On the $\varepsilon$-ascent chromatic index of complete graphs

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(Communicated by Jerrold Griggs)

An edge ordering of a graph $G = (V, E)$ is an injection $f : E \rightarrow \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. A path in $G$ for which the edge ordering $f$ increases along its edge sequence is called an $f$-ascent; an $f$-ascent is maximal if it is not contained in a longer $f$-ascent. The depression $\varepsilon(G)$ of $G$ is the smallest integer $k$ such that any edge ordering $f$ has a maximal $f$-ascent of length at most $k$.

Applying the concept of ascents to edge colourings rather than edge orderings, we consider the problem of determining the minimum number $\chi_\varepsilon(K_n)$ of colours required to edge colour $K_n$, $n \geq 4$, such that the length of a shortest maximal ascent is equal to $\varepsilon(K_n) = 3$. We obtain new upper and lower bounds for $\chi_\varepsilon(K_n)$, which enable us to determine $\chi_\varepsilon(K_n)$ exactly for $n = 7$ and $n \equiv 2 \pmod{4}$ and to bound $\chi_\varepsilon(K_{4m})$ by $4m \leq \chi_\varepsilon(K_{4m}) \leq 4m + 1$.

1. Introduction

Following [Schurch 2013a; 2013b], we consider the following question:

Question 1. For $n \geq 4$, what is the smallest integer $r(n)$ for which there exists a proper edge colouring of $K_n$ in colours $1, \ldots, r(n)$ such that a shortest maximal path of increasing edge labels has length three?

Schurch showed that $r(n) \leq 2n - 3$ for all $n \geq 4$. This bound enabled him to determine $r(n)$ for $n \in \{4, 5\}$ and to show that $7 \leq r(6) \leq 8$. In Section 2 we give a lower bound for $r(n)$ and in Section 3 we improve the general upper bound to

$$r(n) \leq \left\lfloor \frac{3n - 3}{2} \right\rfloor.$$ 

We then improve this bound for even values of $n$. Consequently, we obtain $r(7) = 9$, $r(n) = n + 1$ if $n \equiv 2 \pmod{4}$, and $n \leq r(n) \leq n + 1$ if $n \equiv 0 \pmod{4}$ and $n \geq 8$.

MSC2010: 05C15, 05C78, 05C38.

Keywords: edge ordering of a graph, increasing path, depression, edge colouring.

Breytenbach was a second year undergraduate student, enrolled for the degree BSc in Mathematical Sciences (stream Computer Science) at Stellenbosch University, while this paper was being prepared. The paper earned him extra credit for the Foundations of Abstract Mathematics I course. Mynhardt was supported by an NSERC discovery grant.
We begin with a short historical account of the background to this problem. An edge ordering of a finite, simple graph $G$ is an injection $f : E(G) \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. Denote the set of all edge orderings of $G$ by $\mathcal{F}(G)$. A path $v_1, \ldots, v_k$ (where $v_k \neq v_1$) in $G$ such that $f(v_1) < \cdots < f(v_k)$ is called an $f$-ascent; an $f$-ascent is maximal if it is not contained in a longer $f$-ascent. The height $H(f)$ of an edge ordering $f$ is the length of a longest $f$-ascent, and the flatness of $f$, denoted by $h(f)$, is the length of a shortest maximal $f$-ascent of $G$.

Chvátal and Komlós [1971] posed the problem of determining $\alpha(K_n) = \min_{f \in \mathcal{F}(K_n)} \{H(f)\}$ of the complete graph $K_n$. This is a difficult problem and $\alpha(K_n)$ is known only for $1 \leq n \leq 8$ (see [Burger et al. 2005; Chvátal and Komlós 1971]). The parameter $\alpha(G)$ for complete and other finite graphs was also investigated in [Bialostocki and Roditty 1987; Burger et al. 2005; Calderbank et al. 1984; Graham and Kleitman 1973; Mynhardt et al. 2005; Roditty et al. 2001; Yuster 2001].

For an arbitrary finite graph $G$, Cockayne et al. [2006] considered the problem of determining $\varepsilon(G) = \max_{f \in \mathcal{F}(G)} [h(f)]$, that is, the maximum length, taken over all edge orderings $f \in \mathcal{F}(G)$, of a shortest maximal $f$-ascent. The parameter $\varepsilon(G)$ is known as the depression of $G$ and its computation is likewise a difficult problem. Another interpretation of the depression of $G$ is that any edge ordering $f$ of $G$ has a maximal $f$-ascent of length at most $\varepsilon(G)$, and $\varepsilon(G)$ is the smallest integer for which this statement is true. Graphs with depression two were characterized in [Cockayne et al. 2006], while trees with depression three were characterized in [Mynhardt 2008]. Graphs with no adjacent vertices of degree three or higher that have depression three were characterized in [Mynhardt and Schurch 2013]. Further work on depression can be found in [Cockayne and Mynhardt 2006; Gaber-Rosenblum and Roditty 2009; Schurch and Mynhardt 2014; 2014; Schurch 2013a; 2013b].

An edge ordering of $G$ is also a proper edge colouring — a labelling of the edges of $G$ such that adjacent edges have different labels. The minimum number of labels, also called colours, is called the edge chromatic number or the chromatic index $\chi'(G)$. It is well known (see [Chartrand et al. 2011, Section 10.2], for example) that $\chi'(K_n) = n - 1$ if $n$ is even and $\chi'(K_n) = n$ if $n$ is odd. A 1-factor of $G$ is a 1-regular spanning subgraph of $G$, and $G$ is 1-factorable if $E(G)$ can be partitioned into 1-factors. If $G$ is 1-factorable, then $G$ is $r$-regular for some $r$ and $\chi'(G) = r$. König's theorem (see [Chartrand et al. 2011, Theorem 10.15]) states that every $r$-regular bipartite graph is 1-factorable. In particular, the chromatic index of the complete bipartite graph $K_{n,n}$ is given by $\chi'(K_{n,n}) = n$.

Noticing that the labels of some edges in an edge ordering of $G$ may be unimportant when determining $\varepsilon(G)$, Schurch applied the concept of ascents to edge
We begin with a simple lower bound for $\chi_\varepsilon(G)$ the $\varepsilon$-ascent chromatic index of $G$, denoted $\chi_\varepsilon(G)$. Unlike the case for general graphs, the depression of $K_n$ is easy to determine: $\varepsilon(K_1) = 0, \varepsilon(K_2) = 1, \varepsilon(K_3) = 2$ and $\varepsilon(K_n) = 3$ for all $n \geq 4$ (see [Cockayne et al. 2006]); that is, there does not exist an edge ordering or an edge colouring of $K_n$ such that a shortest maximal ascent has length four or more. Note that $\chi_\varepsilon(K_1) = 0$, $\chi_\varepsilon(K_2) = 1$, $\chi_\varepsilon(K_3) = 3$, and determining $\chi_\varepsilon(K_n)$ for $n \geq 4$ is equivalent to finding the smallest integer $r(n)$ such that there exists a proper edge colouring $c$ of $K_n$ in colours $1, \ldots, r(n)$ with $h(c) = 3$, as formulated in Question 1.

2. Lower bound for the $\varepsilon$-ascent chromatic index of $K_n$

We begin with a simple lower bound for $\chi_\varepsilon(K_n)$, which slightly improves the bound in [Schurch 2013b, Proposition 8] in the special case where $G = K_n$.

**Theorem 1.** If $n \geq 4$, then

$$\chi_\varepsilon(K_n) \geq \begin{cases} n & \text{if } n \equiv 0 \pmod{4}, \\ n + 1 & \text{if } n \equiv 1, 2 \pmod{4}, \\ n + 2 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

**Proof.** Let $c$ be a proper edge colouring of $K_n$ in colours $1, \ldots, r$ such that $h(c) = 3$. Such a colouring exists because $\varepsilon(K_n) = 3$ if $n \geq 4$. For $i = 1, \ldots, r$, define

$$E_i = \{e \in E(K_n) : c(e) = i\}.$$ 

Then $|E_i| \leq \lfloor n/2 \rfloor$ for each $i$. Also, no vertex $v$ is incident with an edge $e \in E_1$ and an edge $e' \in E_r$, otherwise $e, e'$ is a maximal $\varepsilon$-ascent of length two, which contradicts $h(c) = 3$. Thus $|E_1 \cup E_r| \leq \lfloor n/2 \rfloor$ and $E_1 \cup E_r$ is an independent set of edges, that is, $E_1 \cup E_r, E_2, \ldots, E_{r-1}$ is also a proper edge colouring of $K_n$. Hence $r \geq \chi'(K_n) + 1$. In particular,

$$\chi_\varepsilon(K_n) \geq \begin{cases} n & \text{if } n \equiv 0 \pmod{4}, \\ n + 1 & \text{if } n \equiv 1 \pmod{4}. \end{cases}$$

Assume $n \equiv 2 \pmod{4}$; say $n = 4p + 2$. Then $K_n$ has $(2p + 1)(4p + 1)$ edges. Suppose $r = \chi'(K_n) + 1 = n$. The upper bound

$$|E_1 \cup E_r|, |E_2|, \ldots, |E_{r-1}| \leq \left\lfloor \frac{n}{2} \right\rfloor$$

implies that

$$|E_1 \cup E_r| = |E_2| = \cdots = |E_{r-1}| = \left\lfloor \frac{n}{2} \right\rfloor = 2p + 1.$$ 

Since $|E_1| + |E_r| = 2p + 1$, an odd number, $|E_1| \neq |E_r|$. Without loss of generality say $|E_1| = k$, where $k \leq p$, and $|E_r| = 2p + 1 - k$. Suppose $e \in E_2$ is not adjacent to
any edge in \( E_1 \). Since \( |E_1 \cup E_r| = 2p + 1 = \lfloor n/2 \rfloor \), \( e \) is adjacent to an edge \( e' \in E_r \).

But then \( e, e' \) is a maximal \( c \)-ascent of length two, which contradicts \( h(c) = 3 \). Therefore each edge in \( E_2 \) is adjacent to an edge in \( E_1 \), and since \( c \) is a proper edge colouring, \( |E_2| \leq 2|E_1| = 2k \leq 2p < \lfloor n/2 \rfloor \), a contradiction. Thus \( r \geq n + 1 \) as required.

Assume \( n \equiv 3 \pmod{4} \); say \( n = 4p + 3 \). Then \( |E(K_n)| = (4p + 3)(2p + 1) \). Suppose \( r = \chi'(K_n) + 1 = n + 1 \). As in the case \( n \equiv 2 \pmod{4} \), we obtain that \( |E_1 \cup E_r| = |E_2| = \cdots = |E_{r-1}| = \lfloor n/2 \rfloor = 2p + 1 \) and that each edge in \( E_2 \) is adjacent to an edge in \( E_1 \). There is one vertex \( v \) that is not incident with any edge in \( E_1 \cup E_r \), but an edge in \( E_2 \) incident with \( v \) also needs to be adjacent to an edge in \( E_1 \). We obtain a contradiction as above and the result follows. □

3. Upper bounds for the \( \varepsilon \)-ascent chromatic index of \( K_n \)

In Section 3.1 we provide a new general upper bound for \( \chi_\varepsilon(K_n) \). We improve this bound for even values of \( n \) in Sections 3.2 (the case \( n \equiv 0 \pmod{4} \)) and 3.3 (the case \( n \equiv 2 \pmod{4} \)).

3.1. A general bound. For \( n \geq 6 \), we now describe an edge colouring \( c \) of \( K_n \) in \( \lfloor (3n - 3)/2 \rfloor \) colours, as illustrated in Figure 1 for \( n \in \{6, 7\} \), and prove in Theorem 3 that \( h(c) = 3 \). Let \( V(K_n) = \{v_0, \ldots, v_{n-1}\} \) and \( p = \lceil n/2 \rceil \).

- For \( i \in \{0, \ldots, p - 1\} \) and \( j \in \{i + 1, \ldots, n - 1\} \), let \( c(v_i v_j) = i + j \).
- For \( i \in \{p, \ldots, n - 2\} \) and \( j \in \{i + 1, \ldots, n - 1\} \), let \( c(v_i v_j) = i + j - 2p \).

**Lemma 2.** For all \( n \geq 6 \), the colouring \( c \) defines a proper edge colouring of \( K_n \) in \( \lfloor (3n - 3)/2 \rfloor \) colours.

![Figure 1. Edge colourings of \( K_6 \) and \( K_7 \) with flatness three.](image-url)
Proof. Suppose that \( c(v_i v_j) = c(v_i v_{j'}) \) for some \( j < j' \). After a brief reflection, we deduce that \( i + j = i + j' - 2p \). But \( i + j \geq i \) and
\[
i + j' - 2p \leq i + n - 1 - 2[n/2] \leq i - 1,
\]
hence \( c(v_i v_j) > c(v_i v_{j'}) \), contradicting our assumption.

Since the smallest colour is \( 0 + 1 = 1 \) and the largest colour is
\[
p - 1 + n - 1 = \left\lceil \frac{n}{2} \right\rceil + n - 2 = \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1 = \left\lceil \frac{3n-3}{2} \right\rceil,
\]
the colouring \( c \) uses exactly \( \lfloor (3n-3)/2 \rfloor \) colours. \( \square \)

**Theorem 3.** For all \( n \geq 6 \), the colouring \( c \) of \( K_n \) has flatness equal to three.

Proof. To prove that \( h(c) = 3 \), it is sufficient to prove this:

**Statement.** For any \( v_l \in V(K_n) \) and edges \( e = v_j v_i \) and \( f = v_i v_k \) such that \( c(e) < c(f) \), there exists
(Sa) an edge \( g = v_j' v_j \), \( j' \notin \{i, j, k \} \), such that \( c(g) < c(e) \), or
(Sb) an edge \( g = v_k v_{k'} \), \( k' \notin \{i, j, k \} \), such that \( c(f) < c(g) \).

Hence suppose there exist indices \( i, j, k \in I = \{0, \ldots, n-1\} \) such that for edges \( e = v_j v_i \) and \( f = v_i v_k \), we have \( c(e) < c(f) \), but neither (Sa) nor (Sb) holds. Then
\[
c(v_j' v_j) > c(e) \quad \text{for all } j' \in I - \{i, j, k\}, \tag{1}\]
and
\[
c(v_k v_{k'}) < c(f) \quad \text{for all } k' \in I - \{i, j, k\}. \tag{2}\]

We consider three cases, depending on the values of \( i \) and \( j \).

**Case 1:** \( j \leq p - 1 \). Then, regardless of the values of \( i \) and \( j' \), \( c(v_j' v_j) = j + j' \) and \( c(e) = i + j \). By (1), \( j' > i \) for all \( j' \in I - \{i, j, k\} \). Hence \( i \leq 2 \). But \( p \geq 3 \) since \( n \geq 6 \), and therefore \( i \leq p - 1 \). Now \( i + j = c(e) < c(f) = i + k \) implies that \( j < k \). Therefore one of the following three subcases holds:

(i) \( j = 0, k = 1 \) and \( i = 2 \),
(ii) \( j = 0 \) and \( k > i = 1 \),
(iii) \( i = 0 \) and \( k > j > 0 \).

If (i) holds, then \( c(v_j v_k) = 1 \). Since \( n \geq 6 \), there exists \( k' \in I - \{0, 1, 2\} \) such that \( c(v_k v_{k'}) = k + k' \geq 1 \geq 4 > c(f) = i + k = 3 \), contradicting (2). If (ii) holds, then \( c(f) = 1 + k \). If \( k \leq p - 1 \), then \( v_k \) is adjacent to \( v_p \), where \( p \notin \{0, 1, k\} \), and \( c(v_k v_p) = k + p > c(f) \), contradicting (2); while if \( k \geq p \), then \( v_k \) is adjacent to \( v_2 \) and \( c(v_2 v_k) = k + 2 > c(f) \), again a contradiction. If (iii) holds, then \( c(e) = j < k = c(f) \). If \( k \leq p - 1 \), then \( j < p - 1 \) and \( v_k \) is adjacent to \( v_p \), where \( p \notin \{0, j, k\} \), giving a contradiction as in (ii). If \( k \geq p \), then there exists \( \ell \in \{1, 2\} - \{j\} \) such that \( c(v_k v_\ell) = k + \ell > k \), once again a contradiction.
Case 2: \( j \geq p \) and \( i \leq p - 1 \). Then \( c(e) = i + j \). Since \( i \leq p - 1 \) and \( n \geq 6 \), there exists \( j' \in I \setminus \{i, j, k\} \) such that \( j' \geq p \). Then \( c(v_jv_j) = j + j' - 2p > i + j \) by (1); that is, \( i < j' - 2p \leq 0 \), which is impossible.

Case 3: \( \min\{i, j\} \geq p \). Then \( c(e) = i + j - 2p \). Suppose there exists \( j' \in I \setminus \{i, j, k\} \) such that \( j' \geq p \). Then \( c(v_jv_j) = j + j' - 2p \) and thus \( j' > i \) by (1). Since \( i, j' \geq p \),

\[
c(f) = c(v_iv_k) = \begin{cases} i + k & \text{if } k \leq p - 1, \\ i + k - 2p & \text{if } k \geq p, 
\end{cases}
\]

and

\[
c(v_kv_j) = \begin{cases} j' + k & \text{if } k \leq p - 1, \\ j' + k - 2p & \text{if } k \geq p. 
\end{cases}
\]

Thus, regardless of the value of \( k \), \( c(v_kv_j) > c(f) \). Since \( j' \in I \setminus \{i, j, k\} \), this contradicts (2). Hence there does not exist \( j' \in I \setminus \{i, j, k\} \) such that \( j' \geq p \). Since \( n \geq 6 \), we have \( \{|p, \ldots, n-1|\} \geq 3 \). We deduce that \( n \in \{6, 7\} \) and \( \{p, \ldots, n-1\} = \{i, j, k\} \) so that \( c(e) = i + j - 2p \) and \( c(f) = i + k - 2p \), where \( j < k \) since \( c(e) < c(f) \). For either value of \( n \), \( c(f) \leq 3 \) and \( k \geq 4 \). Let \( j' = 0 < p \). Then \( j' \in I \setminus \{i, j, k\} \) and \( c(v_jv_k) = j' + k = k \geq 4 > 3 \geq c(f) \), again contradicting (2). \( \Box \)

The following corollary to Lemma 2 and Theorem 3 improves Theorem 17 of [Schurch 2013b].

**Corollary 4.** For \( n \geq 6 \), we have \( \chi_e(K_n) \leq \lfloor (3n - 3)/2 \rfloor \).

Combining Theorem 1 and Corollary 4 we improve Proposition 20 of [Schurch 2013b] and also obtain the new value \( \chi_e(K_7) \).

**Corollary 5.**

\( \chi_e(K_6) = 7 \) and \( \chi_e(K_7) = 9 \).

### 3.2. The case \( n \equiv 0 \pmod{4} \)

Our next result is an improved upper bound for \( \chi_e(K_n) \) in the case where \( n \equiv 0 \pmod{4} \) and \( n \geq 8 \). Say \( n = 4m \) and \( V(K_n) = \{u_0, \ldots, u_{2m-1}, v_0, \ldots, v_{2m-1}\} \). Let \( G \) and \( H \) be the subgraphs of \( K_n \) induced by \( \{u_0, \ldots, u_{2m-1}\} \) and \( \{v_0, \ldots, v_{2m-1}\} \), respectively. Then \( G \cong H \cong K_{2m} \) and each of them is \((2m-1)\)-edge colourable. We describe a colouring \( c_1 \) of \( K_n \) in the colours \( 1, \ldots, 4m + 1 \) as follows.

- In \( G \), let \( c_1 \) be any proper edge colouring of \( K_{2m} \) in the \( 2m - 1 \) colours \( \{1, 2\} \cup \{m + 3, \ldots, 3m - 1\} \).
- In \( H \), let \( c_1 \) be any proper edge colouring of \( K_{2m} \) in the \( 2m - 1 \) colours \( \{4m, 4m + 1\} \cup \{m + 3, \ldots, 3m - 1\} \).
- We still need to colour the edges of the complete bipartite graph \( F \cong K_{2m,2m} \) induced by the edges \( u_i v_j \), with \( i, j \in \{0, \ldots, 2m-1\} \). But \( \chi'(K_{2m,2m}) = 2m \) and there are \( 2m \) unused colours \( 3, \ldots, m + 2 \) and \( 3m, \ldots, 4m - 1 \). Colour the edges of \( F \) with these colours.
It is clear that $c_1$ is a proper edge colouring of $K_{4m}$ in $4m + 1$ colours.

**Theorem 6.** For all $m \geq 2$, the colouring $c_1$ of $K_{4m}$ has flatness equal to three.

**Proof.** Let $F$, $G$ and $H$ be the subgraphs of $K_{4m}$ defined above and let $e$, $f \in E(K_{4m})$ be adjacent edges such that $c_1(e) < c_1(f)$. We show that (Sa) or (Sb) holds, as stated in the proof of Theorem 3. We consider three cases, depending on the choice of $e$ and $f$.

**Case 1:** $\{e, f\} \cap E(F) = \emptyset$. Assume first $e, f \in E(G)$; say $e = u_ju_i$ and $f = u_iu_k$. Then $c_1(e) < c_1(f) \leq 3m - 1$, and $u_k$ is adjacent to some vertex $v_\ell \in V(H)$ such that $c_1(u_kv_\ell) = 4m - 1 > c_1(f)$. Hence (Sb) holds. Similarly, if $e, f \in E(H)$, say $e = v_jv_i$ and $f = v_iv_k$, then $c_1(f) > c_1(e) \geq m + 3$, and $v_j$ is adjacent to some vertex $u_\ell \in V(G)$ such that $c_1(v_\ell u_\ell) = 3 < c_1(e)$. Hence (Sa) holds.

**Case 2:** $\{|e, f\} \cap E(F)| = 1$. By symmetry we may assume that $e \in E(F)$; say $e = u_iu_j$. If $f \in E(G)$, say $f = u_iu_k$, then $c_1(e) \in \{3, \ldots, m + 2\}$ and $c_1(f) \in \{m + 3, m + 4, \ldots, 3m - 1\}$. Since $m \geq 2$, $u_k$ is adjacent to at least two vertices $v_{i_1}, v_{i_2}$ of $H$ such that $c_1(u_kv_{i_\ell}) \in \{3m, \ldots, 4m - 1\}$ for $\ell = 1, 2$, and we may choose a subscript $t_\ell$, say $t_1$, such that $t_1 \neq j$. Then $v_j, u_i, u_k, v_{i_1}$ is a $c_1$-ascent of length three and (Sb) holds. On the other hand, if $f \in E(H)$, say $f = v_jv_k$, then $c_1(e) \geq 3$. In this case $u_i$ is adjacent to a vertex $u_\ell$ such that $c_1(u_\ell u_i) \in \{1, 2\}$ and (Sa) holds.

**Case 3:** $\{|e, f\} \subseteq E(F)$. First, if $e = u_iu_j$ and $f = v_ju_k$, then there exists at least one index $\ell \in \{0, \ldots, 2m - 1\} - \{i, k\}$ such that $c_1(u_\ell u_i) \in \{1, 2\}$. Then $u_\ell, u_i, v_j, u_k$ is a $c_1$-ascent of length three and (Sa) holds. Finally, if $e = v_ju_j$ and $f = u_jv_k$, then there exists at least one index $\ell \in \{0, \ldots, 2m - 1\} - \{i, k\}$ such that $c_1(v_\ell v_k) \in \{4m, 4m + 1\}$. Then $v_i, u_j, v_k, u_\ell$ is a $c_1$-ascent of length three and (Sb) holds.

Combining Theorems 1 and 6 we narrow down $\chi_\varepsilon(K_n)$ to two possible values in infinitely many cases.

**Corollary 7.** For all $n \geq 8$ and $n \equiv 0 \pmod{4}$, we have $n \leq \chi_\varepsilon(K_n) \leq n + 1$.

### 3.3. The case $n \equiv 2 \pmod{4}$

We now assume that $n \equiv 2 \pmod{4}$ and $n \geq 10$. Say $n = 4m + 2$ and $V(K_n) = \{u_0, \ldots, u_{2m}, v_0, \ldots, v_{2m}\}$. Let $G$ and $H$ be the subgraphs of $K_n$ induced by $\{u_0, \ldots, u_{2m}\}$ and $\{v_0, \ldots, v_{2m}\}$, respectively. Then $G \cong H \cong K_{2m+1}$ and each of them is $(2m + 1)$-edge colourable. We describe an edge colouring $c_2$ of $K_n$ in the colours $1, \ldots, 4m + 3$. This colouring is similar to the colouring $c_1$ above, but not quite as straightforward. See Figure 2 for a partial colouring of $K_{10}$.

- In $G$, let $c_2$ be any proper edge colouring of $K_{2m+1}$ in the $2m + 1$ colours $\{1, 2\} \cup \{m + 3, \ldots, 3m + 1\}$.
- In $H$, let $c_2$ be any proper edge colouring of $K_{2m+1}$ in the $2m + 1$ colours $\{4m + 2, 4m + 3\} \cup \{m + 3, \ldots, 3m + 1\}$.
We still need to colour the edges of the complete bipartite graph $F \cong K_{2m+1,2m+1}$ induced by the edges $u_i v_j$, with $i,j \in \{0, \ldots, 2m\}$. By König’s theorem, $F$ is 1-factorable. Note that for each colour $k$ in the edge colouring of $G$ there is exactly one vertex that is not incident with an edge coloured $k$, and conversely, for each vertex $u_i$ there is exactly one colour that does not occur as colour of an edge incident with $u_i$. A similar remark holds for $H$. Without loss of generality, say colour 2 does not occur at $u_0$, colour 1 does not appear at $u_{2m}$, colour $4m + 3$ does not appear at $v_0$ and colour $4m + 2$ does not appear at $v_{2m}$. Since colour 2 does not occur at $u_0$, all other colours of the colouring do and thus there exists a vertex $v_t \in V(G)$ such that $c_2(u_0 v_s) = 1$. Since colour $4m + 2$ does not appear at $v_{2m}$, there exists a vertex $v_t \in V(H)$ such that $c_2(v_{2m} v_t) = 4m + 3$.

- Colour the edges $u_0 v_0$ and $u_{2m} v_{2m}$ of $F$ with colours 2 and $4m + 2$, respectively.
- For $i, j \in \{1, \ldots, 2m-1\}$ and $k \in \{m+3, \ldots, 3m+1\}$, colour $u_i v_j$ with colour $k$ if and only if no edge incident with $u_i$ in $G$ or with $v_j$ in $H$ is coloured $k$.

We have now coloured a 1-factor $F_0$ of $F$, and $F - F_0$ is a $2m$-regular bipartite graph, which is 1-factorable by König’s theorem. Let $F'_1$ be a 1-factor of $F - F_0$ that contains the edge $v_0 u_s$. If $u_{2m} v_t \notin F'_1$, let $F_1 = F'_1$, and if $u_{2m} v_t \in F'_1$, let $u_i v_j \in F'_1 - \{v_0 u_s, u_{2m} v_t\}$ and define $F_1 = (F'_1 - \{u_i v_j, u_{2m} v_t\}) \cup \{u_i v_j, u_{2m} v_t\}$. Now $F - F_0 - F_1$ is 1-factorable. Let $F_2$ be a 1-factor of $F - F_0 - F_1$ that contains $u_{2m} v_t$.

- Colour the edges in $F_1$ with colour 3 and the edges in $F_2$ with colour $4m + 1$.

Colouring $F - F_0 - F_1 - F_2$ with the $2m - 2$ unused colours 4, $\ldots$, $m + 2$ and $3m + 2, \ldots, 4m$ yields a proper edge colouring of $K_{4m+2}$.

Figure 2. Part of the edge colouring $c_2$ of $K_{10}$. 
**Theorem 8.** For all $m \geq 2$, the colouring $c_2$ of $K_{4m+2}$ has flatness equal to three.

**Proof.** Let $F$, $J$, $G$ and $H$ be the subgraphs of $K_{4m+2}$ defined above and let $e, f \in E(K_{4m+2})$ be adjacent edges such that $c_2(e) < c_2(f)$. We show that (Sa) or (Sb) holds, as stated in the proof of Theorem 3. If $\{e, f\} \cap E(F) = \emptyset$, the proof follows similar to Case 1 in the proof of Theorem 6. We consider two further cases.

**Case 1:** $|\{e, f\} \cap E(H)| = 1$. By symmetry we may assume that $e \in E(F)$; say $e = u_i v_j$. First suppose that $f \in E(G)$, say $f = u_i u_k$. Since $c_2(f) > c_2(e) \geq 2$, $c_2(f) \in \{m+3, \ldots, 3m+1\}$. As in Case 2 of the proof of Theorem 3, (Sb) holds. Now suppose $f = v_j v_k \in E(H)$. If $c_2(e) = 2$, then $i = j = 0$ and $c_2(u_0 u_s) = 1$. If $c_2(e) \neq 2$ then $c_2(e) > 2$ and there exists an index $\ell$ such that $c_2(u_i u_\ell) \in \{1, 2\}$. Thus $u_s, u_i, v_j, v_k$ or $u_\ell, u_i, v_j, v_k$ is a $c_2$-ascent of length three and (Sa) holds.

**Case 2:** $\{e, f\} \subseteq E(F)$. Suppose $e = u_i v_j$ and $f = v_j u_k$. If $e = u_0 v_0$ and $f = v_0 u_s$, then $c_2(e) = 2$ and $c_2(f) = 3$. Therefore there exists a vertex $u_\ell$ such that $c_2(u_s u_\ell) \in \{m+3, \ldots, 3m+1\}$ and (Sb) holds. If $e = u_0 v_0$ and $k \neq s$, then $u_s, u_0, v_0, v_k$ is a $c_2$-ascent of length three and (Sa) holds. For all other choices of $e = u_i v_j$ and $f = v_j u_k$ it follows as in Case 3 of the proof of Theorem 3 that (Sa) or (Sb) holds. Suppose $e = v_i u_j$ and $f = u_j v_k$. If $e = v_i u_2m$ and $f = u_2m v_2m$, then $c_2(e) = 4m + 1$ and $c_2(f) = 4m + 2$. There exists a vertex $v_\ell$ such that $c_2(v_\ell v_i) \in \{m+3, \ldots, 3m+1\}$ and thus (Sa) holds. If $f = u_2m v_2m$ and $i \neq t$, then $v_i, v_2m, u_2m, v_t$ is a $c_2$-ascent of length three and (Sb) holds. All other cases are dealt with as in Case 3 of the proof of Theorem 3.

Combining Theorems 1 and 8 and Corollary 4 determines $\chi_\ell(K_n)$ for all $n \equiv 2 \pmod{10}, n \geq 6$.

**Corollary 9.** For all $n \geq 6$ and $n \equiv 2 \pmod{10}$, we have $\chi_\ell(K_n) = n + 1$.

**4. Conclusion**

In Theorem 1 we proved a lower bound for $\chi_\ell(K_n)$, and in Corollary 4 we improved the previously known general upper bound for $\chi_\ell(K_n)$ from $2n - 3$ to $\lfloor (3n - 3)/2 \rfloor$. Corollary 7 improves this bound for $n \equiv 0 \pmod{4}$ and allows us to bound $\chi_\ell(K_{4m})$ by $4m \leq \chi_\ell(K_n) \leq 4m + 1$. Finally, Corollary 9 determines $\chi_\ell(K_n)$ for all $n \equiv 2 \pmod{4}, n \geq 6$. Based on the results for even $n$ and the values $\chi_\ell(K_5) = 7$ and $\chi_\ell(K_7) = 9$, we formulate the following conjecture.

**Conjecture 10.** For all $n \geq 4$, we have $\chi_\ell(K_n) = \chi'(K_n) + 2$.

**Acknowledgements**

Jean Breytenbach wishes to thank Professor Jan van Vuuren of the Department of Industrial Engineering, Stellenbosch University, for fuelling his interest in graph the-
ory. Both authors hereby also express their gratitude towards Professor van Vuuren for introducing them and providing a wonderful research environment to work in.

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Received: 2013-07-22   Accepted: 2013-10-26

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