k-sets) are considered to be vertices, then each (k, l)-subpartition can be thought of as an edge in an l-uniform hypergraph on these vertices. As a result, a family of (k, l)-subpartitions is a hypergraph, and our definition of a dominating set for a family of (k, l)-subpartitions matches the definition of a dominating set for a hypergraph.

Lemma 3. *Let n, k, l be positive integers with $n \geq kl$, $l \geq 2$ and let $\mathcal{P} \subseteq U^n_{l,k}$ be an intersecting family of (k, l)-subpartitions. Assume that there does not exist a k-set that occurs as a block in every (k, l)-subpartition in $\mathcal{P}$. Then*

$$|\mathcal{P}| \leq l^2 U(n - 2k, l - 2, k). \tag{1}$$

Proof. Let $\{P_1, \dots, P_l\}$ be a (k, l)-subpartition in $\mathcal{P}$, and for $i \in \{1, \dots, l\}$, let $\mathcal{P}_i$ be the set of all (k, l)-subpartitions in $\mathcal{P}$ that contain the block P_i but none of $P_1, \dots, P_{i-1}$. By assumption, P_i does not appear in every (k, l)-subpartition in $\mathcal{P}$, so there exists some (k, l)-subpartition Q that does not contain P_i. The subpartitions in $\mathcal{P}_i$ and Q must be intersecting, so each member of $\mathcal{P}_i$ must contain P_i as well as one of the l blocks from Q. Thus, we can bound the size of $\mathcal{P}_i$ by

$$|\mathcal{P}_i| \leq l U(n - 2k, l - 2, k).$$

Further, since $\{P_1, \dots, P_l\}$ is a dominating set for the family of (k, l)-subpartitions, we have that

$$\bigcup_{i \in \{1,\dots,l\}} \mathcal{P}_i = \mathcal{P}.$$

It follows that

$$|\mathcal{P}| \leq l|\mathcal{P}_i| \leq l^2 U(n - 2k, l - 2, k). \qquad \square$$

Note that Lemma 3 certainly applies for all $n \geq kl$; however, if the size of n is small enough relative to k and l, then we can improve our bound on such an intersecting family $\mathcal{P}$. Note that in the case of $n = kl$, we may use the lemma as considered in [Meagher and Moura 2005].

Lemma 4. *Let n, k, l be positive integers with $kl + 1 \leq n \leq k(l + 1) - 1$, $l \geq 2$, and let $\mathcal{P} \subseteq U^n_{l,k}$ be an intersecting family of (k, l)-subpartitions. Assume that there does not exist a k-set that occurs as a block in every (k, l)-subpartition in $\mathcal{P}$. Then*

$$|\mathcal{P}| \leq l(l - 1)U(n - 2k, l - 2, k). \tag{2}$$

Proof. Under the restriction on the size of n, there are at most $l - 1$ blocks in Q that do not contain an element from P_i. The remainder of the proof follows similarly. □

We also adapt a similar lemma for the t-intersecting case.

Lemma 5. *Let n, k, l, t be positive integers with $1 \leq t \leq l - 1$, and let $\mathcal{P} \subseteq U^n_{l,k}$ be a t-intersecting family of (k, l)-subpartitions. Assume that there does not exist a k-set that occurs as a block in every (k, l)-subpartition in $\mathcal{P}$. Then*

$$|\mathcal{P}| \leq (l - t + 1)\binom{l}{t}U\big(n - (t + 1)k, l - (t + 1), k\big). \tag{3}$$

Proof. As in the proof of Lemma 3, let $\{P_1, \ldots, P_l\}$ be a (k, l)-subpartition in $\mathcal{P}$, and for $i \in \{1, \ldots, l\}$, define the set $\mathcal{P}_i$ similarly. Note that if we order the $\mathcal{P}_i$ sets, then any (k, l)-subpartition in $\mathcal{P}_i$ where $i > l - t + 1$ must contain at least one of the blocks $\{P_1, \ldots, P_{l-t+1}\}$ since the (k, l)-subpartitions here must be t-intersecting with $\{P_1, \ldots, P_l\}$. The block P_i does not appear in every (k, l)-subpartition in $\mathcal{P}$, so there exists some (k, l)-subpartition Q that does not contain P_i. Any (k, l)-subpartition $P \in \mathcal{P}_i$ must be t-intersecting with Q, so there are $\binom{l}{t}$ ways to choose the t blocks from Q that are also in P. Thus, we can bound the size of $\mathcal{P}_i$ by

$$|\mathcal{P}_i| \leq \binom{l}{t}U\big(n - (t + 1)k, l - (t + 1), k\big).$$

Further, since

$$\bigcup_{i \in \{1,\ldots,l-t+1\}} \mathcal{P}_i = \mathcal{P},$$

it follows that

$$|\mathcal{P}| \leq (l - t + 1)\binom{l}{t}U\big(n - (t + 1)k, l - (t + 1), k\big). \qquad \square$$

3. Proof of Theorem 1

We can use (1) or (2), based on the size of n, and compare these bounds with that of (**). Informally, we may think of these as bounds on the size of *noncanonical* families of (k, l)-subpartitions. If the size of the canonical family is larger than these bounds, then we know that the canonical families are the largest and that equality holds if and only if the intersecting family is canonical.

Proof of Theorem 1. Let $\mathcal{P}$ be a noncanonical family of intersecting (k,l)-subpartitions. We shall show that

$$|\mathcal{P}| < \frac{1}{l-1}\binom{n-k}{k}U(n-2k, l-2, k). \tag{4}$$

It can be verified from (**) and (†) that the right-hand side of this inequality is the size of a canonical intersecting family of (k,l)-subpartitions; thus, proving this inequality proves Theorem 1.

Case 1: $kl+1 \le n \le k(l+1)-1$

If we bound n as such, then by (2),

$$|\mathcal{P}| \le l(l-1)U(n-2k, l-2, k),$$

and using (4), we only need to prove that

$$l(l-1)^2 < \binom{n-k}{k}. \tag{5}$$

Since $n \ge kl+1$, and using that $k \ge 3$, by Pascal's rule,

$$\binom{n-k}{k} \ge \binom{k(l-1)+1}{k} \ge \binom{3(l-1)+1}{3} = \frac{(3l-2)(3l-3)(3l-4)}{3!}.$$

Thus, (5) can be reduced to checking the inequality

$$l(l-1)^2 < \frac{(3l-2)(3l-3)(3l-4)}{3!}.$$

It can be verified, using the increasing function test, that this holds for all $l \ge 2$.

Case 2: $n \ge k(l+1)$

Similar to the previous case, using (1) and (4), we only need to show that

$$l^2(l-1) < \binom{n-k}{k}. \tag{6}$$

As before, taking $n \ge k(l+1)$, $k \ge 3$, and using Pascal's rule, we find

$$\binom{n-k}{k} \ge \binom{kl}{k} \ge \binom{3l}{3} = \frac{3l(3l-1)(3l-2)}{3!}.$$

So, (6) can be rewritten as

$$l^2(l-1) < \frac{3l(3l-1)(3l-2)}{3!},$$

and we find that this also holds for all $l \ge 2$.

Thus, (4) holds for all values of n, completing the proof of Theorem 1. □

4. Proof of Theorem 2

Theorem 2 incorporates the t-intersection property, proving a more general EKR-type theorem for (k, l)-subpartitions. Here, the precise lower bound on n for determining when only the canonical families are the largest is unknown — but we shall see that if $k \geq t+2$, it suffices to take $n \geq k(l+t)$ (though this bound is not optimal).

Proof of Theorem 2. From (*) and (†), the size of a canonical t-intersecting family of (k, l)-subpartitions is

$$U(n-tk, l-t, k) = \frac{1}{l-t}\binom{n-tk}{k} U\big(n-(t+1)k, l-(t+1), k\big). \tag{7}$$

As before, let $\mathcal{P}$ be a noncanonical family of t-intersecting (k, l)-subpartitions. If there is a block that is contained in every (k, l)-subpartition of $\mathcal{P}$, then it can be removed from every such subpartition in $\mathcal{P}$. This does not change the size of the family, but reduces n by k and each of l and t by 1. Now we only need to show that this new family is smaller than the canonical $(t-1)$-intersecting family of $(k, l-1)$-subpartitions from $[n-k]$ (the size of which is equal to $U\big(n-(t-1)k, l-(t-1), k\big)$. As such, we may assume that there are no blocks common to every (k, l)-subpartition in $\mathcal{P}$, and we can apply (3).

To prove this theorem, we need to prove that for n sufficiently large,

$$(l-t+1)(l-t)\binom{l}{t} < \binom{n-tk}{k}. \tag{8}$$

Clearly, this inequality is strict if n is sufficiently large relative to t, l and k. □

Consider the case where $k \geq t+2$. If $n \geq k(l+t)$, then (8) holds when

$$(l-t+1)(l-t)\binom{l}{t} \leq \binom{lk}{k}.$$

Since $k \geq t+2$, we have that

$$\binom{lk}{k} = \left(\frac{lk}{k}\right)\left(\frac{lk-1}{k-2}\right)\binom{lk-2}{k-2} > (l-t+1)(l-t)\binom{l}{t},$$

so (8) holds indeed. We do not attempt to find the function $n_0(k, l, t)$ that produces the exact lower bound on n, but such a lower bound is needed, as shown by the example in [Meagher and Moura 2005, Section 5].

5. Extensions

There are versions of the EKR theorem for many different objects. In this final section, we shall outline how this method can be generalized to these different objects.

In general, when considering an EKR-type theorem, there is a set of objects with some notion of intersection. We shall consider the case when each object is comprised of k *atoms*, and two objects are intersecting if they both contain a common atom. If the objects are k-sets, then the atoms are the elements from $\{1, \dots, n\}$, and each k-set contains exactly k atoms. For matchings, the atoms are edges from the complete graph on $2n$ vertices, and a k-matching has k atoms. In this paradigm, if the largest set of intersecting objects is the set of all the objects that contain a fixed atom, then an EKR-type theorem holds.

We can apply the method in this paper to this more general situation. Assume we have a set of objects and that each object contains exactly k distinct atoms from a set of n atoms (there may be many additional rules on which sets of atoms constitute an object). Let $P(n, k)$ be the total number of objects, $P(n-1, k-1)$ the number of objects that contain a fixed atom, and $P(n-2, k-2)$ the number of objects that contain two fixed atoms.

Using the same argument as in this paper, if for some type of object (as above)

$$k^2 P(n-2, k-2) < P(n-1, k-1),$$

then an EKR-type theorem holds for these objects. It is very interesting to note that if the ratio between $P(n-1, k-1)$ and $P(n-2, k-2)$ is sufficiently large, then an EKR-type theorem holds.

For example, this can be applied to k-sets. In this case, the equation is

$$k^2 \binom{n-2}{k-2} < \binom{n-1}{k-1},$$

which holds if and only if

$$k^2(k-1) + 1 < n.$$

This proves the standard EKR theorem, but with a very bad lower bound on n.

For a second example, consider length-n integer sequences with entries from $\{0, 1, \dots, q-1\}$. In this case the atoms are ordered pairs (i, a), where the entry in position i of the sequence is a. Two sequences "intersect" if they have the same entry in the same position. Each sequence contains exactly n atoms, so in this case $k = n$. The values of $P(n-1, n-1)$ and $P(n-2, n-2)$ are q^{n-1} and q^{n-2}, respectively. Thus an EKR-type theorem for integer sequences holds if $n^2 q^{n-2} < q^{n-1}$, or equivalently if $n^2 < q$. Once again we have a simple proof of an EKR-type theorem, but with an unnecessary bound on n.

Finally, consider the blocks in a t-(n, m, λ) design. The blocks are m-sets, so they are t-intersecting if they contain a common set of t-elements. It is straight-forward to calculate the number of blocks that contain any s-set where $s \leq t$ is

$$\lambda \frac{\binom{n-s}{t-s}}{\binom{m-s}{t-s}}.$$

Thus we have that the EKR theorem holds for intersecting blocks in a t-(n, m, λ)

design if

$$m^2 \frac{\lambda\binom{n}{t}\binom{m}{2}}{\binom{m}{t}\binom{n}{2}} \le \frac{\lambda\binom{n}{t}\binom{m}{1}}{\binom{m}{t}\binom{n}{1}},$$

which reduces to

$$m^3 - m^2 + 1 < n.$$

This is the same bound found by Rands [1982]. Moreover, this method can be applied to s-intersecting blocks in a design; again we get the same bound as in [Rands 1982].

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