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Numerical integration of rational bubble functions with multiple singularities

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We derive an effective quadrature scheme via a partitioned Duffy transformation for a class of Zienkiewicz-like rational bubble functions proposed by J. Guzmán and M. Neilan. This includes a detailed construction of the new quadrature scheme, followed by a proof of exponential error convergence. Briefly discussed is the functions application to the finite element method when used to solve Stokes flow and elasticity problems. Numerical experiments which support the theoretical results are also provided.

1. Introduction

The finite element method is one of the most popular and well studied numerical methods used to approximate solutions of partial different equations (PDEs). Its formulation is built upon the variational formulation of the PDE, where the infinite-dimensional problem is restricted to a finite-dimensional setting. What distinguishes the finite element method from other Galerkin methods is that the finite-dimensional space contains piecewise polynomials with respect to a partition (usually rectangles or triangles in two dimensions) of the domain. When performing the finite element method the need to integrate these piecewise polynomials over the partition arises. Solving these integrals directly would prove computationally costly and sometimes extremely difficult. We instead use a variety of numerical integration techniques. One of the most popular of these techniques is Gaussian quadrature. The method approximates the value of the integral via a weighted sum of function values at points within the domain of integration. This is already a mature theory for polynomial basis functions with highly developed implementation techniques and error analysis [Brezzi and Fortin 1991].

J. Guzmán and M. Neilan [2014a; 2014b] proposed a new family of finite methods to approximate two-dimensional Stokes flow and planar elasticity. Varying from the traditional finite element framework, the authors supplemented the usual finite

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element spaces (i.e., piecewise polynomials) with a class of divergence-free rational bubble functions. With the inclusion of these rational functions, Guzmán and Neilan were able to derive finite element methods with several desirable properties (e.g., exactly divergence-free velocity approximations for Stokes and symmetric and conforming stresses for elasticity). Assuming that the integrals are computed exactly, the authors derived several results including stability estimates of the numerical methods and optimal order error estimates. However in practice, these integrals are not computed exactly, and it is not clear how numerical integration will effect these theoretical results. The issue arises from the fact that traditional quadrature rules utilize interpolating polynomials to approximate the function and Taylor's formula to estimate the error [Burden and Faries 2011]. Thus in order to obtain accurate error estimates, our function must be sufficiently smooth. However, the rational functions in [Guzmán and Neilan 2014a; 2014b] are singular. Therefore the behavior of the error is unpredictable. We numerically verify this assertion in Section 5.

One of the traditional methods for computing the integrals of singular functions is the Duffy transformation [1982]. As described in [Lyness and Cools 1994], a mapping from the original triangular domain to the unit square is constructed. The singularity is effectively "stretched" out via its mapping to one of the edges of the square. Since the singularity is no longer present, the square can then be numerically integrated via a standard quadrature rule. This method will not work though for the divergence-free rational bubble functions described in this paper due to the presence of two singularities. While it would effectively eliminate one of the singularities, the remaining singularity would still render standard quadrature methods ineffective.

In the paper, we tackle this issue with a modified application of the Duffy transformation. We subdivide the triangle into four subtriangles and then perform a Duffy transformation on each of these subtriangles, which can essentially remove all the problematic singularities. We can then construct a quadrature rule on the unit square, which can then be mapped back to and used on our original domain. We do not address the effect of the error estimates obtained from this new scheme on the finite element methods in [Guzmán and Neilan 2014a; 2014b] as it is beyond the scope of this paper.

The remainder of this paper is organized as follows. Section 2 contains some preliminaries and the function spaces in which the analysis will be performed. Well known results from vector calculus which are used extensively in the analysis are also provided. In Section 3, the procedure for the partitioned Duffy transformation is established. In Section 4, a quadrature scheme derived from the partitioned Duffy transformation is given. A proof of exponential error convergence for this quadrature scheme is also provided. In Section 5, we present numerical experiments on the unit triangle which support our findings.

2. Preliminaries

In this paper, standard space and norm notations are adopted. If $G : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping with argument $x \in \mathbb{R}^n$, we denote by $DG(x)$ the Jacobian, that is,

$$DG_{ij}(x) = \frac{\partial G_i}{\partial x_j}(x) \quad (i = 1, 2, \dots m,\ j = 1, 2, \dots n).$$

For a differentiable function $g : \mathbb{R}^n \to \mathbb{R}$, we denote by $\nabla g : \mathbb{R}^n \to \mathbb{R}^n$ the gradient of g which is given by

$$\nabla g(x) = \frac{\partial g}{\partial x_1}(x) e_1 + \cdots + \frac{\partial g}{\partial x_n}(x) e_n,$$

where e_i is the orthogonal unit column vector pointing in the coordinates direction x_i. We note that $\nabla g = (Dg)^t$ is the transpose of Dg. The Hessian matrix of a twice differentiable function g is denoted by $D^2 g : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and is defined as

$$D^2 g(x) = D \nabla g(x), \tag{2-1}$$

where the operator D in (2-1) is applied row-wise. Namely, the Hessian matrix is given by

$$(D^2 g)_{ij}(x) = \frac{\partial^2 g}{\partial x_i \partial x_j}(x) \quad (i, j = 1, 2, \dots, n).$$

For an open and bounded set D with Lipschitz continuous boundary ∂D, we denote by $L^p(D)$ $(1 \le p \le \infty)$ the complete normed linear space

$$L^p(D) := \{\text{measurable functions } v : \textstyle\int_D |v|^p \, dx < \infty\} \quad (1 \le p < \infty),$$
$$L^\infty(D) := \{\text{measurable functions } v : \operatorname{ess\,sup}_D |v| < \infty\}.$$

The corresponding norms are then given by

$$\|v\|_{L^p(D)} := \left(\int_D |v|^p \, dx \right)^{1/p}, \quad \|v\|_{L^\infty(D)} := \operatorname*{ess\,sup}_D |v|.$$

The Sobolev spaces $W^{k,p}(D)$ are defined as

$$W^{k,p}(D) = \{u \in L^p(D) : D^\alpha u \in L^p(D) \text{ for all } |\alpha| \le k\},$$

with norms

$$\|u\|_{W^{k,p}(D)} = \left(\sum_{|\alpha| \le k} \int_D |D^\alpha u|^p \, dx \right)^{1/p} \quad (1 \le p < \infty)$$

and

$$\|u\|_{W^{k,\infty}(D)} = \sum_{|\alpha|\le k} \operatorname*{ess\,sup}_D |D^\alpha u|.$$

In the case $p = 2$ and $k \ge 1$, we set $H^k(D) = W^{k,2}(D)$ and $\|\cdot\|_{H^k(D)} = \|\cdot\|_{W^{k,2}(D)}$. We note that $H^k(D)$ is a Hilbert space.

We denote the dual space of $W^{k,p}(D)$ by $W^{-k,p'}(D)$, where p' satisfies

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The associated norm is defined by

$$\|\varphi\|_{W^{-k,p'}(D)} = \sup_{v\in W^{k,p}(D)\setminus\{0\}} \varphi(v)/\|v\|_{W^{k,p}(D)}. \tag{2-2}$$

We denote by $\mathcal{T}_h$ a shape-regular triangulation of the domain Ω with $h_T = \operatorname{diam}(T)$ for all $T \in \mathcal{T}_h$ and $h := \max_{T\in\mathcal{T}_h} h_T$. Given $T \in \mathcal{T}_h$, we denote by $\{e^{(i)}\}_{i=1}^3$ the three edges of T and by $\{\lambda^{(i)}\}_{i=1}^3$ the three barycentric coordinates labeled such that $\lambda^{(i)}|_{e^{(i)}} = 0$. The vertices of T are denoted by $\{a^{(i)}\}_{i=1}^3$ labeled such that $\lambda^{(i)}(a^{(i)}) = \delta_{i,j}$. We set $b_T := \lambda^{(1)}\lambda^{(2)}\lambda^{(3)} \in \mathcal{P}_3(T)$ to be the cubic bubble and $b^{(i)} = \lambda^{(i+1)}\lambda^{(i+2)} \in \mathcal{P}_2(T)$ (mod 3) to be the quadratic edge bubble associated with edge $e^{(i)}$. For each triangle $T \in \mathcal{T}_h$, the three *rational edge bubbles* $\{B^{(i)}\}_{i=1}^3$ associated with T are then given by

$$\begin{aligned} B^{(i)} &:= \frac{b_T b^{(i)}}{(\lambda^{(i)}+\lambda^{(i+1)})(\lambda^{(i)}+\lambda^{(i+2)})} && \text{if } 0 \le \lambda^{(i)} \le 1,\ 0 \le \lambda^{(i+1)}, \lambda^{(i+2)} < 1, \\ B^{(i)}(a^{(i+1)}) &= B^{(i)}(a^{(i+2)}) = 0 && \text{otherwise.} \end{aligned} \tag{2-3}$$

The graphs of the three rational bubble functions are depicted in Figure 1 on the reference triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$. In this case, the barycentric coordinates reduce to $\hat\lambda^{(1)} = x_1$, $\hat\lambda^{(2)} = x_2$ and $\hat\lambda^{(3)} = 1 - x_1 - x_2$. Therefore, the three rational bubble functions on the reference triangle are given by

$$B^{(1)} = \frac{x_1 x_2^2 (1-x_1-x_2)^2}{(x_1+x_2)(1-x_2)}, \quad B^{(2)} = \frac{x_2 x_1^2 (1-x_1-x_2)^2}{(x_1+x_2)(1-x_1)},$$
$$B^{(3)} = \frac{x_1^2 x_2^2 (1-x_1-x_2)}{(1-x_1)(1-x_2)}. \tag{2-4}$$

In [Guzmán and Neilan 2014a] (see also [Ciarlet 1978, pp. 347–348]), the following lemma pertaining to the rational bubble functions was established.

Lemma 2.1. *For each $T \in \mathcal{T}_h$, the following hold ($i = 1, 2, 3$):*

$$B^{(i)} \in C^1(\bar T) \cap W^{2,\infty}(T), \quad B^{(i)}|_{\partial T} = 0, \quad \nabla B^{(i)}(a^{(j)}) = 0 \ (j = 1, 2, 3). \tag{2-5}$$

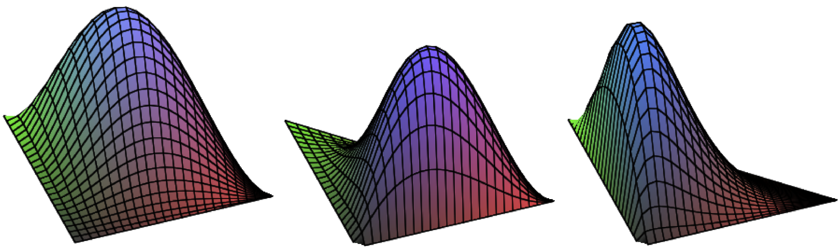

Figure 1. The graphs of the three bubble functions on the reference triangle with vertices $(0,0)$ (left), $(0,1)$ (middle) and $(1,0)$ (right).

We end this section by stating some well known vector calculus results, which will be used extensively in the analysis below.

Theorem 2.2 (inverse function theorem [Spivak 1998]). *Suppose that $G:\mathbb{R}^n\to\mathbb{R}^n$ is continuously differentiable in an open set containing a point a with* $\det(DG(a))\neq 0$. *Then there is an open set V containing a and an open set W containing $G(a)$ such that $G:V\to W$ has a continuous inverse $G^{-1}:W\to V$ which is differentiable for all $y\in W$. Moreover, there holds*

$$D(G^{-1})(y)=[DG(G^{-1}(y))]^{-1}. \tag{2-6}$$

Lemma 2.3 (Bramble–Hilbert lemma [Ciarlet 1978]). *Let D be an open subset of $\mathbb{R}^n$ $(n\geq 1)$ with a Lipschitz-continuous boundary. For some $k\geq 0$ and some number $p\in[0,\infty]$, let φ be a continuous linear form on the space $W^{k+1,p}(D)$ with the property that*

$$\varphi(p)=0 \quad \textit{for all } p\in P_k(D),$$

where $P_k(D)$ is the set of all polynomials up to order k on D. Then there exists a positive constant $C>0$ depending on D such that for all $v\in W^{k+1,p}(D)$,

$$|\varphi(v)|\leq C\|\varphi\|_{W^{-k-1,p'}(D)}|v|_{W^{k+1,p}(D)},$$

where $\|\cdot\|_{W^{-k-1,p'}(D)}$ is defined by (2-2).

Theorem 2.4 (Sard's theorem [Spivak 1998]). *Let $G:\mathbb{R}^2\to\mathbb{R}^2$ be a continuously differentiable mapping. Let X be the set of points x in $\mathbb{R}^2$ at which the Jacobian matrix $DG(x)$ has rank less than 2. Then $G(X)$ has Lebesgue measure 0 in $\mathbb{R}^2$.*

Corollary 2.5. *Let $V,U\subset\mathbb{R}^2$ be two open and bounded sets, and let $G:V\to U$ be a continuously differentiable mapping that is surjective onto U; that is, $G(V)=U$.*

Define the set $X = \{x \in V : DG(x)$ *does not have full rank*$\}$. *Then for any continuous function* $f \in C^0(U)$,

$$\int_U f(y)\,dy = \int_{G(V)\setminus G(X)} f(y)\,dy.$$

Proof. Since $U = G(V)$, we have

$$\int_U f(y)\,dy = \int_{G(V)} f(y)\,dy = \int_{G(V)\setminus G(X)} f(y)\,dy + \int_{G(X)} f(y)\,dy.$$

From Sard's lemma, $G(x)$ has Lebesgue measure 0. Therefore $\int_{G(X)} f(y)\,dy = 0$, and so

$$\int_U f(y)\,dy = \int_{G(V)} f(y)\,dy = \int_{G(V)\setminus G(X)} f(y)\,dy. \qquad \square$$

3. A partitioned Duffy transform

In this section, we describe a partitioned Duffy transform which essentially removes the singularities of the rational bubble functions defined by (2-3). Basically the strategy is to subdivide each triangle into four subtriangles by a red refinement and then apply the Duffy transform to each subtriangle that shares a vertex with the parent triangle. To describe this procedure in further detail, we require some notation.

Denote by $\widehat{T}$ the unit triangle with vertices $\hat{a}^{(1)} := (1, 0)$, $\hat{a}^{(2)} := (0, 1)$ and $\hat{a}^{(3)} := (0, 0)$, and let $\{\widehat{K}^{(i)}\}_{i=1}^{4}$ be the four subtriangles of $\widehat{T}$ obtained by connecting the three midpoints of each edge of $\widehat{T}$ (see Figure 2), where $\hat{a}^{(i)}$ is a vertex of $\widehat{K}^{(i)} (i = 1, 2, 3)$. We denote the three vertices of $\widehat{K}^{(i)}$ by $\{\hat{b}_j^{(i)}\}_{j=1}^{3}$ oriented in a counterclockwise fashion and labeled such that

$$\hat{a}^{(i)} = \hat{b}_i^{(i)} \quad (i = 1, 2, 3).$$

The vertices $\{\hat{b}_j^{(4)}\}_{j=1}^{3}$ can be labeled arbitrarily. Define $\widehat{F}_i : \widehat{T} \to \widehat{K}^{(i)}$ to be the affine mapping such that $\widehat{F}_i(\hat{a}^{(i)}) = \hat{b}^{(i)}$ $(i = 1, 2, 3)$; that is,

$$\widehat{F}_1(y) = \bigl(\tfrac{1}{2}(y_1 + 1), \tfrac{1}{2} y_2\bigr), \tag{3-1}$$

$$\widehat{F}_2(y) = \bigl(\tfrac{1}{2}(1 - y_1 - y_2), \tfrac{1}{2}(y_1 + 1)\bigr), \tag{3-2}$$

$$\widehat{F}_3(y) = \bigl(\tfrac{1}{2} y_2, -\tfrac{1}{2}(y_1 + y_2 - 1)\bigr). \tag{3-3}$$

In the case $i = 4$, $\widehat{F}_i$ can be any one of the possible affine mappings that takes $\widehat{T}$ onto $\widehat{K}^{(4)}$. Denote by $\widehat{Q} := (0, 1)^2$ the unit square and define the Duffy transform $\widehat{S} : \widehat{Q} \to \widehat{T}$ as

$$\widehat{S}(\hat{s}) := (\hat{s}_1, \hat{s}_2(1 - \hat{s}_1))^t. \tag{3-4}$$

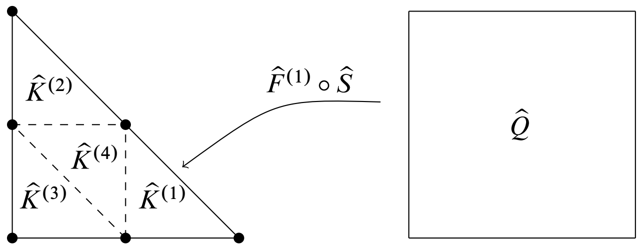

Figure 2. A pictorial description of the notation used.

Finally for a function $f : \widehat{T} \to \mathbb{R}$, we set

$$\hat{f}_i(\hat{s}) := f(\widehat{F}_i(\widehat{S}(\hat{s})))\ (i = 1, 2, 3) \quad \text{and} \quad \hat{f}_4(\hat{y}) = f(F_4(\hat{y})). \tag{3-5}$$

We note that $\hat{f}_i : \widehat{Q} \to \mathbb{R}$ $(i = 1, 2, 3)$, whereas $\hat{f}_4 : \widehat{T} \to \mathbb{R}$.

Lemma 3.1. *For $i = 1, 2, 3$, define $\widehat{G}_i = F_i \circ \widehat{S} : \widehat{Q} \to \widehat{K}^{(i)}$. Then there holds*

$$(\widehat{\nabla_{\hat{x}} f})_i(\hat{s}) = (D_{\hat{s}} G_i(\hat{s}))^{-t} \nabla_{\hat{s}} \hat{f}_i(\hat{s}),$$

for any function f satisfying $\hat{f}_i \in C^1(\widehat{Q})$. Here, $\nabla_{\hat{x}}$ and $\nabla_{\hat{s}}$ denotes the gradient with respect to $\hat{x}$ and $\hat{s}$, respectively $D_{\hat{s}} = (\nabla_{\hat{s}})^t$, and $(D_{\hat{s}} G_i(\hat{s}))^{-t}$ denote the inverse matrix of the transpose of $D_{\hat{s}} G_i(\hat{s})$.

Proof. For ease of notation, we omit the subscript i in the arguments below.

By (3-5) and the definition of $\widehat{G}$, we have $\hat{f}(\hat{s}) = f(\widehat{G}(\hat{s}))$. Now let $\hat{x} = G(\hat{s})$ so that $\hat{f}(\hat{s}) = f(\hat{x})$ and $\hat{s} = G^{-1}(\hat{x})$. We then have

$$\hat{s}_k = (G^{-1})_k(\hat{x}) \quad \text{and} \quad \frac{\partial \hat{s}_k}{\partial \hat{x}_j} = \frac{\partial}{\partial \hat{x}_j}(G^{-1})_k(\hat{x}).$$

Letting $D_{\hat{x}} G^{-1}(\hat{x})$ be the Jacobian of G^{-1}, we have

$$\frac{\partial \hat{s}_k}{\partial \hat{x}_j} = (DG^{-1})_{kj}(\hat{x}). \tag{3-6}$$

Therefore by the chain rule and (3-6), we have

$$\begin{aligned}\left(\frac{\partial f}{\partial \hat{x}_j} \circ G\right)(\hat{s}) &= \sum_{k=1}^{2} \frac{\partial \hat{f}}{\partial \hat{s}_k}(\hat{s}) \frac{\partial \hat{s}_k}{\partial \hat{x}_j} = \sum_{k=1}^{2} \frac{\partial \hat{f}}{\partial \hat{s}_k}(\hat{s}) (D_{\hat{x}} G^{-1}(\hat{x}))_{kj} \\ &= \sum_{k=1}^{2} \big((D_{\hat{x}} G^{-1}(\hat{x}))_{jk}\big)^t \frac{\partial \hat{f}}{\partial \hat{s}_k}(\hat{s}) = \big((D_{\hat{x}} G^{-1})^t(\hat{x}) \nabla_{\hat{s}} \hat{f}(\hat{s})\big)_j.\end{aligned}$$

It then follows that

$$(\nabla_x f \circ G)(\hat{s}) = (D_{\hat{x}} G^{-1}(\hat{x}))^t \nabla_{\hat{s}} \hat{f}(\hat{s}). \tag{3-7}$$

Now by the implicit function theorem (see Theorem 2.2), we have

$$D_{\hat{x}} G^{-1}(\hat{x}) = [DG(G^{-1}(\hat{x}))]^{-1} = [DG(\hat{s})]^{-1}.$$

Therefore by (3-7) and (3-5), we have

$$\widehat{\nabla_{\hat{x}} f}(\hat{s}) = (\nabla_{\hat{x}} f \circ G)(\hat{s}) = (D_{\hat{s}} G(\hat{s}))^{-t} \nabla_{\hat{s}} \hat{f}(\hat{s}). \qquad \square$$

Lemma 3.2. *Let $\widehat{G}_i = \widehat{F}_i \circ \widehat{S} : \widehat{Q} \to \widehat{K}^{(i)}$. Then,*

$$\widehat{(D^2_{\hat{x}} f)}_i(\hat{s}) = D_{\hat{s}}\big((D_{\hat{s}} G_i(\hat{s}))^{-t} \nabla_{\hat{s}} \hat{f}_i(\hat{s})\big) D_{\hat{s}} G_i(\hat{s})^{-1}.$$

Proof. Again, we omit the subscript i in the proof for ease of notation.

From Lemma 3.1 we have

$$\left(\frac{\partial f}{\partial \hat{x}_j} \circ G\right)(\hat{s}) = \sum_{k=1}^{2} \big(D_{\hat{s}} G(\hat{s})\big)^{-t}_{jk} \frac{\partial \hat{f}}{\partial \hat{s}_k}(\hat{s}).$$

Set

$$r_{jk}(\hat{s}) := (D_{\hat{s}} G(\hat{s}))^{-t}_{jk} \frac{\partial \hat{f}}{\partial \hat{s}_k}(\hat{s}) \quad \text{so that} \quad \frac{\partial f}{\partial \hat{x}_j}(\hat{x}) = \sum_{k=1}^{2} r_{jk}(\hat{s}).$$

Then by the chain rule, (3-6), and the inverse function theorem, we have

$$\begin{aligned}\left(\frac{\partial^2 f}{\partial \hat{x}_j \partial \hat{x}_l} \circ G\right)(\hat{s}) &= \sum_{k=1}^{2} \sum_{m=1}^{2} \frac{\partial r_{jk}}{\partial \hat{s}_m}(\hat{s}) \frac{\partial \hat{s}_m}{\partial \hat{x}_l} = \sum_{k=1}^{2} \sum_{m=1}^{2} \frac{\partial r_{jk}}{\partial \hat{s}_m}(\hat{s}) (D_{\hat{x}} G^{-1})_{ml}(\hat{x}) \\ &= \sum_{k=1}^{2} \sum_{m=1}^{2} \frac{\partial r_{jk}}{\partial \hat{s}_m}(\hat{s}) (D_{\hat{x}} G)^{-1}_{ml}(\hat{s}).\end{aligned}$$

Now since

$$r_{jk}(\hat{s}) = (D_{\hat{s}} G(\hat{s}))^{-t}_{jk} \frac{\partial \hat{f}}{\partial \hat{s}_k}(\hat{s}),$$

we have

$$\Big(\frac{\partial^2 f}{\partial \hat{x}_j \partial \hat{x}_l} \circ G\Big)(\hat{s}) = \sum_{k=1}^{2} \sum_{m=1}^{2} \frac{\partial}{\partial \hat{s}_m} \bigg((D_{\hat{s}} G)^{-t}_{jk}(\hat{s}) \frac{\partial \hat{f}}{\partial \hat{s}_k}(\hat{s}) \bigg) ((D_{\hat{s}} G)^{-1}_{ml}(\hat{s}))$$
$$= \sum_{m=1}^{2} \frac{\partial}{\partial \hat{s}_m} ((D_{\hat{s}} G)^{-t}(\hat{s}) \nabla_{\hat{s}} \hat{f}(\hat{s}))_j ((D_{\hat{s}} G)^{-1}_{ml}(\hat{s}))$$
$$= \sum_{m=1}^{2} \big(D_{\hat{s}}((D_{\hat{s}} G)^{-t}(\hat{s}) \nabla_{\hat{s}} \hat{f}(\hat{s})) \big)_{jm} ((D_{\hat{s}} G)^{-1}_{ml}(\hat{s}))$$
$$= \Big(\big(D_{\hat{s}}((D_{\hat{s}} G)^{-t}(\hat{s}) \nabla_{\hat{s}} \hat{f}(\hat{s})) \big) ((D_{\hat{s}} G)^{-1}(\hat{s})) \Big)_{jl}.$$

It then follows that

$$(\widehat{D^2_{\hat{x}} f})_i(\hat{s}) = (D^2_{\hat{x}} f \circ \widehat{G}_i)(\hat{s}) = D_{\hat{s}}((D_{\hat{s}} G_i(\hat{s}))^{-t} \nabla_{\hat{s}} \hat{f}_i(\hat{s})) D_{\hat{s}} G_i(\hat{s})^{-1}. \qquad \square$$

We are now ready to state the main result of this section.

Lemma 3.3. *Let $B^{(j)}$ be the rational edge bubble (2-3) defined on the reference triangle $\widehat{T}$ with vertices* $(1,0)$, $(0,1)$, *and* $(0,0)$. *Then* $(i = 1,2,3)$,

$$\widehat{B^{(j)}}_i \in C^\infty(\widehat{Q}), \quad (\widehat{\nabla_{\hat{x}} B^{(j)}})_i \in [C^\infty(\widehat{Q})]^2, \quad (\widehat{D^2_{\hat{x}} B^{(j)}})_i \in [C^\infty(\widehat{Q})]^{2\times 2},$$

and

$$\widehat{B^{(j)}}_4 \in C^\infty(\widehat{T}), \quad (\widehat{\nabla_{\hat{x}} B^{(j)}})_4 \in [C^\infty(\widehat{T})]^2, \quad (\widehat{D^2_{\hat{x}} B^{(j)}})_4 \in [C^\infty(\widehat{T})]^{2\times 2}.$$

Here, $\nabla_{\hat{x}}$ *denotes the gradient with respect to $\hat{x}$ and $D^2_{\hat{x}}$ denotes the Hessian with respect to $\hat{x}$.*

Remark 3.4. Essentially, Lemma 3.3 states that if we map the rational bubble functions' derivatives to the unit square via the partitioned Duffy transform, then the resulting function is C^∞.

Proof. Due to the symmetry of the rational edge bubbles, it suffices to prove the result for the function

$$B(\hat{x}) := \frac{\hat{x}_1^2 \hat{x}_2^2 (1 - \hat{x}_1 - \hat{x}_2)}{(1 - \hat{x}_1)(1 - \hat{x}_2)} \in C^1(\widehat{T}) \cap W^{2,\infty}(\widehat{T}) \tag{3-8}$$

(see (2-4)). Since $B(\hat{x})$ has singularities only at the vertices $(1,0)$ and $(0,1)$, we have $B|_{K_4} \in C^\infty(\widehat{K}_4)$. It is then trivial to see that

$$\widehat{B}_4 \in C^\infty(\widehat{T}), \quad (\widehat{\nabla_{\hat{x}} B})_4 \in [C^\infty(\widehat{T})]^2, \quad (\widehat{D^2_{\hat{x}} B})_4 \in [C^\infty(\widehat{T})]^{2\times 2}.$$

Next, a direct calculation shows that

$$\widehat{B}_1(\hat{s}) = B(F_1(S(\hat{s}))) = B\big(\tfrac{1}{2}(\hat{s}_1+1), \tfrac{1}{2}(\hat{s}_2(1-\hat{s}_1))\big) \\ = \frac{\hat{s}_2^2(\hat{s}_2-1)(\hat{s}_1-1)^2(\hat{s}_1+1)^2}{8(2-\hat{s}_2+\hat{s}_2\hat{s}_1)} \in C^\infty(\widehat{Q}), \tag{3-9a}$$

$$\widehat{B}_2(\hat{s}) = B(F_2(S(\hat{s}))) = B\big(\tfrac{1}{2}(1-\hat{s}_1-\hat{s}_2(1-\hat{s}_1)), \tfrac{1}{2}(\hat{s}_1+1)\big) \\ = \frac{1}{8}\frac{(s_1-1)s_2(s_1+1)^2(s_2-1)(1-s_1-s_2+s_2s_1)}{8(1+s_1+s_2-s_2s_1)} \in C^\infty(\widehat{Q}), \tag{3-9b}$$

$$\widehat{B}_3(\hat{s}) = B(F_3(S(\hat{s}))) = B\big(\tfrac{1}{2}\hat{s}_2(1-\hat{s}_1), -\tfrac{1}{2}(\hat{s}_1+\hat{s}_2(1-\hat{s}_1))+\tfrac{1}{2}\big) \\ = \frac{\hat{s}_2^2(\hat{s}_1-1)^2(1-\hat{s}_1-\hat{s}_2+\hat{s}_2\hat{s}_1)^2(1+\hat{s}_1)}{8(2-\hat{s}_2+\hat{s}_2\hat{s}_1)(1+\hat{s}_1+\hat{s}_2-\hat{s}_2\hat{s}_1)} \in C^\infty(\widehat{Q}). \tag{3-9c}$$

Then from Lemma 3.1, we have

$$(\widehat{\nabla_{\hat{x}} B})_i(\hat{s}) = (D_{\hat{s}}G_i(\hat{s}))^{-t}\nabla_{\hat{s}}\widehat{B}(\hat{s}), \tag{3-10}$$

where $\nabla_{\hat{s}}$ denotes the gradient with respect to $\hat{s}$, $D_{\hat{s}} = (\nabla_{\hat{s}})^t$, and $(D_{\hat{s}}G_i(\hat{s}))^{-t}$ denotes the inverse matrix of the transpose of $D_{\hat{s}}G_i(\hat{s})$. Using the identity $DF_i = 2|K_i| = 1/2$ and the chain rule, we have

$$DG_i(\hat{s}) = D(F_i(S(\hat{s}))) = DF_i(S(\hat{s}))DS(\hat{s}) = \tfrac{1}{2}DS(\hat{s}).$$

It then follows that $(DG_i(\hat{s}))^{-t} = 2(DS(\hat{s}))^{-t}$; that is,

$$(DG_i(\hat{s}))^{-t} = \begin{pmatrix} 2 & 2\hat{s}_2/(1-\hat{s}_1) \\ 0 & 2/(1-\hat{s}_1) \end{pmatrix}. \tag{3-11}$$

By (3-9), we see that the derivatives $\partial\widehat{B}_i/\partial\hat{s}_2$ $(i=1,2,3)$ all have a factor $(1-\hat{s}_1)^2$. In particular, we may write

$$\nabla_{\hat{s}}\widehat{B}_i = \begin{pmatrix} g_i^{(1)}(\hat{s})(1-\hat{s}_1) \\ g_i^{(2)}(\hat{s})(1-\hat{s}_1)^2 \end{pmatrix} \tag{3-12}$$

for some $g_i^{(1)}, g_i^{(2)} \in C^\infty(\widehat{Q})$. Combining (3-12) with (3-10) and (3-11) we see that $(\widehat{\nabla_{\hat{x}} B})_i \in [C^\infty(\widehat{Q})]^2$.

Continuing, we use Lemma 3.2 and the inverse function theorem to obtain

$$(\widehat{D^2_{\hat{x}} B})_i(\hat{s}) = D_{\hat{s}}\big((D_{\hat{s}}G_i(\hat{s}))^{-t}\nabla_{\hat{s}}\widehat{B}(\hat{s})\big)D_{\hat{s}}G_i(\hat{s})^{-1}. \tag{3-13}$$

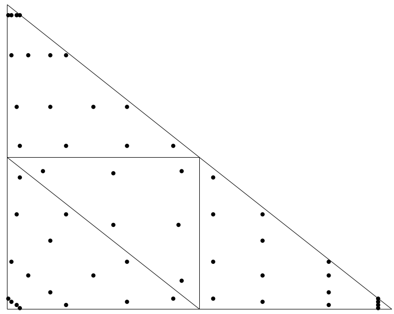

Figure 3. The location of the nodes $\hat{x}^{(j)}$ with $L = 16$ and $M = 6$.

By (3-12) and (3-11), we have

$$D_{\hat{s}}\big((D_{\hat{s}}G_i(\hat{s}))^{-t}\nabla_{\hat{s}}\widehat{B}(\hat{s})\big) = \begin{pmatrix} g_i^{(1,1)}(\hat{s}) & g_i^{(1,2)}(\hat{s})(1-\hat{s}_1) \\ g_i^{(2,1)}(\hat{s}) & g_i^{(2,2)}(\hat{s})(1-\hat{s}_1) \end{pmatrix},$$

for some $g^{(i,j)}(\hat{s}) \in C^\infty(\widehat{Q})$. It then follows from the definition of $D_{\hat{s}}G_i(\hat{s})^{-1}$ (see (3-11)) and (3-13) that

$$(\widehat{D^2_{\hat{x}}B})_i(\hat{s}) \in [C^\infty(\widehat{Q})]^{2\times 2}. \qquad \square$$

4. A quadrature rule based upon the partitioned Duffy transform

We now build quadrature schemes for the integral $\int_{\widehat{T}} f(\hat{x})\,d\hat{x}$ based upon the partitioned Duffy transform described above. To this end, we let $\{\hat{s}^{(j)}, \hat{\theta}^{(j)}\}_{j=1}^{L}$ be a tensor product Gaussian quadrature rule on the unit square $\widehat{Q}$, and we let $\{\hat{y}^{(j)}, \hat{\varrho}^{(j)}\}_{j=1}^{M}$ be a quadrature rule on the unit triangle $\widehat{T}$. We then map the quadrature points and weights on $\widehat{Q}$ to the subtriangles $\widehat{K}^{(i)}$ $(i = 1, 2, 3)$ by the formulas $\hat{x}^{((i-1)L+j)} = \widehat{F}_i(\widehat{S}(\hat{s}^{(j)}))$ and

$$\hat{\omega}^{((i-1)L+j)} = \tfrac{1}{2}(1-\hat{s}_1^{(j)})\hat{\theta}^{(j)} \quad (j = 1, 2, \dots, L).$$

We map the quadrature points and weights $\{\hat{y}^{(j)}, \hat{\varrho}^{(j)}\}_{j=1}^{M}$ to $\widehat{K}^{(4)}$ by

$$\hat{x}^{3L+j} = \widehat{F}_4(\hat{y}^{(j)}) \quad \text{and} \quad \hat{\omega}^{(3L+j)} = \frac{1}{2}\hat{\varrho}^{(j)} \quad (j = 1, 2, \dots, M).$$

The new quadrature scheme on $\widehat{T}$ is then given by $\{\hat{x}^{(j)}, \hat{\omega}^{(j)}\}_{j=1}^{3L+M}$ (see Figure 3).

Remark 4.1. By Figure 3, we see that the quadrature points are clustered near the vertices of the (macro) triangle $\widehat{T}$. On the other hand, the weights defined by $\hat{\omega}^{((i-1)L+j)} = \frac{1}{2}(1-\hat{s}_1^{(j)})\hat{\theta}^{(j)}$ are small near the vertices since the line $\hat{s}_1 = 1$ on $\widehat{Q}$ is mapped to each of the vertices of $\widehat{T}$.

Note by a change of variables and Sard's theorem, we have

$$
\begin{aligned}
\int_{\widehat{T}} f(\hat{x})\,d\hat{x} &= \sum_{i=1}^{4} \int_{\widehat{K}^{(i)}} f(\hat{x})\,d\hat{x} = \sum_{i=1}^{4} \int_{\widehat{T}} f(\hat{F^{(i)}}(\hat{y}))|D\widehat{F}^{(i)}(\hat{y})|\,d\hat{y} \\
&= \sum_{i=1}^{4} 2|\widehat{K}^{(i)}| \int_{\widehat{T}} f(\widehat{F}_i(\hat{y}))\,d\hat{y} \\
&= \frac{1}{2}\sum_{i=1}^{3} \int_{\widehat{T}} f(\widehat{F}_i(\hat{y}))\,d\hat{y} + \frac{1}{2}\int_{\widehat{T}} \hat{f}_4(\hat{y})\,d\hat{y} \\
&= \frac{1}{2}\sum_{i=1}^{3} \int_{\widehat{Q}} f\big(\widehat{F}_i(\widehat{S}(\hat{s}))\big)|D_{\hat{s}}\widehat{S}(\hat{s})|\,d\hat{s} + \frac{1}{2}\int_{\widehat{T}} \hat{f}_4(\hat{y})\,d\hat{y} \\
&= \frac{1}{2}\sum_{i=1}^{3} \int_{\widehat{Q}} \hat{f}_i(\hat{s})(1-\hat{s}_1)\,d\hat{s} + \frac{1}{2}\int_{\widehat{T}} \hat{f}_4(\hat{y})\,d\hat{y}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\sum_{j=1}^{3L+M} \hat{\omega}^{(j)} f(\hat{x}^{(j)}) \\
&\qquad = \sum_{i=1}^{3}\sum_{j=1}^{L} \hat{\omega}^{(i-1)L+j} f(\hat{x}^{((i-1)L+j)}) + \sum_{j=1}^{M} \hat{\omega}^{(3L+j)} f(\hat{x}^{(3L+j)}) \\
&\qquad = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{L}(1-\hat{s}^{(j)})\theta^{(j)} f(\widehat{F}_i(\widehat{S}(\hat{s}^{(j)}))) + \frac{1}{2}\sum_{j=1}^{M} \hat{\varrho}^{(j)} f(\widehat{F}_4(\hat{y}^{(j)})) \\
&\qquad = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{L} \theta^{(j)} \hat{f}_i(\hat{s}^{(j)})(1-\hat{s}_1^{(j)}) + \frac{1}{2}\sum_{j=1}^{M} \hat{\varrho}^{(j)} \hat{f}_4(\hat{y}^{(j)}).
\end{aligned}
$$

It then follows from these two identities that the error can be written as

$$
\begin{aligned}
E_{\widehat{T}}(f) &:= \int_{\widehat{T}} f(\hat{x})\,d\hat{x} - \sum_{j=1}^{3L+M} \hat{\omega}^{(j)} f(\hat{x}^{(j)}) \qquad (4\text{-}1) \\
&= \frac{1}{2}\sum_{i=1}^{3}\bigg(\int_{\widehat{Q}} \hat{f}_i(\hat{s})(1-\hat{s}_1)\,d\hat{s} - \sum_{j=1}^{L} \theta^{(j)} \hat{f}_i(\hat{s}^{(j)})(1-\hat{s}_1^{(j)}) \bigg) \\
&\qquad + \frac{1}{2}\bigg(\int_{\widehat{T}} \hat{f}_4(\hat{y})\,d\hat{y} - \sum_{j=1}^{M} \hat{\varrho}^{(j)} \hat{f}_4(\hat{y}^{(j)}) \bigg).
\end{aligned}
$$

Theorem 4.2. *Let $B^{(j)}$ be any of the three rational edge bubbles defined on the reference triangle. Then for any multi-index $\alpha = (\alpha_1, \alpha_2)$ with $0 \le |\alpha| \le 2$, there exists $C_\alpha > 0$ and $\delta_\alpha > 0$ such that*

$$\left| E_{\widehat{T}}\left(\frac{\partial^{|\alpha|} B^{(j)}}{\partial \hat{x}^\alpha}\right)\right| \le C_\alpha(\exp(-\delta_\alpha M) + \exp(-\delta_\alpha L)).$$

Proof. This result follows from (4-1), Lemma 3.3, and standard estimates of Gaussian quadrature [Sauter and Schwab 2011, pp. 324–325]. □

We now discuss the quadrature rule on an arbitrary triangle $T \in \mathcal{T}_h$. This is done in a natural way. Namely, letting $F_T : \widehat{T} \to T$ denote the affine transformation, we define the quadrature scheme $\{x_T^{(j)}, \omega_T^{(j)}\}_{j=1}^{3L+M}$ by $x_T^{(j)} = F_T(\hat{x}^{(j)})$ and $\omega_T^{(j)} = 2|T|\hat{\omega}^{(j)}$. The error of the scheme is then given by

$$E_T(f) := \int_T f(x)\,dx - \sum_{j=1}^{3L+M} \omega_T^{(j)} f(x_T^{(j)}).$$

Using the Bramble–Hilbert lemma, we can obtain the following result.

Theorem 4.3. *Suppose that the quadrature schemes*

$$\{\hat{s}^{(j)}, \hat{\theta}^{(j)}\}_{j=1}^{L} \quad \textit{and} \quad \{\hat{y}^{(j)}, \hat{\varrho}^{(j)}\}_{j=1}^{M}$$

are exact for polynomials of degree at most m on $\widehat{Q}$ and $\widehat{T}$, respectively. For a given triangle $T \in \mathcal{T}_h$, let f be a continuous function on $\overline{T}$ and $\hat{f}_i \in H^{m+1}(\widehat{Q})$ and $\hat{f}_4 \in H^{m+1}(\widehat{T})$, where

$$\hat{f}_i(\hat{s}) = f\big(F_T(\widehat{F}_i(\widehat{S}(\hat{s})))\big) \quad \textit{and} \quad \hat{f}_4(\hat{y}) = f(F_T(\widehat{F}_4(\hat{y}))).$$

Then,

$$E_T(f) \le C h_T^2 \left(\sum_{i=1}^{3} |\hat{f}_i|_{H^{m+1}(\widehat{Q})} + |\hat{f}_4|_{H^{m+1}(\widehat{T})} \right).$$

Proof. Let $\hat{f} \in C^0(\widehat{T})$ be defined as $\hat{f}(\hat{x}) = f(F_T(\hat{x}))$ so that $\hat{f}_i(\hat{s}) = \hat{f}(\widehat{F}_i(\widehat{S}(\hat{s})))$ and $\hat{f}_4(\hat{y}) = \hat{f}(\widehat{F}_4(\hat{y}))$. Then by a change of variables and (4-1), we have

$$\begin{aligned} E_T(f) &= 2|T| E_{\widehat{T}}(\hat{f}) \\ &= |T| \left(\sum_{i=1}^{3} \left(\int_{\widehat{Q}} \hat{f}_i(\hat{s})(1-\hat{s}_1)\,d\hat{s} - \sum_{j=1}^{L} \hat{\theta}^{(j)} \hat{f}_i(\hat{s}^{(j)})(1-\hat{s}_1^{(j)}) \right) \right. \\ &\qquad \left. + \int_{\widehat{T}} \hat{f}_4(\hat{y})\,d\hat{y} - \sum_{j=1}^{M} \hat{\varrho}^{(j)} \hat{f}_4(\hat{y}^{(j)}) \right). \end{aligned}$$

It then follows from the Bramble–Hilbert lemma that

$$E_T(f) \le C|T| \left(\sum_{i=1}^{3} |(1-\hat{s}_1)\hat{f}_i|_{H^{m+1}(\hat{Q})} + |\hat{f}_4|_{H^{m+1}(\hat{T})} \right)$$
$$\le Ch_T^2 \left(\sum_{i=1}^{3} |\hat{f}_i|_{H^{m+1}(\hat{Q})} + |\hat{f}_4|_{H^{m+1}(\hat{T})} \right). \qquad \square$$

Corollary 4.4. *Let $B^{(j)}$ be any of the three rational edge bubbles defined on an arbitrary triangle $T \in \mathcal{T}_h$. Then for any multi-index $\alpha = (\alpha_1, \alpha_2)$ with $0 \le |\alpha| \le 2$, there exists $C_\alpha > 0$ and $\delta_\alpha > 0$ such that*

$$\left| E_T\left(\frac{\partial^{|\alpha|} B^{(j)}}{\partial \hat{x}^\alpha} \right) \right| \le h_T^2 C_\alpha (\exp(-\delta_\alpha M) + \exp(-\delta_\alpha L)).$$

Proof. This follows directly from Theorem 4.2 and Theorem 4.3. $\square$

5. Numerical experiments on a single triangle

In this section, we implement the quadrature scheme discussed in the previous section on the reference triangle $\widehat{T}$ and validate the results of Theorem 4.2. In all of the numerical experiments, we approximate the integral of the third function in (2-4), that is,

$$B(x) := B^{(3)}(x) = \frac{x_1^2 x_2^2 (1 - x_1 - x_2)}{(1-x_1)(1-x_2)}.$$

For comparison, we first implement some standard Gauss–Legendre quadrature schemes for the rational function and its first and second derivatives. Using the mathematical software package Maple, we find the exact value of the integrals to be

$$\int_{\hat{T}} B(\hat{x})\, d\hat{x} = -\tfrac{1}{6}\pi^2 + \tfrac{593}{360} \approx 0.00228815482278675 01,$$
$$\int_{\hat{T}} \frac{\partial B}{\partial \hat{x}_1}\, d\hat{x} = 0, \qquad \int_{\hat{T}} \frac{\partial^2 B}{\partial \hat{x}_1^2}\, d\hat{x} = -\tfrac{1}{6}.$$

The numerical results are depicted in Table 1. As can be seen from the tables, the errors behave sporadically. At best, the errors converge algebraically, but certainly not exponentially. Moreover, even for high order quadrature rules, we are only able to recover four digits of accuracy. This also proves true for Gaussian quadrature applied to the function's first and second derivatives (see Table 1). We can attribute these poor results to the two singularities at the vertices $(1, 0)$ and $(0, 1)$.

Next, we implement the quadrature scheme using the Duffy transform described in Section 4. Of particular interest are the integrals on the subtriangles containing

	L	degree	approx. integral	absolute error	relative error	rate
$\widehat{B}$	1	1	$4.62\cdot10^{-3}$	$2.34\cdot10^{-3}$	1.023	
	3	2	$-9.74\cdot10^{-2}$	$9.96\cdot10^{-2}$	4.35	3.41
	6	4	$2.36\cdot10^{-3}$	$7.83\cdot10^{-5}$	$3.42\cdot10^{-2}$	10.31
	7	5	$2.29\cdot10^{-3}$	$3.51\cdot10^{-6}$	$1.53\cdot10^{-3}$	20.15
	16	8	$2.28\cdot10^{-3}$	$1.71\cdot10^{-6}$	$7.51\cdot10^{-4}$	0.86
	19	9	$2.28\cdot10^{-3}$	$2.81\cdot10^{-7}$	$1.22\cdot10^{-4}$	10.54
	28	11	$2.28\cdot10^{-3}$	$1.21\cdot10^{-7}$	$5.29\cdot10^{-5}$	2.17
	37	13	$2.30\cdot10^{-3}$	$2.08\cdot10^{-5}$	$9.11\cdot10^{-3}$	18.48
1st derivative of $\widehat{B}$	1	1	$-2.08\cdot10^{-2}$	$2.08\cdot10^{-2}$		
	3	2	$8.78\cdot10^{-1}$	$8.78\cdot10^{-1}$		5.39
	6	4	$-2.02\cdot10^{-3}$	$2.02\cdot10^{-3}$		8.76
	7	5	$-1.27\cdot10^{-3}$	$1.27\cdot10^{-3}$		2.08
	16	8	$1.05\cdot10^{-4}$	$1.05\cdot10^{-4}$		5.29
	19	9	$1.02\cdot10^{-4}$	$1.02\cdot10^{-4}$		0.24
	28	11	$8.88\cdot10^{-6}$	$8.88\cdot10^{-6}$		12.18
	37	13	$4.12\cdot10^{-4}$	$4.12\cdot10^{-4}$		22.97
2nd derivative of $\widehat{B}$	1	1	$5.80\cdot10^{-2}$	$2.18\cdot10^{-1}$	-1.31	
	3	2	-7.13	6.97	-41.83	4.99
	6	4	$-1.39\cdot10^{-1}$	$2.69\cdot10^{-2}$	$-1.61\cdot10^{-1}$	8.01
	7	5	$-1.45\cdot10^{-1}$	$2.09\cdot10^{-2}$	-1.25	1.12
	16	8	$-1.62\cdot10^{-1}$	$4.06\cdot10^{-3}$	$-2.43\cdot10^{-2}$	3.49
	19	9	$-1.62\cdot10^{-1}$	$3.70\cdot10^{-3}$	$-2.22\cdot10^{-2}$	0.78
	28	11	$-1.65\cdot10^{-1}$	$8.41\cdot10^{-4}$	$-5.05\cdot10^{-3}$	7.37
	37	13	$-1.75\cdot10^{-1}$	$8.60\cdot10^{-3}$	$-5.16\cdot10^{-2}$	13.91

Table 1. Gaussian quadrature results for the function $\widehat{B}$ of (3-8) and its first two derivatives. Rates of convergence are with respect to the relative error and the number of points L.

the singularities. For the sake of brevity we omit $\widehat{B}$ and its derivatives over the subtriangle $\widehat{K}_4 = \{(0,5,0),(0.5,0.5),(0.5,0)\}$; that is, we approximate the integrals

$$\int_{\widehat{K}_1} B(\hat{x})\,dx = -\tfrac{2}{3}\ln 2 + \tfrac{6019}{5760} - \tfrac{1}{12}\pi^2 + \tfrac{1}{2}\ln^2 2 \approx 6.266309395\times10^{-4},$$

$$\int_{\widehat{K}_1} \frac{\partial B}{\partial \hat{x}_1}\,dx = -\tfrac{17}{96} + \tfrac{1}{4}\ln 2 \approx -3.7955381933\times10^{-3},$$

$$\int_{\widehat{K}_1} \frac{\partial^2 B}{\partial \hat{x}_1^2}\,dx = -\tfrac{35}{24} + \ln 4 \approx -7.20389722\times10^{-2},$$

	L	degree	approx. integral	absolute error	relative error	rate
$\widehat{B}$	4	2	$6.66 \cdot 10^{-4}$	$3.90 \cdot 10^{-5}$	$6.22 \cdot 10^{-2}$	
	9	3	$6.27 \cdot 10^{-4}$	$4.00 \cdot 10^{-7}$	$6.38 \cdot 10^{-4}$	5.65
	16	4	$6.27 \cdot 10^{-4}$	$9.40 \cdot 10^{-9}$	$1.50 \cdot 10^{-5}$	6.52
	25	5	$6.27 \cdot 10^{-4}$	$2.23 \cdot 10^{-10}$	$3.57 \cdot 10^{-7}$	8.38
	36	6	$6.27 \cdot 10^{-4}$	$5.48 \cdot 10^{-12}$	$8.74 \cdot 10^{-9}$	10.17
	49	7	$6.27 \cdot 10^{-4}$	$1.38 \cdot 10^{-13}$	$2.20 \cdot 10^{-10}$	11.95
	64	8	$6.27 \cdot 10^{-4}$	$9.10 \cdot 10^{-17}$	$1.45 \cdot 10^{-13}$	27.42
1st derivative of $\widehat{B}$	4	2	$-3.58 \cdot 10^{-3}$	$2.12 \cdot 10^{-4}$	$-5.60 \cdot 10^{-2}$	
	9	3	$-3.79 \cdot 10^{-3}$	$3.76 \cdot 10^{-6}$	$-9.92 \cdot 10^{-4}$	4.97
	16	4	$-3.79 \cdot 10^{-3}$	$7.95 \cdot 10^{-8}$	$-2.09 \cdot 10^{-5}$	6.70
	25	5	$-3.79 \cdot 10^{-3}$	$1.82 \cdot 10^{-9}$	$-4.81 \cdot 10^{-7}$	8.45
	36	6	$-3.79 \cdot 10^{-3}$	$4.41 \cdot 10^{-11}$	$-1.16 \cdot 10^{-8}$	10.21
	49	7	$-3.79 \cdot 10^{-3}$	$1.10 \cdot 10^{-12}$	$-2.90 \cdot 10^{-10}$	11.97
	64	8	$-3.79 \cdot 10^{-3}$	$2.81 \cdot 10^{-14}$	$-7.40 \cdot 10^{-12}$	13.73
2nd derivative of $\widehat{B}$	4	2	$-7.07 \cdot 10^{-2}$	$1.24 \cdot 10^{-3}$	$-1.72 \cdot 10^{-2}$	
	9	3	$-7.20 \cdot 10^{-2}$	$2.34 \cdot 10^{-5}$	$-3.24 \cdot 10^{-4}$	4.89
	16	4	$-7.20 \cdot 10^{-2}$	$5.10 \cdot 10^{-7}$	$-7.08 \cdot 10^{-6}$	6.64
	25	5	$-7.20 \cdot 10^{-2}$	$2.91 \cdot 10^{-10}$	$-1.65 \cdot 10^{-7}$	8.41
	36	6	$-7.20 \cdot 10^{-2}$	$7.34 \cdot 10^{-12}$	$-4.04 \cdot 10^{-9}$	10.18
	49	7	$-7.20 \cdot 10^{-2}$	$1.88 \cdot 10^{-13}$	$-1.01 \cdot 10^{-10}$	11.94
	64	8	$-7.20 \cdot 10^{-2}$	$4.99 \cdot 10^{-15}$	$-2.62 \cdot 10^{-12}$	13.70

Table 2. Quadrature results using the Duffy transform for the function $\widehat{B}$ of (3-8) and its first two derivatives. The domain of integration is the triangle $\widehat{K}_1$ (the next two tables deal with $\widehat{K}_2$ and $\widehat{K}_3$). Rates of convergence are with respect to the relative error and the number of points L.

$$\int_{\widehat{K}_2} B(\hat{x})\,dx = -\tfrac{2}{3}\ln 2 + \tfrac{6019}{5760} - \tfrac{1}{12}\pi^2 + \tfrac{1}{2}\ln^2 2 \approx 6.266309395 \times 10^{-4},$$

$$\int_{\widehat{K}_2} \frac{\partial^2 B}{\partial \hat{x}_1}\,dx = 0,$$

$$\int_{\widehat{K}_2} \frac{\partial^2 B}{\partial \hat{x}_1^2}\,dx = -\tfrac{1}{12} \approx -0.0833333333,$$

$$\int_{\widehat{K}_3} B(\hat{x})\,dx \approx 8.096731144 \times 10^{-5},$$

$$\int_{\widehat{K}_3} \frac{\partial B}{\partial \hat{x}_1}\,dx = \tfrac{1}{6}\ln 2 - \tfrac{11}{96} \approx 9.411967600 \times 10^{-4},$$

	L	degree	approx. integral	absolute error	relative error	rate
$\widehat{B}$	4	2	$6.66\cdot10^{-4}$	$3.89\cdot10^{-5}$	$6.22\cdot10^{-2}$	
	9	3	$6.27\cdot10^{-4}$	$3.99\cdot10^{-7}$	$6.38\cdot10^{-4}$	5.64
	16	4	$6.26\cdot10^{-4}$	$9.40\cdot10^{-9}$	$1.50\cdot10^{-5}$	6.51
	25	5	$6.26\cdot10^{-4}$	$2.23\cdot10^{-11}$	$3.57\cdot10^{-7}$	8.36
	36	6	$6.26\cdot10^{-4}$	$5.47\cdot10^{-12}$	$8.74\cdot10^{-9}$	10.17
	49	7	$6.26\cdot10^{-4}$	$1.37\cdot10^{-13}$	$2.20\cdot10^{-10}$	11.94
	64	8	$6.26\cdot10^{-4}$	$3.52\cdot10^{-15}$	$1.45\cdot10^{-13}$	13.71
1st derivative of $\widehat{B}$	4	2	$8.61\cdot10^{-4}$	$8.61\cdot10^{-4}$		
	9	3	$4.25\cdot10^{-6}$	$4.25\cdot10^{-6}$		6.54
	16	4	$1.19\cdot10^{-7}$	$1.19\cdot10^{-7}$		6.20
	25	5	$3.43\cdot10^{-9}$	$3.43\cdot10^{-9}$		7.96
	36	6	$9.89\cdot10^{-11}$	$9.89\cdot10^{-11}$		9.72
	49	7	$2.86\cdot10^{-13}$	$2.86\cdot10^{-3}$		11.48
	64	8	$8.33\cdot10^{-15}$	$8.33\cdot10^{-15}$		13.24
2nd derivative of $\widehat{B}$	4	2	$-8.21\cdot10^{-2}$	$1.00\cdot10^{-3}$	$-1.44\cdot10^{-2}$	
	9	3	$-8.32\cdot10^{-2}$	$4.55\cdot10^{-5}$	$-5.46\cdot10^{-4}$	4.03
	16	4	$-8.33\cdot10^{-2}$	$1.63\cdot10^{-6}$	$-1.95\cdot10^{-5}$	5.78
	25	5	$-8.33\cdot10^{-2}$	$5.64\cdot10^{-8}$	$-6.77\cdot10^{-7}$	7.53
	36	6	$-8.33\cdot10^{-2}$	$1.90\cdot10^{-9}$	$-2.29\cdot10^{-8}$	9.28
	49	7	$-8.33\cdot10^{-2}$	$6.34\cdot10^{-11}$	$-7.61\cdot10^{-10}$	11.04
	64	8	$-8.33\cdot10^{-2}$	$2.08\cdot10^{-13}$	$-2.49\cdot10^{-11}$	12.79

Table 3. Quadrature results over the domain of integration $\widehat{K}_2$. See caption of Table 2 for details.

$$\int_{\widehat{K}_3} \frac{\partial^2 B}{\partial \hat{x}_1^2}\, dx = \tfrac{5}{8} - \tfrac{8}{9}\ln 2 \approx 8.869172836 \times 10^{-3}$$

by the quadrature scheme $\sum_{j=1}^{L} \hat{\omega}^{(j)} B(\hat{x}^{(j)})$. The numerical results and the errors are listed in the tables below. Our error now converges in an exponential manner for our initial function as well as its first and second derivative. These results are in agreement with Theorem 4.2.

6. Conclusion

In this paper, we have created an effective Gaussian quadrature scheme for a specific class of divergence free rational functions. We also managed to derive error estimates as well as show exponential error convergence, with numerical experiments confirming our results. While the findings of this paper appear to support the finite element method proposed in [Guzmán and Neilan 2014a], there

	L	degree	approx. integral	absolute error	relative error	rate
$\widehat{B}$	4	2	$6.35 \cdot 10^{-5}$	$1.73 \cdot 10^{-5}$	$2.14 \cdot 10^{-1}$	
	9	3	$8.08 \cdot 10^{-5}$	$1.35 \cdot 10^{-7}$	$1.67 \cdot 10^{-3}$	5.98
	16	4	$8.09 \cdot 10^{-5}$	$3.60 \cdot 10^{-9}$	$4.45 \cdot 10^{-5}$	6.30
	25	5	$8.09 \cdot 10^{-5}$	$1.12 \cdot 10^{-10}$	$1.38 \cdot 10^{-6}$	7.77
	36	6	$8.09 \cdot 10^{-5}$	$3.06 \cdot 10^{-12}$	$3.78 \cdot 10^{-8}$	9.88
	49	7	$8.09 \cdot 10^{-5}$	$4.43 \cdot 10^{-13}$	$5.47 \cdot 10^{-10}$	13.73
	64	8	$8.09 \cdot 10^{-5}$	$3.56 \cdot 10^{-13}$	$4.39 \cdot 10^{-10}$	.82
1st derivative of $\widehat{B}$	4	2	$9.47 \cdot 10^{-4}$	$6.04 \cdot 10^{-6}$	$6.42 \cdot 10^{-3}$	
	9	3	$9.42 \cdot 10^{-4}$	$1.04 \cdot 10^{-6}$	$1.10 \cdot 10^{-3}$	4.16
	16	4	$9.41 \cdot 10^{-4}$	$1.00 \cdot 10^{-8}$	$1.07 \cdot 10^{-5}$	8.06
	25	5	$9.41 \cdot 10^{-4}$	$1.09 \cdot 10^{-10}$	$1.16 \cdot 10^{-7}$	10.12
	36	6	$9.41 \cdot 10^{-4}$	$9.74 \cdot 10^{-12}$	$1.03 \cdot 10^{-8}$	6.64
	49	7	$9.41 \cdot 10^{-4}$	$3.97 \cdot 10^{-13}$	$4.22 \cdot 10^{-10}$	10.37
	64	8	$9.41 \cdot 10^{-4}$	$1.38 \cdot 10^{-14}$	$1.46 \cdot 10^{-11}$	12.58
2nd derivative of $\widehat{B}$	4	2	$9.00 \cdot 10^{-3}$	$1.34 \cdot 10^{-4}$	$1.51 \cdot 10^{-2}$	
	9	3	$8.87 \cdot 10^{-3}$	$2.95 \cdot 10^{-6}$	$3.33 \cdot 10^{-4}$	4.70
	16	4	$8.86 \cdot 10^{-3}$	$6.96 \cdot 10^{-8}$	$7.85 \cdot 10^{-6}$	6.51
	25	5	$8.86 \cdot 10^{-3}$	$6.21 \cdot 10^{-9}$	$7.00 \cdot 10^{-7}$	5.41
	36	6	$8.86 \cdot 10^{-3}$	$2.85 \cdot 10^{-11}$	$3.22 \cdot 10^{-8}$	8.44
	49	7	$8.86 \cdot 10^{-3}$	$1.11 \cdot 10^{-12}$	$1.25 \cdot 10^{-9}$	10.53
	64	8	$8.86 \cdot 10^{-3}$	$4.01 \cdot 10^{-14}$	$4.53 \cdot 10^{-11}$	12.42

Table 4. Quadrature results over the domain of integration $\hat{K}_2$. See caption of Table 2 for details.

are still a number of conditions such as V_h [Ciarlet 1978, p. 174], ellipticity and determining global error estimates which must be worked out. This will be the subject of ongoing research.

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