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# Convergence of the maximum zeros of a class of Fibonacci-type polynomials

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Let  $a$  be a positive integer and let  $k$  be an arbitrary, fixed positive integer. We define a generalized Fibonacci-type polynomial sequence by  $G_{k,0}(x) = -a$ ,  $G_{k,1}(x) = x - a$ , and  $G_{k,n}(x) = x^k G_{k,n-1}(x) + G_{k,n-2}(x)$  for  $n \geq 2$ . Let  $g_{k,n}$  represent the maximum real zero of  $G_{k,n}$ . We prove that the sequence  $\{g_{k,2n}\}$  is decreasing and converges to a real number  $\beta_k$ . Moreover, we prove that the sequence  $\{g_{k,2n+1}\}$  is increasing and converges to  $\beta_k$  as well. We conclude by proving that  $\{\beta_k\}$  is decreasing and converges to  $a$ .

## 1. Introduction

Let  $\alpha$ ,  $\beta$ , and  $k$  be integers, with  $\alpha \neq 0$ . Consider a Fibonacci-type polynomial sequence given by the recurrence relation  $G_{k,0} = -\alpha$ ,  $G_{k,1} = x - \beta$ , and for  $n \geq 2$ ,

$$G_{k,n}(x) = x^k G_{k,n-1}(x) + G_{k,n-2}(x). \quad (1)$$

We should point out that the classical Fibonacci polynomial sequence  $F_n$  is obtained when  $\alpha = -1$ ,  $\beta = 0$ , and  $k = 1$ . Moreover, the Lucas polynomial sequence  $L_n$  is obtained when  $\alpha = -2$ ,  $\beta = 0$ , and  $k = 1$ . Hoggatt and Bicknell [1973] give explicit forms for the zeros of  $F_n$  and  $L_n$ . Even though finding explicit formulas for other Fibonacci-type polynomial sequences has been a challenge, several results about the properties of the zeros of some specific cases are known. For example, G. Moore [1994] and H. Prodinger [1996] studied the asymptotic behavior of the maximal zeros of  $G_{1,n}$  when  $\alpha = \beta = k = 1$ , and Yu, Wang and He [Yu et al. 1996] generalized Moore's result for  $\alpha = \beta = a$ , where  $a$  is any positive integer. F. Mátyás [1998] studied the same problem for  $\alpha = a$ ,  $a \neq 0$  and  $\beta = \pm a$ . More recently, Wang and He [2004] generalized their previous result for any two integers  $\alpha$  and  $\beta$  with  $\alpha \neq 0$ . We also mention the works of P. E. Ricci [1995] and Mátyás [1998] for boundedness results of the zeros of  $G_{1,n}$ . In addition, Molina and Zeleke [2007; 2009] studied the asymptotic behavior of the zeros of  $G_{k,n}$  when  $\alpha = \beta = 1$  and  $k$  is an arbitrary integer.

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Moore [1994] proved that when  $\alpha = \beta = k = 1$ , the maximum zeros of the odd-indexed polynomials converge to  $\frac{3}{2}$  from below and the maximum roots of the even-indexed polynomials converge to  $\frac{3}{2}$  from above. In that article, a remark was made about the possibilities of investigating asymptotic behaviors of maximum zeros of other Fibonacci-type polynomial sequences. In [Miller and Zeleke 2013], the first author and Zeleke studied the maximum real zeros of the Fibonacci-type polynomial sequence where  $\alpha = \beta = a$ ,  $a$  is a positive integer, and  $k = 2$ . They provided asymptotic results for the maximum real zeros numerically as well as analytically. We extend those results by allowing  $k$  to be an arbitrary, fixed positive integer. The proof techniques expand those used in [Miller and Zeleke 2013] and [Molina and Zeleke 2009].

Before delving into the technical results, we provide a numerical example to motivate our work.

**Example.** Consider the Fibonacci-type polynomial sequence given by the recurrence relation  $G_{k,0} = -2$ ,  $G_{k,1} = x - 2$ , and for  $n \geq 2$ ,

$$G_{k,n}(x) = x^k G_{k,n-1}(x) + G_{k,n-2}(x).$$

In the context of the generalized Fibonacci-type polynomial sequences we study in this paper, this example corresponds to the case when  $a = 2$ . For a fixed positive integer  $k$  and a natural number  $n$ , let  $g_{k,n}$  represent the maximum real root of the polynomial  $G_{k,n}$ . The first six terms in the sequences of the maximum real roots for  $k = 2$ ,  $k = 3$ , and  $k = 4$  are shown in the following three columns, respectively.

$g_{2,1} = 2$	$g_{3,1} = 2$	$g_{4,1} = 2$
$g_{2,2} \doteq 2.359304086$	$g_{3,2} \doteq 2.190327947$	$g_{4,2} \doteq 2.102374082$
$g_{2,3} \doteq 2.350513611$	$g_{3,3} \doteq 2.188965777$	$g_{4,3} \doteq 2.102149889$
$g_{2,4} \doteq 2.350789278$	$g_{3,4} \doteq 2.188978002$	$g_{4,4} \doteq 2.102150474$
$g_{2,5} \doteq 2.350780807$	$g_{3,5} \doteq 2.188977893$	$g_{4,5} \doteq 2.102150473$
$g_{2,6} \doteq 2.350781067$	$g_{3,6} \doteq 2.188977894$	$g_{4,6} \doteq 2.102150473$

For each sequence, the subsequence created by the odd-indexed (i.e.,  $n$  is odd) maximum real roots is increasing. And, the subsequence created by the even-indexed (i.e.,  $n$  is even) maximum real roots is decreasing. In fact, each of the sequences  $k$  converge to a real number which is dependent on  $k$ . We call this real number  $\beta_k$ . We should mention  $\beta_k$  is also dependent on our choice of  $a$  and for this example,  $a = 2$ . For the sequences above, we have

$$\beta_2 \doteq 2.350781059, \quad \beta_3 \doteq 2.188977894, \quad \beta_4 \doteq 2.102150473.$$

It is also the case that  $\{\beta_k\}$  converges to 2 and it is not a coincidence that this is the value of  $a$ .

## 2. Formulas

At this time, we introduce a few handy formulas that were established in [Molina and Zeleke 2009]. The formulas in the following lemma allow us to write  $G_{k,n}(x)$  in terms of smaller indexed functions.

**Lemma 2.1.** *For  $n \geq 1$ , the following recursive formulas are true:*

$$G_{k,2n+2}(x) = (x^{2k} + 1)G_{k,2n}(x) + x^{2k}G_{k,2n-2}(x) + \dots + x^{2k}G_{k,2}(x) + x^kG_{k,1}(x),$$

$$G_{k,2n+1}(x) = (x^{2k} + 1)G_{k,2n-1}(x) + x^{2k}G_{k,2n-3}(x) + \dots + x^{2k}G_{k,1}(x) + x^kG_{k,0}(x).$$

The formula that we present in the next lemma provides a type of shift from one indexed polynomial evaluated at  $g_{k,n}$  to another indexed polynomial evaluated at  $g_{k,n}$ . The proof can be found in [Molina and Zeleke 2009, Lemma 4].

**Lemma 2.2.** *For  $n \geq m$ ,  $G_{k,n+m}(g_{k,n}) = (-1)^{m+1}G_{k,n-m}(g_{k,n})$ .*

## 3. Preliminary results

We're now ready to study the maximum real roots,  $g_{k,n}$ , for the generalized Fibonacci-type polynomial sequence defined by  $G_{k,0}(x) = -a$ ,  $G_{k,1}(x) = x - a$ , and  $G_{k,n}(x) = x^kG_{k,n-1}(x) + G_{k,n-2}(x)$  for  $n \geq 2$ , where  $a$  is a positive integer and  $k$  is an arbitrary, fixed positive integer.

**Proposition 3.1.** *If  $n \geq 2$ , then  $g_{k,n} \in (a, a + 1)$ .*

*Proof.* For  $n \geq 2$ , we will show  $G_{k,n}(a) < 0$  and  $G_{k,n}(x) > 0$  for  $x \in [a + 1, \infty)$ ; thus, our conclusion will follow. We'll begin by showing  $G_{k,n}(a) < 0$  by induction. Since  $G_{k,0}(a) = -a$  and  $G_{k,1}(a) = a - a = 0$ , we have  $G_{k,2}(a) = a^k(0) - a = -a < 0$ . Now suppose  $G_{k,m}(a) < 0$  for all  $m$  such that  $2 \leq m \leq n$ . By (1) and the inductive hypothesis,  $G_{k,n+1}(a) = a^kG_{k,n}(a) + G_{k,n-1}(a) < 0$ . Hence,  $G_{k,n}(a) < 0$  for  $n \geq 2$ .

For the remainder of the proof, let  $x \in [a + 1, \infty)$ . We again use induction. Notice

$$G_{k,1}(x) = x - a \geq a + 1 - a > 0, \quad \text{and}$$

$$G_{k,2}(x) = x^k(x - a) - a \geq (a + 1)^k(a + 1 - a) - a = (a + 1)^k - a > 0.$$

Now suppose  $G_{k,m}(x) > 0$  for all  $m$  such that  $2 \leq m \leq n$ . By (1) and the inductive hypothesis, it follows that  $G_{k,n+1}(x) = x^kG_{k,n}(x) + G_{k,n-1}(x) > 0$ . Hence,  $G_{k,n}(x) > 0$  for  $x \in [a + 1, \infty)$  and  $n \geq 2$ .

Therefore,  $g_{k,n} \in (a, a + 1)$  for  $n \geq 2$ . □

**Proposition 3.2.** *Let  $a$  be a positive integer and let  $\beta_k$  be a positive real number that satisfies the equation  $G_{k,2}(x) = -(a - x)^2/a$ ; that is,  $\beta_k$  is a zero of  $T_k(x) = ax^k - a^2x^{k-1} + x - 2a$ . Then*

$$G_{k,n}(\beta_k) = \frac{-(a - \beta_k)^n}{a^{n-1}} \quad \text{for all } n \geq 0.$$

*Proof.* We prove this proposition by induction. The result is true for  $n = 0$  and  $n = 1$  by simple computation. It is true for  $n = 2$  by construction. Now assume  $G_{k,n}(\beta_k) = -(a - \beta_k)^n/a^{n-1}$  for all positive integers less than or equal to  $n$ . Then

$$\begin{aligned}
 G_{k,n+1}(\beta_k) &= \beta_k^k G_{k,n}(\beta_k) + G_{k,n-1}(\beta_k) \\
 &= \beta_k^k \left( \frac{-(a - \beta_k)^n}{a^{n-1}} \right) + \frac{-(a - \beta_k)^{n-1}}{a^{n-2}} \\
 &= \frac{-(a - \beta_k)^{n-1}}{a^{n-2}} \left( \frac{\beta_k^k (a - \beta_k)}{a} + 1 \right) \\
 &= \frac{-(a - \beta_k)^{n-1}}{a^{n-2}} \left( \frac{a\beta_k^k (a - \beta_k) + a^2}{a^2} \right) \\
 &= \frac{-(a - \beta_k)^{n-1}}{a^n} (a\beta_k^k (a - \beta_k) + a^2) \\
 &= \frac{-(a - \beta_k)^{n-1}}{a^n} (-a(\beta_k^k (\beta_k - a) - a)) \\
 &= \frac{-(a - \beta_k)^{n-1}}{a^n} \left( -a \left( \frac{-(a - \beta_k)^2}{a} \right) \right) \\
 &= \frac{-(a - \beta_k)^{n-1}}{a^n} (a - \beta_k)^2 \\
 &= \frac{-(a - \beta_k)^{n+1}}{a^n}.
 \end{aligned}$$

Therefore, our result is true for all nonnegative integers.  $\square$

We remind the reader that whenever  $\beta_k$  is used in this article, it will be dependent on the choice of  $a$ .

**Corollary 3.3.**  $\lim_{n \rightarrow \infty} G_{k,n}(\beta_k) = 0.$

*Proof.* Before we begin, we kindly remind the reader that  $k \geq 1$  and this assumption is continued throughout our work unless stated otherwise. Now the first fact we establish for this proof is that  $\beta_k \in (a, a + 1)$ . To show this, we will again consider  $T_k(x) = ax^k - a^2x^{k-1} + x - 2a$ . It is easily verified that  $T_k(a) < 0 < T_k(a + 1)$ . Moreover,  $T_k$  is strictly increasing on the interval  $[a, \infty)$ , which will be shown by examining the first derivative of  $T_k$ . Notice

$$\begin{aligned}
 T'_k(x) &= kax^{k-1} - (k-1)a^2x^{k-2} + 1 \\
 &= ax^{k-2}(kx - ka + a) + 1 \\
 &= ax^{k-2}(k(x - a) + a) + 1 \\
 &> 0
 \end{aligned}$$

for all  $x \in [a, \infty)$ . Thus,  $\beta_k \in (a, a + 1)$ . Therefore,

$$\lim_{n \rightarrow \infty} G_{k,n}(\beta_k) = \lim_{n \rightarrow \infty} \frac{-(a - \beta_k)^n}{a^{n-1}} = 0. \quad \square$$

#### 4. Analysis of $G'_{k,3}(x)$

In order to prove our main result on the convergence of the maximum zeros, we will need a lower bound on the values  $G'_{k,n}(g_{k,n})$ . This section will provide a lower bound of  $G'_{k,3}(x)$  on the interval  $[g_{k,3}, \infty)$ . We begin with a couple of lemmas to help us achieve this lower bound.

**Lemma 4.1.** *For  $k \geq 3$ ,  $G''_{k,3}(x)$  has exactly one zero in the interval  $(0, \infty)$ .*

*Proof.* Let  $k \geq 3$  and recall  $G_{k,3}(x) = x^{2k+1} - ax^{2k} - ax^k + x - a$ . Thus,

$$\begin{aligned} G''_{k,3}(x) &= (2k + 1)(2k)x^{2k-1} - 2ka(2k - 1)x^{2k-2} - k(k - 1)ax^{k-2} \\ &= kx^{k-2}(2(2k + 1)x^{k+1} - 2a(2k - 1)x^k - a(k - 1)) \\ &= kx^{k-2}f(x), \end{aligned}$$

where  $f(x) = 2(2k + 1)x^{k+1} - 2a(2k - 1)x^k - a(k - 1)$ . We can see that 0 is a zero of  $G''_{k,3}$ . In order to show  $G''_{k,3}$  has only one zero in  $(0, \infty)$ , we will show that  $f(x)$  has exactly one zero in  $(0, \infty)$ . To do so, consider

$$\begin{aligned} f'(x) &= 2(2k + 1)(k + 1)x^k - 2a(2k - 1)kx^{k-1} \\ &= 2x^{k-1}((2k + 1)(k + 1)x - a(2k - 1)k). \end{aligned}$$

The critical numbers of  $f$  are

$$c_1 = 0 \quad \text{and} \quad c_2 = \frac{a(2k - 1)k}{(2k + 1)(k + 1)}.$$

Using this information, it can be verified that  $f$  is decreasing on  $(0, c_2)$  and increasing on  $(c_2, \infty)$ . Pairing this with  $f(0) = -a(k - 1) < 0$  and  $\lim_{x \rightarrow \infty} f(x) = \infty$ , we conclude  $f$ , and hence  $G''_{k,3}$ , has exactly one zero in  $(0, \infty)$ . Therefore, our conclusion holds. □

**Lemma 4.2.** *For  $k \geq 3$ ,  $G'_{k,3}(x)$  has exactly two zeros in the interval  $(0, \infty)$ .*

*Proof.* Let  $k \geq 3$  and recall  $G_{k,3}(x) = x^{2k+1} - ax^{2k} - ax^k + x - a$ . Thus,

$$G'_{k,3}(x) = (2k + 1)x^{2k} - 2kax^{2k-1} - kax^{k-1} + 1.$$

Using the intermediate value theorem and the inequalities  $G'_{k,3}(0) = 1 > 0$ ,  $G'_{k,3}(1) = k(2 - 3a) + 2 \leq -1 < 0$ , and  $\lim_{x \rightarrow \infty} G'_{k,3}(x) = \infty$ , we can conclude  $G'_{k,3}(x)$  has at least two zeros in  $(0, \infty)$ . To show there can be no more than two zeros in  $(0, \infty)$ , we will explore the possibility of  $G'_{k,3}(x)$  having at least three zeros in  $(0, \infty)$ . If

$G'_{k,3}(x)$  has at least three zeros in  $(0, \infty)$ , then  $G''_{k,3}$  would have at least two zeros in  $(0, \infty)$  by Rolle's theorem, but, by [Lemma 4.1](#), we know this cannot be the case. Thus,  $G'_{k,3}(x)$  has exactly two zeros in  $(0, \infty)$  and since  $G'_{k,3}(0) \neq 0$ , those two zeros are indeed in  $(0, \infty)$ .  $\square$

We are now ready to obtain a lower bound on  $G'_{k,3}(x)$  for  $x \in [g_{k,3}, \infty)$ .

**Proposition 4.3.** *If  $k \geq 1$  and  $x \in [g_{k,3}, \infty)$ , then  $G'_{k,3}(x) > 1$ .*

*Proof.* Let  $x \in [g_{k,3}, \infty)$ . We break our proof into cases.

**Case 1:** Consider  $k = 1$ . We then have

- $G_{1,3}(x) = x^3 - ax^2 - ax + x - a$ ,
- $G'_{1,3}(x) = 3x^2 - 2ax - a + 1$ , and
- $G''_{1,3}(x) = 6x - 2a$ .

Since  $G''_{1,3}(x) > 0$  for  $x \in (a/3, \infty)$ , we know  $G'_{1,3}$  is increasing on  $(a/3, \infty)$ . Thus,  $1 \leq G'_{1,3}(a) < G'_{1,3}(x)$  when  $x \in [g_{1,3}, \infty)$  as  $g_{1,3} > a$  by [Proposition 3.1](#).

**Case 2:** Consider  $k = 2$ . We then have

- $G_{2,3}(x) = x^5 - ax^4 - ax^2 + x - a$ ,
- $G'_{2,3}(x) = 5x^4 - 4ax^3 - 2ax + 1$ , and
- $G''_{2,3}(x) = 2(10x^3 - 6ax^2 - a)$ .

Since  $G''_{2,3}(x) > 0$  for  $x \in (a, \infty)$ , we know  $G'_{2,3}$  is increasing on  $(a, \infty)$ . Again notice  $g_{2,3} > a$  by [Proposition 3.1](#). Applying the mean value theorem, we know there exists  $c \in (a, g_{2,3})$  such that

$$G'_{2,3}(c) = \frac{G_{2,3}(g_{2,3}) - G_{2,3}(a)}{g_{2,3} - a}.$$

It follows that when  $x \in [g_{2,3}, \infty)$ ,

$$G'_{2,3}(x) > G'_{2,3}(c) = \frac{G_{2,3}(g_{2,3}) - G_{2,3}(a)}{g_{2,3} - a} = \frac{0 - G_{2,3}(a)}{g_{2,3} - a} = \frac{a^3}{g_{2,3} - a} > 1.$$

**Case 3:** Consider  $k \geq 3$ . By [Lemma 4.1](#), we know  $G''_{k,3}(x)$  has one positive root, call it  $r$ , and, by [Lemma 4.2](#), we know  $G'_{k,3}(x)$  has two positive roots, call them  $s$  and  $t$ , where  $s < t$ . Moreover, by Rolle's theorem,  $s < r < t$ . Notice that

- $G'_{k,3}(0) = 1 > 0$ ,
- $G'_{k,3}(1) = k(2 - 3a) + 2 \leq -1 < 0$ ,
- $\lim_{x \rightarrow \infty} G'_{k,3}(x) = \infty$ , and
- $G''_{k,3}$  is positive on  $(r, \infty)$ .

Thus,  $s < 1 < t$ . Moreover,  $G'_{k,3}$  is negative on  $(s, t)$  and  $G'_{k,3}$  is positive and increasing on  $(t, \infty)$ , and, by the mean value theorem, there exists  $c \in [1, g_{k,3}]$  such that

$$G'_{k,3}(c) = \frac{G_{k,3}(g_{k,3}) - G_{k,3}(1)}{g_{k,3} - 1} = \frac{0 - (2 - 3a)}{g_{k,3} - 1} = \frac{3a - 2}{g_{k,3} - 1} \geq 1.$$

Hence,  $c > t$ , and thus  $g_{k,3} > t$ . Therefore, if  $x \in [g_{k,3}, \infty)$ , then

$$G'_{k,3}(x) > G'_{k,3}(c) \geq 1.$$

Therefore, our conclusion holds for all cases. □

We're now ready to prove that all of the first derivatives of the polynomials are bounded below by 1 as well as explore the characteristics of the maximum zeros. We break this up into two sections, one with the odd-indexed polynomials and the other with the even-indexed polynomials.

### 5. Odd-indexed polynomials

We will use the following two propositions to help establish our results. The proofs are left to the reader as they are similar to those found in [Molina and Zeleke 2009, Lemmas 6 and 7].

**Proposition 5.1.** *The maximum zeros of the odd-indexed polynomials  $G_{k,2n+1}$  form a strictly increasing sequence.*

**Proposition 5.2.** *If  $n \geq 0$ , then the derivative of  $G_{k,2n+1}(x)$  is bounded below by 1 for  $x \in [g_{k,2n+1}, \infty)$ .*

**Proposition 5.3.** *If  $n \geq 0$ , then  $g_{k,2n+1} < \beta_k$  for each  $k \geq 1$ .*

*Proof.* By Proposition 3.2 and for  $n \geq 1$ ,

$$G_{k,2n+1}(\beta_k) = \frac{-(a - \beta_k)^{2n+1}}{a^{2n}} > 0$$

as  $\beta_k \in (a, a + 1)$ . Our goal is to show that

$$G'_{k,2n+1}(x) > G'_{k,2n-1}(x) > \dots > G'_{k,3}(x) > G'_{k,1}(x) = 1$$

for  $x \in [\beta_k, \infty)$  as it will then follow that  $g_{k,2n+1} < \beta_k$ . Now, since  $G_{k,3}(x) \leq 0$  on  $[a, g_{k,3}]$ , it must be the case that  $\beta_k > g_{k,3}$ . Proposition 5.2 gives

$$G'_{k,3}(x) > G'_{k,1}(x) = 1$$

on  $[g_{k,3}, \infty)$ . Thus,

$$G'_{k,3}(x) > G'_{k,1}(x) = 1$$

on  $[\beta_k, \infty)$  as  $[\beta_k, \infty) \subseteq [g_{k,3}, \infty)$ . We note that the rest of the proof follows a similar format to the induction argument used in [Proposition 5.2](#) with  $[\beta_k, \infty)$  replacing  $[g_{k,2n+1}, \infty)$ .  $\square$

## 6. Even-indexed polynomials

**Proposition 6.1.** *If  $n \geq 1$ , then the derivative of  $G_{k,2n}(x)$  is bounded below by 1 for  $x \in [g_{k,2n-1}, \infty)$ .*

*Proof.* We will make use of induction to obtain our result. Let  $x \in [g_{k,2n-1}, \infty)$ . For  $n = 1$ , we have

$$G'_{k,2}(x) = (k+1)x^k - akx^{k-1} = x^{k-1}((k+1)x - ak) > 1.$$

By (1), we have

$$\begin{aligned} G_{k,2n}(x) &= x^k G_{k,2n-1}(x) + G_{k,2n-2}(x), \quad \text{and} \\ G'_{k,2n}(x) &= x^k G'_{k,2n-1}(x) + kx^{k-1} G_{k,2n-1}(x) + G'_{k,2n-2}(x). \end{aligned}$$

From [Proposition 5.1](#), we know  $kx^{k-1} G_{k,2n-1}(x) \geq 0$  as  $x \in [g_{k,2n-1}, \infty)$ . So,

$$G'_{k,2n}(x) \geq x^k G'_{k,2n-1}(x) + G'_{k,2n-2}(x).$$

Now suppose  $G'_{k,2n-2}(x) \geq 1$ . Then

$$\begin{aligned} G'_{k,2n}(x) &\geq x^k G'_{k,2n-1}(x) + G'_{k,2n-2}(x) \\ &> G'_{k,2n-2}(x) \quad (\text{as } x^k G'_{k,2n-1}(x) > 1 \text{ by } \text{Proposition 5.2}) \\ &\geq 1 \quad (\text{by the induction hypothesis}). \end{aligned}$$

Therefore, the derivative of the even-indexed polynomials are bounded below by 1 for  $x \in [g_{k,2n-1}, \infty)$ .  $\square$

Referring back to [Proposition 5.3](#), we should note that the result in [Proposition 6.1](#) also holds for  $x \in [\beta_k, \infty)$  as  $[\beta_k, \infty) \subseteq [g_{k,2n-1}, \infty)$ .

**Proposition 6.2.** *The maximum zeros of the even-indexed polynomials form a decreasing sequence that is bounded below by  $\beta_k$ .*

*Proof.* Let  $n \geq 1$ . By [Proposition 3.2](#),

$$G_{k,2n}(\beta_k) = \frac{-(a - \beta_k)^{2n}}{a^{2n-1}} < 0.$$

Thus,  $\beta_k < g_{k,2n}$ . We proceed by induction to show the maximum zeros of the even-indexed polynomials form a decreasing sequence. Notice that

$$G_{k,4}(x) = x^k G_{k,3}(x) + G_{k,2}(x)$$

implies

$$G_{k,4}(g_{k,2}) = g_{k,2}^k G_{k,3}(g_{k,2}) + G_{k,2}(g_{k,2}) = g_{k,2}^k G_{k,3}(g_{k,2}) > 0$$

by utilizing [Proposition 5.3](#). Since  $G_{k,4}$  is increasing on  $[\beta_k, \infty)$  as well, we conclude that  $g_{k,2} > g_{k,4}$ . Now assume  $g_{k,2} > g_{k,4} > \dots > g_{k,2n}$ . By [Lemma 2.2](#),  $G_{k,2n-2}(g_{k,2n}) = -G_{k,2n+2}(g_{k,2n})$ . Since  $g_{k,2n-2} > g_{k,2n}$  (induction hypothesis),  $G_{k,2n-2}$  is increasing on  $[\beta_k, \infty)$ , and  $G_{k,2n-2}(g_{k,2n-2}) = 0$ , it follows that

$$G_{k,2n-2}(g_{k,2n}) < 0 \quad \text{and} \quad G_{k,2n+2}(g_{k,2n}) > 0,$$

and, since  $G_{k,2n+2}(x)$  is increasing on  $[\beta_k, \infty)$ , we have  $g_{k,2n} > g_{k,2n+2}$ . Therefore,  $g_{k,2} > g_{k,4} > \dots > \beta_k$ . □

### 7. Main results

**Theorem 7.1.** *The sequence of odd-indexed zeros is increasing and converges to  $\beta_k$ , and the sequence of even-indexed zeros is decreasing and converges to  $\beta_k$  as well.*

*Proof.* By [Proposition 5.1](#) and [Proposition 5.3](#), we have shown the maximum zeros of the odd-indexed polynomials form an increasing sequence bounded above by  $\beta_k$ , and, by [Proposition 6.2](#), we know the maximum zeros of the even-indexed polynomials form a decreasing sequence bounded below by  $\beta_k$ . In order to show both of the sequences converge to  $\beta_k$ , we will show that  $\lim_{n \rightarrow \infty} g_{k,n} = \beta_k$ . The mean value theorem tells us there exists a real number  $c$  between  $g_{k,n}$  and  $\beta_k$  such that

$$|G'_{k,n}(c)| = \left| \frac{G_{k,n}(\beta_k) - G_{k,n}(g_{k,n})}{\beta_k - g_{k,n}} \right| = \left| \frac{G_{k,n}(\beta_k)}{\beta_k - g_{k,n}} \right|.$$

Since  $G'_{k,n}(c) \geq 1$ ,  $|\beta_k - g_{k,n}| \leq |G_{k,n}(\beta_k)|$ . By utilizing [Corollary 3.3](#), which states  $\lim_{n \rightarrow \infty} G_{k,n}(\beta_k) = 0$ , we can say  $\lim_{n \rightarrow \infty} g_{k,n} = \beta_k$ . Therefore, the sequence of odd-indexed zeros and the sequence of even-indexed zeros converge to  $\beta_k$ . □

**Theorem 7.2.** *The sequence  $\{\beta_k\}$  is decreasing and converges to  $a$ .*

*Proof.* We begin by referring the reader back to  $T_k(x)$  as defined in [Proposition 3.2](#). Recall that  $T_k$  is increasing on  $[a, \infty)$  and  $\beta_k \in (a, a + 1)$  is a zero of  $T_k$ . Using the fact that  $\beta_k$  is a zero of  $T_k$ , we have  $a\beta_k^k - a^2\beta_k^{k-1} = 2a - \beta_k$ . Then

$$\begin{aligned} T_{k+1}(\beta_k) &= a\beta_k^{k+1} - a^2\beta_k^k + \beta_k - 2a = \beta_k(a\beta_k^k - a^2\beta_k^{k-1}) + \beta_k - 2a \\ &= \beta_k(2a - \beta_k) + \beta_k - 2a = (\beta_k - 1)(2a - \beta_k) \\ &> 0. \end{aligned}$$

Thus,  $\beta_{k+1} < \beta_k$ , which verifies that  $\{\beta_k\}$  is decreasing. Now let  $\varepsilon > 0$ . Then

$$\begin{aligned}
\lim_{k \rightarrow \infty} T_k(a + \varepsilon) &= \lim_{k \rightarrow \infty} [a(a + \varepsilon)^k - a^2(a + \varepsilon)^{k-1} + (a + \varepsilon) - 2a] \\
&= \lim_{k \rightarrow \infty} [a(a + \varepsilon)^{k-1}(a + \varepsilon - a) + a + \varepsilon - 2a] \\
&= \lim_{k \rightarrow \infty} [\varepsilon a(a + \varepsilon)^{k-1} + \varepsilon - a] \\
&= \infty.
\end{aligned}$$

We then know that there exists  $j \in \mathbb{Z}$  such that  $T_j(a + \varepsilon) > 0$  and so  $\beta_j \in (a, a + \varepsilon)$ . Therefore,  $\lim_{k \rightarrow \infty} \beta_k = a$ .  $\square$

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[Rebecca.Miller-1@ou.edu](mailto:Rebecca.Miller-1@ou.edu)

*Department of Mathematics, University of Oklahoma, 601 Elm Avenue, Room 423, Norman, OK 73019, United States*

[kkarber1@uco.edu](mailto:kkarber1@uco.edu)

*Department of Mathematics and Statistics, University of Central Oklahoma, 100 North University Drive, Edmond, OK 73034, United States*

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Sat Gupta	U of North Carolina, Greensboro, USA <a href="mailto:sgupta@uncg.edu">sgupta@uncg.edu</a>	Robert W. Robinson	University of Georgia, USA <a href="mailto:rwr@cs.uga.edu">rwr@cs.uga.edu</a>
Jim Haglund	University of Pennsylvania, USA <a href="mailto:jhaglund@math.upenn.edu">jhaglund@math.upenn.edu</a>	Filip Saidak	U of North Carolina, Greensboro, USA <a href="mailto:f_saidak@uncg.edu">f_saidak@uncg.edu</a>
Johnny Henderson	Baylor University, USA <a href="mailto:johnny_henderson@baylor.edu">johnny_henderson@baylor.edu</a>	James A. Sellers	Penn State University, USA <a href="mailto:sellersj@math.psu.edu">sellersj@math.psu.edu</a>
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