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On the ε-ascent chromatic index of complete graphs

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An edge ordering of a graph $G = (V, E)$ is an injection $f : E \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. A path in G for which the edge ordering f increases along its edge sequence is called an f-ascent; an f-ascent is maximal if it is not contained in a longer f-ascent. The depression $\varepsilon(G)$ of G is the smallest integer k such that any edge ordering f has a maximal f-ascent of length at most k. Applying the concept of ascents to edge colourings rather than edge orderings, we consider the problem of determining the minimum number $\chi_\varepsilon(K_n)$ of colours required to edge colour K_n, $n \geq 4$, such that the length of a shortest maximal ascent is equal to $\varepsilon(K_n) = 3$. We obtain new upper and lower bounds for $\chi_\varepsilon(K_n)$, which enable us to determine $\chi_\varepsilon(K_n)$ exactly for $n = 7$ and $n \equiv 2 \pmod 4$ and to bound $\chi_\varepsilon(K_{4m})$ by $4m \leq \chi_\varepsilon(K_{4m}) \leq 4m + 1$.

1. Introduction

Following [Schurch 2013a; 2013b], we consider the following question:

Question 1. For $n \geq 4$, what is the smallest integer $r(n)$ for which there exists a proper edge colouring of K_n in colours $1, \ldots, r(n)$ such that a shortest maximal path of increasing edge labels has length three?

Schurch showed that $r(n) \leq 2n - 3$ for all $n \geq 4$. This bound enabled him to determine $r(n)$ for $n \in \{4, 5\}$ and to show that $7 \leq r(6) \leq 8$. In Section 2 we give a lower bound for $r(n)$ and in Section 3 we improve the general upper bound to

$$r(n) \leq \left\lfloor \frac{3n-3}{2} \right\rfloor.$$

We then improve this bound for even values of n. Consequently, we obtain $r(7) = 9$, $r(n) = n + 1$ if $n \equiv 2 \pmod 4$, and $n \leq r(n) \leq n + 1$ if $n \equiv 0 \pmod 4$ and $n \geq 8$.

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We begin with a short historical account of the background to this problem. An *edge ordering* of a finite, simple graph G is an injection $f : E(G) \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. Denote the set of all edge orderings of G by $\mathcal{F}(G)$. A path $v_1, \dots, v_k$ (where $v_k \neq v_1$) in G such that $f(v_1) < \cdots < f(v_k)$ is called an *f-ascent*; an f-ascent is *maximal* if it is not contained in a longer f-ascent. The *height* $H(f)$ of an edge ordering f is the length of a longest f-ascent, and the *flatness* of f, denoted by $h(f)$, is the length of a shortest maximal f-ascent of G.

Chvátal and Komlós [1971] posed the problem of determining

$$\alpha(K_n) = \min_{f \in \mathcal{F}(K_n)} \{H(f)\}$$

of the complete graph K_n. This is a difficult problem and $\alpha(K_n)$ is known only for $1 \leq n \leq 8$ (see [Burger et al. 2005; Chvátal and Komlós 1971]). The parameter $\alpha(G)$ for complete and other finite graphs was also investigated in [Bialostocki and Roditty 1987; Burger et al. 2005; Calderbank et al. 1984; Graham and Kleitman 1973; Mynhardt et al. 2005; Roditty et al. 2001; Yuster 2001].

For an arbitrary finite graph G, Cockayne et al. [2006] considered the problem of determining $\varepsilon(G) = \max_{f \in \mathcal{F}(G)}\{h(f)\}$, that is, the maximum length, taken over all edge orderings $f \in \mathcal{F}(G)$, of a shortest maximal f-ascent. The parameter $\varepsilon(G)$ is known as the *depression* of G and its computation is likewise a difficult problem. Another interpretation of the depression of G is that any edge ordering f of G has a maximal f-ascent of length at most $\varepsilon(G)$, and $\varepsilon(G)$ is the smallest integer for which this statement is true. Graphs with depression two were characterized in [Cockayne et al. 2006], while trees with depression three were characterized in [Mynhardt 2008]. Graphs with no adjacent vertices of degree three or higher that have depression three were characterized in [Mynhardt and Schurch 2013]. Further work on depression can be found in [Cockayne and Mynhardt 2006; Gaber-Rosenblum and Roditty 2009; Schurch and Mynhardt 2014; 2014; Schurch 2013a; 2013b].

An edge ordering of G is also a *proper edge colouring* — a labelling of the edges of G such that adjacent edges have different labels. The minimum number of labels, also called *colours*, is called the *edge chromatic number* or the *chromatic index* $\chi'(G)$. It is well known (see [Chartrand et al. 2011, Section 10.2], for example) that $\chi'(K_n) = n - 1$ if n is even and $\chi'(K_n) = n$ if n is odd. A 1-*factor* of G is a 1-regular spanning subgraph of G, and G is 1-*factorable* if $E(G)$ can be partitioned into 1-factors. If G is 1-factorable, then G is r-regular for some r and $\chi'(G) = r$. König's theorem (see [Chartrand et al. 2011, Theorem 10.15]) states that every r-regular bipartite graph is 1-factorable. In particular, the chromatic index of the complete bipartite graph $K_{n,n}$ is given by $\chi'(K_{n,n}) = n$.

Noticing that the labels of some edges in an edge ordering of G may be unimportant when determining $\varepsilon(G)$, Schurch applied the concept of ascents to edge colourings and called the minimum number of colours in a proper edge colouring c of G such that $h(c) = \varepsilon(G)$ the *ε-ascent chromatic index* of G, denoted $\chi_\varepsilon(G)$. Unlike the case for general graphs, the depression of K_n is easy to determine: $\varepsilon(K_1) = 0$, $\varepsilon(K_2) = 1$, $\varepsilon(K_3) = 2$ and $\varepsilon(K_n) = 3$ for all $n \geq 4$ (see [Cockayne et al. 2006]); that is, there does not exist an edge ordering or an edge colouring of K_n such that a shortest maximal ascent has length four or more. Note that $\chi_\varepsilon(K_1) = 0$, $\chi_\varepsilon(K_2) = 1$, $\chi_\varepsilon(K_3) = 3$, and determining $\chi_\varepsilon(K_n)$ for $n \geq 4$ is equivalent to finding the smallest integer $r(n)$ such that there exists a proper edge colouring c of K_n in colours $1, \dots, r(n)$ with $h(c) = 3$, as formulated in Question 1.

2. Lower bound for the ε-ascent chromatic index of K_n

We begin with a simple lower bound for $\chi_\varepsilon(K_n)$, which slightly improves the bound in [Schurch 2013b, Proposition 8] in the special case where $G = K_n$.

Theorem 1. *If $n \geq 4$, then*

$$\chi_\varepsilon(K_n) \geq \begin{cases} n & \text{if } n \equiv 0 \pmod 4, \\ n+1 & \text{if } n \equiv 1, 2 \pmod 4, \\ n+2 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Let c be a proper edge colouring of K_n in colours $1, \dots, r$ such that $h(c) = 3$. Such a colouring exists because $\varepsilon(K_n) = 3$ if $n \geq 4$. For $i = 1, \dots, r$, define

$$E_i = \{e \in E(K_n) : c(e) = i\}.$$

Then $|E_i| \leq \lfloor n/2 \rfloor$ for each i. Also, no vertex v is incident with an edge $e \in E_1$ and an edge $e' \in E_r$, otherwise e, e' is a maximal c-ascent of length two, which contradicts $h(c) = 3$. Thus $|E_1 \cup E_r| \leq \lfloor n/2 \rfloor$ and $E_1 \cup E_r$ is an independent set of edges, that is, $E_1 \cup E_r, E_2, \dots, E_{r-1}$ is also a proper edge colouring of K_n. Hence $r \geq \chi'(K_n) + 1$. In particular,

$$\chi_\varepsilon(K_n) \geq \begin{cases} n & \text{if } n \equiv 0 \pmod 4, \\ n+1 & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Assume $n \equiv 2 \pmod 4$; say $n = 4p + 2$. Then K_n has $(2p + 1)(4p + 1)$ edges. Suppose $r = \chi'(K_n) + 1 = n$. The upper bound

$$|E_1 \cup E_r|, |E_2|, \dots, |E_{r-1}| \leq \left\lfloor \frac{n}{2} \right\rfloor$$

implies that

$$|E_1 \cup E_r| = |E_2| = \cdots = |E_{r-1}| = \left\lfloor \frac{n}{2} \right\rfloor = 2p + 1.$$

Since $|E_1| + |E_r| = 2p + 1$, an odd number, $|E_1| \neq |E_r|$. Without loss of generality say $|E_1| = k$, where $k \leq p$, and $|E_r| = 2p + 1 - k$. Suppose $e \in E_2$ is not adjacent to any edge in E_1. Since $|E_1 \cup E_r| = 2p+1 = \lfloor n/2 \rfloor$, e is adjacent to an edge $e' \in E_r$. But then e, e' is a maximal c-ascent of length two, which contradicts $h(c) = 3$. Therefore each edge in E_2 is adjacent to an edge in E_1, and since c is a proper edge colouring, $|E_2| \le 2|E_1| = 2k \le 2p < \lfloor n/2 \rfloor$, a contradiction. Thus $r \ge n+1$ as required.

Assume $n \equiv 3 \pmod 4$; say $n = 4p+3$. Then $|E(K_n)| = (4p+3)(2p+1)$. Suppose $r = \chi'(K_n) + 1 = n+1$. As in the case $n \equiv 2 \pmod 4$, we obtain that $|E_1 \cup E_r| = |E_2| = \cdots = |E_{r-1}| = \lfloor n/2 \rfloor = 2p+1$ and that each edge in E_2 is adjacent to an edge in E_1. There is one vertex v that is not incident with any edge in $E_1 \cup E_r$, but an edge in E_2 incident with v also needs to be adjacent to an edge in E_1. We obtain a contradiction as above and the result follows. □

3. Upper bounds for the ε-ascent chromatic index of K_n

In Section 3.1 we provide a new general upper bound for $\chi_\varepsilon(K_n)$. We improve this bound for even values of n in Sections 3.2 (the case $n \equiv 0 \pmod 4$) and 3.3 (the case $n \equiv 2 \pmod 4$).

3.1. *A general bound.* For $n \ge 6$, we now describe an edge colouring c of K_n in $\lfloor (3n-3)/2 \rfloor$ colours, as illustrated in Figure 1 for $n \in \{6, 7\}$, and prove in Theorem 3 that $h(c) = 3$. Let $V(K_n) = \{v_0, \ldots, v_{n-1}\}$ and $p = \lceil n/2 \rceil$.

- For $i \in \{0, \ldots, p-1\}$ and $j \in \{i+1, \ldots, n-1\}$, let $c(v_i v_j) = i + j$.
- For $i \in \{p, \ldots, n-2\}$ and $j \in \{i+1, \ldots, n-1\}$, let $c(v_i v_j) = i + j - 2p$.

Lemma 2. *For all $n \ge 6$, the colouring c defines a proper edge colouring of K_n in $\lfloor (3n-3)/2 \rfloor$ colours.*

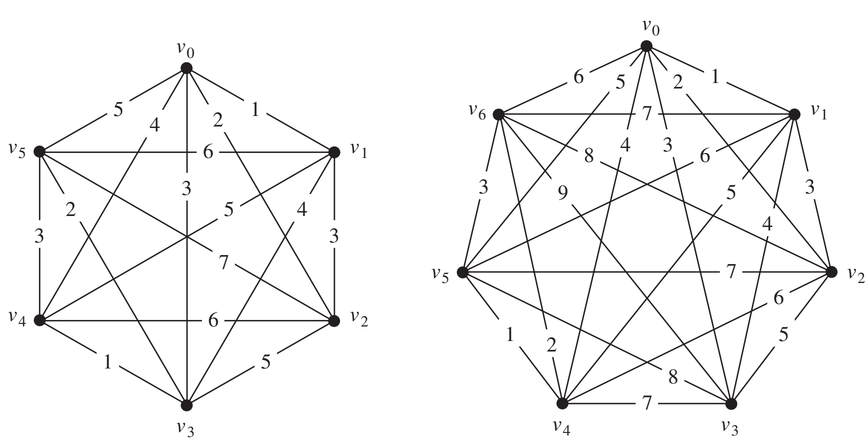

Figure 1. Edge colourings of K_6 and K_7 with flatness three.

Proof. Suppose that $c(v_i v_j) = c(v_i v_{j'})$ for some $j < j'$. After a brief reflection, we deduce that $i + j = i + j' - 2p$. But $i + j \geq i$ and

$$i + j' - 2p \leq i + n - 1 - 2\lceil n/2 \rceil \leq i - 1,$$

hence $c(v_i v_j) > c(v_i v_{j'})$, contradicting our assumption.

Since the smallest colour is $0 + 1 = 1$ and the largest colour is

$$p - 1 + n - 1 = \left\lceil \frac{n}{2} \right\rceil + n - 2 = \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1 = \left\lfloor \frac{3n-3}{2} \right\rfloor,$$

the colouring c uses exactly $\lfloor (3n-3)/2 \rfloor$ colours. □

Theorem 3. *For all $n \geq 6$, the colouring c of K_n has flatness equal to three.*

Proof. To prove that $h(c) = 3$, it is sufficient to prove this:

Statement. *For any $v_i \in V(K_n)$ and edges $e = v_j v_i$ and $f = v_i v_k$ such that $c(e) < c(f)$, there exists*

(Sa) *an edge $g = v_{j'} v_j$, $j' \notin \{i, j, k\}$, such that $c(g) < c(e)$, or*

(Sb) *an edge $g = v_k v_{k'}$, $k' \notin \{i, j, k\}$, such that $c(f) < c(g)$.*

Hence suppose there exist indices $i, j, k \in I = \{0, \ldots, n-1\}$ such that for edges $e = v_j v_i$ and $f = v_i v_k$, we have $c(e) < c(f)$, but neither (Sa) nor (Sb) holds. Then

$$c(v_{j'} v_j) > c(e) \quad \text{for all } j' \in I - \{i, j, k\}, \tag{1}$$

and

$$c(v_k v_{k'}) < c(f) \quad \text{for all } k' \in I - \{i, j, k\}. \tag{2}$$

We consider three cases, depending on the values of i and j.

Case 1: $j \leq p - 1$. Then, regardless of the values of i and j', $c(v_{j'} v_j) = j + j'$ and $c(e) = i + j$. By (1), $j' > i$ for all $j' \in I - \{i, j, k\}$. Hence $i \leq 2$. But $p \geq 3$ since $n \geq 6$, and therefore $i \leq p - 1$. Now $i + j = c(e) < c(f) = i + k$ implies that $j < k$. Therefore one of the following three subcases holds:

(i) $j = 0$, $k = 1$ and $i = 2$,

(ii) $j = 0$ and $k > i = 1$,

(iii) $i = 0$ and $k > j > 0$.

If (i) holds, then $c(v_j v_k) = 1$. Since $n \geq 6$, there exists $k' \in I - \{0, 1, 2\}$ such that $c(v_k v_{k'}) = k + k' \geq k' + 1 \geq 4 > c(f) = i + k = 3$, contradicting (2). If (ii) holds, then $c(f) = 1 + k$. If $k \leq p - 1$, then v_k is adjacent to v_p, where $p \notin \{0, 1, k\}$, and $c(v_k v_p) = k + p > c(f)$, contradicting (2); while if $k \geq p$, then v_k is adjacent to v_2 and $c(v_2 v_k) = k + 2 > c(f)$, again a contradiction. If (iii) holds, then $c(e) = j < k = c(f)$. If $k \leq p - 1$, then $j < p - 1$ and v_k is adjacent to v_p, where $p \notin \{0, j, k\}$, giving a contradiction as in (ii). If $k \geq p$, then there exists $\ell \in \{1, 2\} - \{j\}$ such that $c(v_k v_\ell) = k + \ell > k$, once again a contradiction.

Case 2: $j \geq p$ and $i \leq p-1$. Then $c(e) = i + j$. Since $i \leq p-1$ and $n \geq 6$, there exists $j' \in I - \{i, j, k\}$ such that $j' \geq p$. Then $c(v_{j'}v_j) = j + j' - 2p > i + j$ by (1); that is, $i < j' - 2p \leq 0$, which is impossible.

Case 3: $\min\{i, j\} \geq p$. Then $c(e) = i + j - 2p$. Suppose there exists $j' \in I - \{i, j, k\}$ such that $j' \geq p$. Then $c(v_{j'}v_j) = j + j' - 2p$ and thus $j' > i$ by (1). Since $i, j' \geq p$,

$$c(f) = c(v_i v_k) = \begin{cases} i+k & \text{if } k \leq p-1, \\ i+k-2p & \text{if } k \geq p, \end{cases}$$

and

$$c(v_k v_{j'}) = \begin{cases} j'+k & \text{if } k \leq p-1, \\ j'+k-2p & \text{if } k \geq p. \end{cases}$$

Thus, regardless of the value of k, $c(v_k v_{j'}) > c(f)$. Since $j' \in I - \{i, j, k\}$, this contradicts (2). Hence there does not exist $j' \in I - \{i, j, k\}$ such that $j' \geq p$. Since $n \geq 6$, we have $|\{p, \dots, n-1\}| \geq 3$. We deduce that $n \in \{6, 7\}$ and $\{p, \dots, n-1\} = \{i, j, k\}$ so that $c(e) = i + j - 2p$ and $c(f) = i + k - 2p$, where $j < k$ since $c(e) < c(f)$. For either value of n, $c(f) \leq 3$ and $k \geq 4$. Let $j' = 0 < p$. Then $j' \in I - \{i, j, k\}$ and $c(v_{j'}v_k) = j' + k = k \geq 4 > 3 \geq c(f)$, again contradicting (2). □

The following corollary to Lemma 2 and Theorem 3 improves Theorem 17 of [Schurch 2013b].

Corollary 4. *For $n \geq 6$, we have $\chi_\varepsilon(K_n) \leq \lfloor (3n-3)/2 \rfloor$.*

Combining Theorem 1 and Corollary 4 we improve Proposition 20 of [Schurch 2013b] and also obtain the new value $\chi_\varepsilon(K_7)$.

Corollary 5. $$\chi_\varepsilon(K_6) = 7 \quad \textit{and} \quad \chi_\varepsilon(K_7) = 9.$$

3.2. *The case $n \equiv 0 \pmod 4$*. Our next result is an improved upper bound for $\chi_\varepsilon(K_n)$ in the case where $n \equiv 0 \pmod 4$ and $n \geq 8$. Say $n = 4m$ and $V(K_n) = \{u_0, \dots, u_{2m-1}, v_0, \dots, v_{2m-1}\}$. Let G and H be the subgraphs of K_n induced by $\{u_0, \dots, u_{2m-1}\}$ and $\{v_0, \dots, v_{2m-1}\}$, respectively. Then $G \cong H \cong K_{2m}$ and each of them is $(2m-1)$-edge colourable. We describe a colouring c_1 of K_n in the colours $1, \dots, 4m+1$ as follows.

- In G, let c_1 be any proper edge colouring of K_{2m} in the $2m-1$ colours $\{1, 2\} \cup \{m+3, \dots, 3m-1\}$.
- In H, let c_1 be any proper edge colouring of K_{2m} in the $2m-1$ colours $\{4m, 4m+1\} \cup \{m+3, \dots, 3m-1\}$.
- We still need to colour the edges of the complete bipartite graph $F \cong K_{2m,2m}$ induced by the edges $u_i v_j$, with $i, j \in \{0, \dots, 2m-1\}$. But $\chi'(K_{2m,2m}) = 2m$ and there are $2m$ unused colours $3, \dots, m+2$ and $3m, \dots, 4m-1$. Colour the edges of F with these colours.

It is clear that c_1 is a proper edge colouring of K_{4m} in $4m+1$ colours.

Theorem 6. *For all $m \geq 2$, the colouring c_1 of K_{4m} has flatness equal to three.*

Proof. Let F, G and H be the subgraphs of K_{4m} defined above and let $e, f \in E(K_{4m})$ be adjacent edges such that $c_1(e) < c_1(f)$. We show that (Sa) or (Sb) holds, as stated in the proof of Theorem 3. We consider three cases, depending on the choice of e and f.

Case 1: $\{e, f\} \cap E(F) = \varnothing$. Assume first $e, f \in E(G)$; say $e = u_j u_i$ and $f = u_i u_k$. Then $c_1(e) < c_1(f) \leq 3m-1$, and u_k is adjacent to some vertex $v_\ell \in V(H)$ such that $c_1(u_k v_\ell) = 4m-1 > c_1(f)$. Hence (Sb) holds. Similarly, if $e, f \in E(H)$, say $e = v_j v_i$ and $f = v_i v_k$, then $c_1(f) > c_1(e) \geq m+3$, and v_j is adjacent to some vertex $u_\ell \in V(G)$ such that $c_1(v_j u_\ell) = 3 < c_1(e)$. Hence (Sa) holds.

Case 2: $|\{e, f\} \cap E(F)| = 1$. By symmetry we may assume that $e \in E(F)$; say $e = u_i v_j$. If $f \in E(G)$, say $f = u_i u_k$, then $c_1(e) \in \{3, \ldots, m+2\}$ and $c_1(f) \in \{m+3, m+4, \ldots, 3m-1\}$. Since $m \geq 2$, u_k is adjacent to at least two vertices v_{t_1}, v_{t_2} of H such that $c_1(u_k v_{t_\ell}) \in \{3m, \ldots, 4m-1\}$ for $\ell = 1, 2$, and we may choose a subscript t_ℓ, say t_1, such that $t_1 \neq j$. Then v_j, u_i, u_k, v_{t_1} is a c_1-ascent of length three and (Sb) holds. On the other hand, if $f \in E(H)$, say $f = v_j v_k$, then $c_1(e) \geq 3$. In this case u_i is adjacent to a vertex u_ℓ such that $c_1(u_\ell u_i) \in \{1, 2\}$ and (Sa) holds.

Case 3: $\{e, f\} \subseteq E(F)$. First, if $e = u_i v_j$ and $f = v_j u_k$, then there exists at least one index $\ell \in \{0, \ldots, 2m-1\} - \{i, k\}$ such that $c_1(u_\ell u_i) \in \{1, 2\}$. Then u_ℓ, u_i, v_j, u_k is a c_1-ascent of length three and (Sa) holds. Finally, if $e = v_i u_j$ and $f = u_j v_k$, then there exists at least one index $\ell \in \{0, \ldots, 2m-1\} - \{i, k\}$ such that $c_1(v_k v_\ell) \in \{4m, 4m+1\}$. Then v_i, u_j, v_k, v_ℓ is a c_1-ascent of length three and (Sb) holds. □

Combining Theorems 1 and 6 we narrow down $\chi_\varepsilon(K_n)$ to two possible values in infinitely many cases.

Corollary 7. *For all $n \geq 8$ and $n \equiv 0 \pmod 4$, we have $n \leq \chi_\varepsilon(K_n) \leq n+1$.*

3.3. ***The case $n \equiv 2 \pmod 4$*.** We now assume that $n \equiv 2 \pmod 4$ and $n \geq 10$. Say $n = 4m+2$ and $V(K_n) = \{u_0, \ldots, u_{2m}, v_0, \ldots, v_{2m}\}$. Let G and H be the subgraphs of K_n induced by $\{u_0, \ldots, u_{2m}\}$ and $\{v_0, \ldots, v_{2m}\}$, respectively. Then $G \cong H \cong K_{2m+1}$ and each of them is $(2m+1)$-edge colourable. We describe an edge colouring c_2 of K_n in the colours $1, \ldots, 4m+3$. This colouring is similar to the colouring c_1 above, but not quite as straightforward. See Figure 2 for a partial colouring of K_{10}.

- In G, let c_2 be any proper edge colouring of K_{2m+1} in the $2m+1$ colours $\{1, 2\} \cup \{m+3, \ldots, 3m+1\}$.
- In H, let c_2 be any proper edge colouring of K_{2m+1} in the $2m+1$ colours $\{4m+2, 4m+3\} \cup \{m+3, \ldots, 3m+1\}$.

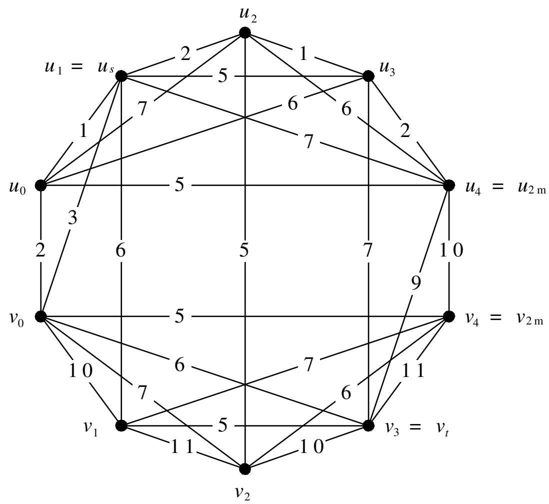

Figure 2. Part of the edge colouring c_2 of K_{10}.

We still need to colour the edges of the complete bipartite graph $F \cong K_{2m+1,2m+1}$ induced by the edges $u_i v_j$, with $i, j \in \{0, \ldots, 2m\}$. By König's theorem, F is 1-factorable. Note that for each colour k in the edge colouring of G there is exactly one vertex that is not incident with an edge coloured k, and conversely, for each vertex u_i there is exactly one colour that does not occur as colour of an edge incident with u_i. A similar remark holds for H. Without loss of generality, say colour 2 does not occur at u_0, colour 1 does not appear at u_{2m}, colour $4m+3$ does not appear at v_0 and colour $4m+2$ does not appear at v_{2m}. Since colour 2 does not occur at u_0, all other colours of the colouring do and thus there exists a vertex $u_s \in V(G)$ such that $c_2(u_0 u_s) = 1$. Since colour $4m+2$ does not appear at v_{2m}, there exists a vertex $v_t \in V(H)$ such that $c_2(v_{2m} v_t) = 4m+3$.

- Colour the edges $u_0 v_0$ and $u_{2m} v_{2m}$ of F with colours 2 and $4m+2$, respectively. For $i, j \in \{1, \ldots, 2m-1\}$ and $k \in \{m+3, \ldots, 3m+1\}$, colour $u_i v_j$ with colour k if and only if no edge incident with u_i in G or with v_j in H is coloured k.

We have now coloured a 1-factor F_0 of F, and $F - F_0$ is a $2m$-regular bipartite graph, which is 1-factorable by König's theorem. Let F_1' be a 1-factor of $F - F_0$ that contains the edge $v_0 u_s$. If $u_{2m} v_t \notin F_1'$, let $F_1 = F_1'$, and if $u_{2m} v_t \in F_1'$, let $u_i v_j \in F_1' - \{v_0 u_s, u_{2m} v_t\}$ and define $F_1 = (F_1' - \{u_i v_j, u_{2m} v_t\}) \cup \{u_i v_t, u_{2m} v_j\}$. Now $F - F_0 - F_1$ is 1-factorable. Let F_2 be a 1-factor of $F - F_0 - F_1$ that contains $u_{2m} v_t$.

- Colour the edges in F_1 with colour 3 and the edges in F_2 with colour $4m+1$. Colouring $F - F_0 - F_1 - F_2$ with the $2m-2$ unused colours $4, \ldots, m+2$ and $3m+2, \ldots, 4m$ yields a proper edge colouring of K_{4m+2}.

Theorem 8. *For all $m \geq 2$, the colouring c_2 of K_{4m+2} has flatness equal to three.*

Proof. Let F, J, G and H be the subgraphs of K_{4m+2} defined above and let $e, f \in E(K_{4m+2})$ be adjacent edges such that $c_2(e) < c_2(f)$. We show that (Sa) or (Sb) holds, as stated in the proof of Theorem 3. If $\{e, f\} \cap E(F) = \varnothing$, the proof follows similar to Case 1 in the proof of Theorem 6. We consider two further cases.

Case 1: $|\{e, f\} \cap E(F)| = 1$. By symmetry we may assume that $e \in E(F)$; say $e = u_i v_j$. First suppose that $f \in E(G)$, say $f = u_i u_k$. Since $c_2(f) > c_2(e) \geq 2$, $c_2(f) \in \{m+3, \ldots, 3m+1\}$. As in Case 2 of the proof of Theorem 3, (Sb) holds. Now suppose $f = v_j v_k \in E(H)$. If $c_2(e) = 2$, then $i = j = 0$ and $c_2(u_0 u_s) = 1$. If $c_2(e) \neq 2$ then $c_2(e) > 2$ and there exists an index ℓ such that $c_2(u_i u_\ell) \in \{1, 2\}$. Thus u_s, u_i, v_j, v_k or u_ℓ, u_i, v_j, v_k is a c_2-ascent of length three and (Sa) holds.

Case 2: $\{e, f\} \subseteq E(F)$. Suppose $e = u_i v_j$ and $f = v_j u_k$. If $e = u_0 v_0$ and $f = v_0 u_s$, then $c_2(e) = 2$ and $c_2(f) = 3$. Therefore there exists a vertex u_ℓ such that $c_2(u_s u_\ell) \in \{m+3, \ldots, 3m+1\}$ and (Sb) holds. If $e = u_0 v_0$ and $k \neq s$, then u_s, u_0, v_0, v_k is a c_2-ascent of length three and (Sa) holds. For all other choices of $e = u_i v_j$ and $f = v_j u_k$ it follows as in Case 3 of the proof of Theorem 3 that (Sa) or (Sb) holds. Suppose $e = v_i u_j$ and $f = u_j v_k$. If $e = v_t u_{2m}$ and $f = u_{2m} v_{2m}$, then $c_2(e) = 4m + 1$ and $c_2(f) = 4m + 2$. There exists a vertex v_ℓ such that $c_2(v_\ell v_t) \in \{m+3, \ldots, 3m+1\}$ and thus (Sa) holds. If $f = u_{2m} v_{2m}$ and $i \neq t$, then v_i, v_{2m}, u_{2m}, v_t is a c_2-ascent of length three and (Sb) holds. All other cases are dealt with as in Case 3 of the proof of Theorem 3. □

Combining Theorems 1 and 8 and Corollary 4 determines $\chi_\varepsilon(K_n)$ for all $n \equiv 2 \pmod{10}$, $n \geq 6$.

Corollary 9. *For all $n \geq 6$ and $n \equiv 2 \pmod{10}$, we have $\chi_\varepsilon(K_n) = n + 1$.*

4. Conclusion

In Theorem 1 we proved a lower bound for $\chi_\varepsilon(K_n)$, and in Corollary 4 we improved the previously known general upper bound for $\chi_\varepsilon(K_n)$ from $2n - 3$ to $\lfloor (3n-3)/2 \rfloor$. Corollary 7 improves this bound for $n \equiv 0 \pmod 4$ and allows us to bound $\chi_\varepsilon(K_{4m})$ by $4m \leq \chi_\varepsilon(K_n) \leq 4m + 1$. Finally, Corollary 9 determines $\chi_\varepsilon(K_n)$ for all $n \equiv 2 \pmod 4$, $n \geq 6$. Based on the results for even n and the values $\chi_\varepsilon(K_5) = 7$ and $\chi_\varepsilon(K_7) = 9$, we formulate the following conjecture.

Conjecture 10. *For all $n \geq 4$, we have $\chi_\varepsilon(K_n) = \chi'(K_n) + 2$.*

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References

[Bialostocki and Roditty 1987] A. Bialostocki and Y. Roditty, "A monotone path in an edge-ordered graph", *Internat. J. Math. Math. Sci.* **10**:2 (1987), 315–320. MR 88b:05087 Zbl 0633.05043

[Burger et al. 2005] A. P. Burger, E. J. Cockayne, and C. M. Mynhardt, "Altitude of small complete and complete bipartite graphs", *Australas. J. Combin.* **31** (2005), 167–177. MR 2005i:05160 Zbl 1080.05046

[Calderbank et al. 1984] A. R. Calderbank, F. R. K. Chung, and D. G. Sturtevant, "Increasing sequences with nonzero block sums and increasing paths in edge-ordered graphs", *Discrete Math.* **50**:1 (1984), 15–28. MR 85k:05062 Zbl 0542.05058

[Chartrand et al. 2011] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & digraphs*, 5th ed., CRC Press, Boca Raton, FL, 2011. MR 2012c:05001 Zbl 1211.05001

[Chvátal and Komlós 1971] V. Chvátal and J. Komlós, "Some combinatorial theorems on monotonicity", *Canad. Math. Bull.* **14** (1971), 151–157. MR 49 #2445 Zbl 0214.23503

[Cockayne and Mynhardt 2006] E. J. Cockayne and C. M. Mynhardt, "A lower bound for the depression of trees", *Australas. J. Combin.* **35** (2006), 319–328. MR 2007j:05118 Zbl 1094.05018

[Cockayne et al. 2006] E. J. Cockayne, G. Geldenhuys, P. J. P. Grobler, C. M. Mynhardt, and J. H. van Vuuren, "The depression of a graph", *Util. Math.* **69** (2006), 143–160. Preprint (2004) available at http://tinyurl.com/GraphDepression2004.

[Gaber-Rosenblum and Roditty 2009] I. Gaber-Rosenblum and Y. Roditty, "The depression of a graph and the diameter of its line graph", *Discrete Math.* **309**:6 (2009), 1774–1778. MR 2010d:05045 Zbl 1205.05066

[Graham and Kleitman 1973] R. L. Graham and D. J. Kleitman, "Increasing paths in edge ordered graphs", *Period. Math. Hungar.* **3** (1973), 141–148. MR 48 #5910 Zbl 0243.05116

[Mynhardt 2008] C. M. Mynhardt, "Trees with depression three", *Discrete Math.* **308**:5-6 (2008), 855–864. MR 2008j:05186 Zbl 1149.05042

[Mynhardt and Schurch 2013] C. M. Mynhardt and M. Schurch, "A class of graphs with depression three", *Discrete Math.* **313**:11 (2013), 1224–1232. MR 3034754 Zbl 1277.05083

[Mynhardt and Schurch 2014] C. M. Mynhardt and M. Schurch, "A construction of a class of graphs with depression three", *Australas. J. Combin.* **58**:2 (2014), 249–263. Zbl 1296.05087

[Mynhardt et al. 2005] C. M. Mynhardt, A. P. Burger, T. C. Clark, B. Falvai, and N. D. R. Henderson, "Altitude of regular graphs with girth at least five", *Discrete Math.* **294**:3 (2005), 241–257. MR 2006a:05079 Zbl 1062.05131

[Roditty et al. 2001] Y. Roditty, B. Shoham, and R. Yuster, "Monotone paths in edge-ordered sparse graphs", *Discrete Math.* **226**:1-3 (2001), 411–417. MR 2001i:05096 Zbl 0961.05040

[Schurch 2013a] M. Schurch, *On the depression of graphs*, Ph.D. thesis, University of Victoria, 2013, available at https://dspace.library.uvic.ca:8443/handle/1828/4527.

[Schurch 2013b] M. Schurch, "Edge colourings and the depression of a graph", *J. Combin. Math. Combin. Comput.* **85** (2013), 195–212. MR 3088160 Zbl 1274.05176

[Schurch and Mynhardt 2014] M. Schurch and C. Mynhardt, "The depression of a graph and k-kernels", *Discuss. Math. Graph Theory* **34**:2 (2014), 233–247. MR 3194034 Zbl 1290.05128

[Yuster 2001] R. Yuster, "Large monotone paths in graphs with bounded degree", *Graphs Combin.* **17**:3 (2001), 579–587. MR 2002k:05135 Zbl 1010.05044

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