On the $\varepsilon$-ascent chromatic index of complete graphs

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An edge ordering of a graph $G = (V, E)$ is an injection $f : E \rightarrow \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. A path in $G$ for which the edge ordering $f$ increases along its edge sequence is called an $f$-ascent; an $f$-ascent is maximal if it is not contained in a longer $f$-ascent. The depression $\varepsilon(G)$ of $G$ is the smallest integer $k$ such that any edge ordering $f$ has a maximal $f$-ascent of length at most $k$. Applying the concept of ascents to edge colourings rather than edge orderings, we consider the problem of determining the minimum number $\chi_\varepsilon(K_n)$ of colours required to edge colour $K_n$, $n \geq 4$, such that the length of a shortest maximal ascent is equal to $\varepsilon(K_n) = 3$. We obtain new upper and lower bounds for $\chi_\varepsilon(K_n)$, which enable us to determine $\chi_\varepsilon(K_n)$ exactly for $n = 7$ and $n \equiv 2 \pmod{4}$ and to bound $\chi_\varepsilon(K_{4m})$ by $4m \leq \chi_\varepsilon(K_{4m}) \leq 4m + 1$.

1. Introduction

Following [Schurch 2013a; 2013b], we consider the following question:

**Question 1.** For $n \geq 4$, what is the smallest integer $r(n)$ for which there exists a proper edge colouring of $K_n$ in colours $1, \ldots, r(n)$ such that a shortest maximal path of increasing edge labels has length three?

Schurch showed that $r(n) \leq 2n - 3$ for all $n \geq 4$. This bound enabled him to determine $r(n)$ for $n \in \{4, 5\}$ and to show that $7 \leq r(6) \leq 8$. In Section 2 we give a lower bound for $r(n)$ and in Section 3 we improve the general upper bound to

$$r(n) \leq \left\lfloor \frac{3n - 3}{2} \right\rfloor.$$ 

We then improve this bound for even values of $n$. Consequently, we obtain $r(7) = 9$, $r(n) = n + 1$ if $n \equiv 2 \pmod{4}$, and $n \leq r(n) \leq n + 1$ if $n \equiv 0 \pmod{4}$ and $n \geq 8$.

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We begin with a short historical account of the background to this problem. An edge ordering of a finite, simple graph $G$ is an injection $f : E(G) \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. Denote the set of all edge orderings of $G$ by $\mathcal{F}(G)$. A path $v_1, \ldots, v_k$ (where $v_k \neq v_1$) in $G$ such that $f(v_1) < \cdots < f(v_k)$ is called an $f$-ascent; an $f$-ascent is maximal if it is not contained in a longer $f$-ascent. The height $H(f)$ of an edge ordering $f$ is the length of a longest $f$-ascent, and the flatness of $f$, denoted by $h(f)$, is the length of a shortest maximal $f$-ascent of $G$.

Chvátal and Komlós [1971] posed the problem of determining $\alpha(K_n) = \min_{f \in \mathcal{F}(K_n)} \{H(f)\}$ of the complete graph $K_n$. This is a difficult problem and $\alpha(K_n)$ is known only for $1 \leq n \leq 8$ (see [Burger et al. 2005; Chvátal and Komlós 1971]). The parameter $\alpha(G)$ for complete and other finite graphs was also investigated in [Bialostocki and Roditty 1987; Burger et al. 2005; Calderbank et al. 1984; Graham and Kleitman 1973; Mynhardt et al. 2005; Roditty et al. 2001; Yuster 2001].

For an arbitrary finite graph $G$, Cockayne et al. [2006] considered the problem of determining $\varepsilon(G) = \max_{f \in \mathcal{F}(G)} \{h(f)\}$, that is, the maximum length, taken over all edge orderings $f \in \mathcal{F}(G)$, of a shortest maximal $f$-ascent. The parameter $\varepsilon(G)$ is known as the depression of $G$ and its computation is likewise a difficult problem. Another interpretation of the depression of $G$ is that any edge ordering $f$ of $G$ has a maximal $f$-ascent of length at most $\varepsilon(G)$, and $\varepsilon(G)$ is the smallest integer for which this statement is true. Graphs with depression two were characterized in [Cockayne et al. 2006], while trees with depression three were characterized in [Mynhardt 2008]. Graphs with no adjacent vertices of degree three or higher that have depression three were characterized in [Mynhardt and Schurch 2013]. Further work on depression can be found in [Cockayne and Mynhardt 2006; Gaber-Rosenblum and Roditty 2009; Schurch and Mynhardt 2014; 2014; Schurch 2013a; 2013b].

An edge ordering of $G$ is also a proper edge colouring—a labelling of the edges of $G$ such that adjacent edges have different labels. The minimum number of labels, also called colours, is called the edge chromatic number or the chromatic index $\chi'(G)$. It is well known (see [Chartrand et al. 2011, Section 10.2], for example) that $\chi'(K_n) = n - 1$ if $n$ is even and $\chi'(K_n) = n$ if $n$ is odd. A 1-factor of $G$ is a 1-regular spanning subgraph of $G$, and $G$ is 1-factorable if $E(G)$ can be partitioned into 1-factors. If $G$ is 1-factorable, then $G$ is $r$-regular for some $r$ and $\chi'(G) = r$. König’s theorem (see [Chartrand et al. 2011, Theorem 10.15]) states that every $r$-regular bipartite graph is 1-factorable. In particular, the chromatic index of the complete bipartite graph $K_{n,n}$ is given by $\chi'(K_{n,n}) = n$.

Noticing that the labels of some edges in an edge ordering of $G$ may be unimportant when determining $\varepsilon(G)$, Schurch applied the concept of ascents to edge
colourings and called the minimum number of colours in a proper edge colouring $c$ of $G$ such that $h(c) = \varepsilon(G)$ the $\varepsilon$-ascent chromatic index of $G$, denoted $\chi_\varepsilon(G)$. Unlike the case for general graphs, the depression of $K_n$ is easy to determine: $\varepsilon(K_1) = 0$, $\varepsilon(K_2) = 1$, $\varepsilon(K_3) = 2$ and $\varepsilon(K_n) = 3$ for all $n \geq 4$ (see [Cockayne et al. 2006]); that is, there does not exist an edge ordering or an edge colouring of $K_n$ such that a shortest maximal ascent has length four or more. Note that $\chi_\varepsilon(K_1) = 0$, $\chi_\varepsilon(K_2) = 1$, $\chi_\varepsilon(K_3) = 3$, and determining $\chi_\varepsilon(K_n)$ for $n \geq 4$ is equivalent to finding the smallest integer $r(n)$ such that there exists a proper edge colouring $c$ of $K_n$ in colours $1, \ldots, r(n)$ with $h(c) = 3$, as formulated in Question 1.

2. Lower bound for the $\varepsilon$-ascent chromatic index of $K_n$

We begin with a simple lower bound for $\chi_\varepsilon(K_n)$, which slightly improves the bound in [Schurch 2013b, Proposition 8] in the special case where $G = K_n$.

**Theorem 1.** If $n \geq 4$, then

$$\chi_\varepsilon(K_n) \geq \begin{cases} n & \text{if } n \equiv 0 \pmod{4}, \\ n + 1 & \text{if } n \equiv 1, 2 \pmod{4}, \\ n + 2 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

**Proof.** Let $c$ be a proper edge colouring of $K_n$ in colours $1, \ldots, r$ such that $h(c) = 3$. Such a colouring exists because $\varepsilon(K_n) = 3$ if $n \geq 4$. For $i = 1, \ldots, r$, define

$$E_i = \{e \in E(K_n) : c(e) = i\}.$$ 

Then $|E_i| \leq \lfloor n/2 \rfloor$ for each $i$. Also, no vertex $v$ is incident with an edge $e \in E_1$ and an edge $e' \in E_r$, otherwise $e, e'$ is a maximal $c$-ascent of length two, which contradicts $h(c) = 3$. Thus $|E_1 \cup E_r| \leq \lfloor n/2 \rfloor$ and $E_1 \cup E_r$ is an independent set of edges, that is, $E_1 \cup E_r, E_2, \ldots, E_{r-1}$ is also a proper edge colouring of $K_n$. Hence $r \geq \chi'(K_n) + 1$. In particular,

$$\chi_\varepsilon(K_n) \geq \begin{cases} n & \text{if } n \equiv 0 \pmod{4}, \\ n + 1 & \text{if } n \equiv 1 \pmod{4}. \end{cases}$$

Assume $n \equiv 2 \pmod{4}$; say $n = 4p + 2$. Then $K_n$ has $(2p + 1)(4p + 1)$ edges. Suppose $r = \chi'(K_n) + 1 = n$. The upper bound

$$|E_1 \cup E_r|, |E_2|, \ldots, |E_{r-1}| \leq \left\lfloor \frac{n}{2} \right\rfloor$$

implies that

$$|E_1 \cup E_r| = |E_2| = \cdots = |E_{r-1}| = \left\lfloor \frac{n}{2} \right\rfloor = 2p + 1.$$ 

Since $|E_1| + |E_r| = 2p + 1$, an odd number, $|E_1| \neq |E_r|$. Without loss of generality say $|E_1| = k$, where $k \leq p$, and $|E_r| = 2p + 1 - k$. Suppose $e \in E_2$ is not adjacent to
any edge in $E_1$. Since $|E_1 \cup E_r| = 2p + 1 = \lfloor n/2 \rfloor$, $e$ is adjacent to an edge $e' \in E_r$. But then $e, e'$ is a maximal $c$-ascent of length two, which contradicts $h(c) = 3$. Therefore each edge in $E_2$ is adjacent to an edge in $E_1$, and since $c$ is a proper edge colouring, $|E_2| \leq 2|E_1| = 2k \leq 2p < \lfloor n/2 \rfloor$, a contradiction. Thus $r \geq n + 1$ as required.

Assume $n \equiv 3 \pmod{4}$; say $n = 4p + 3$. Then $|E(K_n)| = (4p + 3)(2p + 1)$. Suppose $r = \chi'(K_n) + 1 = n + 1$. As in the case $n \equiv 2 \pmod{4}$, we obtain that $|E_1 \cup E_r| = |E_2| = \cdots = |E_{r-1}| = \lfloor n/2 \rfloor = 2p + 1$ and that each edge in $E_2$ is adjacent to an edge in $E_1$. There is one vertex $v$ that is not incident with any edge in $E_1 \cup E_r$, but an edge in $E_2$ incident with $v$ also needs to be adjacent to an edge in $E_1$. We obtain a contradiction as above and the result follows. □

3. Upper bounds for the $\varepsilon$-ascent chromatic index of $K_n$

In Section 3.1 we provide a new general upper bound for $\chi_\varepsilon(K_n)$. We improve this bound for even values of $n$ in Sections 3.2 (the case $n \equiv 0 \pmod{4}$) and 3.3 (the case $n \equiv 2 \pmod{4}$).

3.1. A general bound. For $n \geq 6$, we now describe an edge colouring $c$ of $K_n$ in $\lfloor (3n - 3)/2 \rfloor$ colours, as illustrated in Figure 1 for $n \in \{6, 7\}$, and prove in Theorem 3 that $h(c) = 3$. Let $V(K_n) = \{v_0, \ldots, v_{n-1}\}$ and $p = \lfloor n/2 \rfloor$.

- For $i \in \{0, \ldots, p-1\}$ and $j \in \{i + 1, \ldots, n-1\}$, let $c(v_i v_j) = i + j$.
- For $i \in \{p, \ldots, n-2\}$ and $j \in \{i + 1, \ldots, n-1\}$, let $c(v_i v_j) = i + j - 2p$.

Lemma 2. For all $n \geq 6$, the colouring $c$ defines a proper edge colouring of $K_n$ in $\lfloor (3n - 3)/2 \rfloor$ colours.

Figure 1. Edge colourings of $K_6$ and $K_7$ with flatness three.
Proof. Suppose that \( c(v_i v_j) = c(v_i v_{j'}) \) for some \( j < j' \). After a brief reflection, we deduce that \( i + j = i + j' - 2p \). But \( i + j \geq i \) and

\[
i + j' - 2p \leq i + n - 1 - 2\lceil n/2 \rceil \leq i - 1,
\]

hence \( c(v_i v_j) > c(v_i v_{j'}) \), contradicting our assumption.

Since the smallest colour is \( 0 + 1 = 1 \) and the largest colour is

\[
p - 1 + n - 1 = \lceil n/2 \rceil + n - 2 = \lceil n - 1/2 \rceil + n - 1 = \lceil 3n - 3/2 \rceil,
\]

the colouring \( c \) uses exactly \( \lfloor (3n - 3)/2 \rfloor \) colours. \( \square \)

**Theorem 3.** For all \( n \geq 6 \), the colouring \( c \) of \( K_n \) has flatness equal to three.

**Proof.** To prove that \( h(c) = 3 \), it is sufficient to prove this:

**Statement.** For any \( v_i \in V(K_n) \) and edges \( e = v_j v_i \) and \( f = v_i v_k \) such that \( c(e) < c(f) \), there exists

(Sa) an edge \( g = v_j' v_j, j' \notin \{i, j, k\} \), such that \( c(g) < c(e) \), or

(Sb) an edge \( g = v_k v'_k, k' \notin \{i, j, k\} \), such that \( c(f) < c(g) \).

Hence suppose there exist indices \( i, j, k \in I = \{0, \ldots, n - 1\} \) such that for edges \( e = v_j v_i \) and \( f = v_i v_k \), we have \( c(e) < c(f) \), but neither (Sa) nor (Sb) holds. Then

\[
c(v_j' v_j) > c(e) \quad \text{for all } j' \in I - \{i, j, k\}, \quad (1)
\]

and

\[
c(v_k v'_k) < c(f) \quad \text{for all } k' \in I - \{i, j, k\}. \quad (2)
\]

We consider three cases, depending on the values of \( i \) and \( j \).

**Case 1:** \( j \leq p - 1 \). Then, regardless of the values of \( i \) and \( j' \), \( c(v_j' v_j) = j + j' \) and \( c(e) = i + j \). By (1), \( j' > i \) for all \( j' \in I - \{i, j, k\} \). Hence \( i \leq 2 \). But \( p \geq 3 \) since \( n \geq 6 \), and therefore \( i \leq p - 1 \). Now \( i + j = c(e) < c(f) = i + k \) implies that \( j < k \). Therefore one of the following three subcases holds:

(i) \( j = 0, k = 1 \) and \( i = 2 \),

(ii) \( j = 0 \) and \( k > i = 1 \),

(iii) \( i = 0 \) and \( k > j > 0 \).

If (i) holds, then \( c(v_j v_k) = 1 \). Since \( n \geq 6 \), there exists \( k' \in I - \{0, 1, 2\} \) such that \( c(v_k v_{k'}) = k + k' \geq k' + 1 \geq 4 > c(f) = i + k = 3 \), contradicting (2). If (ii) holds, then \( c(f) = 1 + k \). If \( k \leq p - 1 \), then \( v_k \) is adjacent to \( v_p \), where \( p \notin \{0, 1, k\} \), and \( c(v_k v_p) = k + p > c(f) \), contradicting (2); while if \( k \geq p \), then \( v_k \) is adjacent to \( v_2 \) and \( c(v_2 v_k) = k + 2 > c(f) \), again a contradiction. If (iii) holds, then \( c(e) = j < k = c(f) \). If \( k \leq p - 1 \), then \( j < p - 1 \) and \( v_k \) is adjacent to \( v_p \), where \( p \notin \{0, j, k\} \), giving a contradiction as in (ii). If \( k \geq p \), then there exists \( \ell \in \{1, 2\} - \{j\} \) such that \( c(v_k v_{\ell}) = k + \ell > k \), once again a contradiction.
Thus, regardless of the value of $\chi$, Corollary 5.

The case $n \equiv 0 \pmod{4}$. Our next result is an improved upper bound for $\chi_e(K_n)$ in the case where $n \equiv 0 \pmod{4}$ and $n \geq 8$. Say $n = 4m$ and $V(K_n) = \{u_0, \ldots, u_{2m-1}, v_0, \ldots, v_{2m-1}\}$. Let $G$ and $H$ be the subgraphs of $K_n$ induced by $\{u_0, \ldots, u_{2m-1}\}$ and $\{v_0, \ldots, v_{2m-1}\}$, respectively. Then $G \cong H \cong K_{2m}$ and each of them is $(2m-1)$-edge colourable. We describe a colouring $c_1$ of $K_n$ in the colours $1, \ldots, 4m+1$ as follows.

- In $G$, let $c_1$ be any proper edge colouring of $K_{2m}$ in the $2m-1$ colours $\{1, 2\} \cup \{m+3, \ldots, 3m-1\}$.
- In $H$, let $c_1$ be any proper edge colouring of $K_{2m}$ in the $2m-1$ colours $\{4m, 4m+1\} \cup \{m+3, \ldots, 3m-1\}$.
- We still need to colour the edges of the complete bipartite graph $F \cong K_{2m,2m}$ induced by the edges $u_iv_j$, with $i, j \in \{0, \ldots, 2m-1\}$. But $\chi'(K_{2m,2m}) = 2m$ and there are $2m$ unused colours $3, \ldots, m+2$ and $3m, \ldots, 4m-1$. Colour the edges of $F$ with these colours.
It is clear that $c_1$ is a proper edge colouring of $K_{4m}$ in $4m + 1$ colours.

**Theorem 6.** For all $m \geq 2$, the colouring $c_1$ of $K_{4m}$ has flatness equal to three.

*Proof.* Let $F$, $G$ and $H$ be the subgraphs of $K_{4m}$ defined above and let $e, f \in E(K_{4m})$ be adjacent edges such that $c_1(e) < c_1(f)$. We show that (Sa) or (Sb) holds, as stated in the proof of Theorem 3. We consider three cases, depending on the choice of $e$ and $f$.

**Case 1:** $\{e, f\} \cap E(F) = \emptyset$. Assume first $e, f \in E(G)$; say $e = u_ju_i$ and $f = u_iu_k$. Then $c_1(e) < c_1(f) \leq 3m - 1$, and $u_k$ is adjacent to some vertex $v_\ell \in V(H)$ such that $c_1(u_kv_\ell) = 4m - 1 > c_1(f)$. Hence (Sb) holds. Similarly, if $e, f \in E(H)$, say $e = v_jv_i$ and $f = v_iv_k$, then $c_1(f) > c_1(e) \geq m + 3$, and $v_j$ is adjacent to some vertex $u_\ell \in V(G)$ such that $c_1(v_ju_\ell) = 3 < c_1(e)$. Hence (Sa) holds.

**Case 2:** $|\{e, f\} \cap E(F)| = 1$. By symmetry we may assume that $e \in E(F)$; say $e = u_iu_j$. If $f \in E(G)$, say $f = u_iu_k$, then $c_1(e) \in \{3, \ldots, m + 2\}$ and $c_1(f) \in \{m + 3, m + 4, \ldots, 3m - 1\}$. Since $m \geq 2$, $u_k$ is adjacent to at least two vertices $v_{i_1}, v_{i_2}$ of $H$ such that $c_1(u_kv_{i_\ell}) \in \{3m, \ldots, 4m - 1\}$ for $\ell = 1, 2$, and we may choose a subscripts $t_\ell$, say $t_1$, such that $t_1 \neq j$. Then $v_j, u_i, u_k, v_{i_1}$ is a $c_1$-ascent of length three and (Sb) holds. On the other hand, if $f \in E(H)$, say $f = v_jv_k$, then $c_1(e) \geq 3$. In this case $u_i$ is adjacent to a vertex $u_\ell$ such that $c_1(u_\ell u_i) \in \{1, 2\}$ and (Sa) holds.

**Case 3:** $\{e, f\} \subseteq E(F)$. First, if $e = u_iu_j$ and $f = v_jv_k$, then there exists at least one index $\ell \in \{0, \ldots, 2m - 1\} - \{i, k\}$ such that $c_1(u_\ell u_i) \in \{1, 2\}$. Then $u_\ell, u_i, v_j, u_k$ is a $c_1$-ascent of length three and (Sa) holds. Finally, if $e = v_iu_j$ and $f = u_jv_k$, then there exists at least one index $\ell \in \{0, \ldots, 2m - 1\} - \{i, k\}$ such that $c_1(v_\ell v_k) \in \{4m, 4m + 1\}$. Then $v_i, u_j, v_k, v_\ell$ is a $c_1$-ascent of length three and (Sb) holds. $\square$

Combining Theorems 1 and 6 we narrow down $\chi_\varepsilon(K_n)$ to two possible values in infinitely many cases.

**Corollary 7.** For all $n \geq 8$ and $n \equiv 0 \pmod{4}$, we have $n \leq \chi_\varepsilon(K_n) \leq n + 1$.

### 3.3. The case $n \equiv 2 \pmod{4}$

We now assume that $n \equiv 2 \pmod{4}$ and $n \geq 10$. Say $n = 4m + 2$ and $V(K_n) = \{u_0, \ldots, u_{2m}, v_0, \ldots, v_{2m}\}$. Let $G$ and $H$ be the subgraphs of $K_n$ induced by $\{u_0, \ldots, u_{2m}\}$ and $\{v_0, \ldots, v_{2m}\}$, respectively. Then $G \cong H \cong K_{2m+1}$ and each of them is $(2m+1)$-edge colourable. We describe an edge colouring $c_2$ of $K_n$ in the colours $1, \ldots, 4m + 3$. This colouring is similar to the colouring $c_1$ above, but not quite as straightforward. See Figure 2 for a partial colouring of $K_{10}$.

- In $G$, let $c_2$ be any proper edge colouring of $K_{2m+1}$ in the $2m + 1$ colours $\{1, 2\} \cup \{m + 3, \ldots, 3m + 1\}$.
- In $H$, let $c_2$ be any proper edge colouring of $K_{2m+1}$ in the $2m + 1$ colours $\{4m + 2, 4m + 3\} \cup \{m + 3, \ldots, 3m + 1\}$.
Figure 2. Part of the edge colouring $c_2$ of $K_{10}$.

We still need to colour the edges of the complete bipartite graph $F \cong K_{2m+1,2m+1}$ induced by the edges $u_i v_j$, with $i, j \in \{0, \ldots, 2m\}$. By König’s theorem, $F$ is 1-factorable. Note that for each colour $k$ in the edge colouring of $G$ there is exactly one vertex that is not incident with an edge coloured $k$, and conversely, for each vertex $u_i$ there is exactly one colour that does not occur as colour of an edge incident with $u_i$. A similar remark holds for $H$. Without loss of generality, say colour 2 does not occur at $u_0$, colour 1 does not appear at $u_{2m}$, colour $4m+3$ does not appear at $v_0$ and colour $4m+2$ does not appear at $v_{2m}$. Since colour 2 does not occur at $u_0$, all other colours of the colouring do and thus there exists a vertex $u_s \in V(G)$ such that $c_2(u_0 u_s) = 1$. Since colour $4m+2$ does not appear at $v_{2m}$, there exists a vertex $v_t \in V(H)$ such that $c_2(v_{2m} v_t) = 4m+3$.

- Colour the edges $u_0 v_0$ and $u_{2m} v_{2m}$ of $F$ with colours 2 and 4$m+2$, respectively.

  For $i, j \in \{1, \ldots, 2m-1\}$ and $k \in \{m+3, \ldots, 3m+1\}$, colour $u_i v_j$ with colour $k$ if and only if no edge incident with $u_i$ in $G$ or with $v_j$ in $H$ is coloured $k$.

We have now coloured a 1-factor $F_0$ of $F$, and $F - F_0$ is a $2m$-regular bipartite graph, which is 1-factorable by König’s theorem. Let $F'_1$ be a 1-factor of $F - F_0$ that contains the edge $v_0 u_s$. If $u_{2m} v_t \notin F'_1$, let $F_1 = F'_1$, and if $u_{2m} v_t \in F'_1$, let $u_i v_j \in F'_1 - \{v_0 u_s, u_{2m} v_t\}$ and define $F_1 = (F'_1 - \{u_i v_j, u_{2m} v_t\}) \cup \{u_i v_t, u_{2m} v_j\}$. Now $F - F_0 - F_1$ is 1-factorable. Let $F_2$ be a 1-factor of $F - F_0 - F_1$ that contains $u_{2m} v_t$.

- Colour the edges in $F_1$ with colour 3 and the edges in $F_2$ with colour $4m+1$. Colouring $F - F_0 - F_1 - F_2$ with the $2m - 2$ unused colours $4, \ldots, m+2$ and $3m+2, \ldots, 4m$ yields a proper edge colouring of $K_{4m+2}$. 
Theorem 8. For all $m \geq 2$, the colouring $c_2$ of $K_{4m+2}$ has flatness equal to three.

Proof. Let $F$, $J$, $G$ and $H$ be the subgraphs of $K_{4m+2}$ defined above and let $e, f \in E(K_{4m+2})$ be adjacent edges such that $c_2(e) < c_2(f)$. We show that (Sa) or (Sb) holds, as stated in the proof of Theorem 3. If $\{e, f\} \cap E(F) = \emptyset$, the proof follows similar to Case 1 in the proof of Theorem 6. We consider two further cases.

Case 1: $|\{e, f\} \cap E(F)| = 1$. By symmetry we may assume that $e \in E(F)$; say $e = u_i v_j$. First suppose that $f \in E(G)$, say $f = u_i u_k$. Since $c_2(f) > c_2(e) \geq 2$, $c_2(f) \in \{m+3, \ldots, 3m+1\}$. As in Case 2 of the proof of Theorem 3, (Sb) holds. Now suppose $f = v_j v_k \in E(H)$. If $c_2(e) = 2$, then $i = j = 0$ and $c_2(u_0 u_s) = 1$. If $c_2(e) \neq 2$ then $c_2(e) > 2$ and there exists an index $\ell$ such that $c_2(u_\ell u_\ell) \in \{1, 2\}$. Thus $u_s, u_i, v_j, v_k$ or $u_\ell, u_i, v_j, v_k$ is a $c_2$-ascent of length three and (Sa) holds.

Case 2: $\{e, f\} \subseteq E(F)$. Suppose $e = u_i v_j$ and $f = v_j u_k$. If $e = u_0 v_0$ and $f = v_0 u_s$, then $c_2(e) = 2$ and $c_2(f) = 3$. Therefore there exists a vertex $u_\ell$ such that $c_2(u_\ell u_\ell) \in \{m+3, \ldots, 3m+1\}$ and (Sb) holds. If $e = u_0 v_0$ and $k \neq s$, then $u_s, u_0, v_0, v_k$ is a $c_2$-ascent of length three and (Sa) holds. For all other choices of $e = u_i v_j$ and $f = v_j u_k$ it follows as in Case 3 of the proof of Theorem 3 that (Sa) or (Sb) holds. Suppose $e = v_i u_j$ and $f = u_j v_k$. If $e = v_i u_2m$ and $f = u_{2m} v_{2m}$, then $c_2(e) = 4m + 1$ and $c_2(f) = 4m + 2$. There exists a vertex $v_\ell$ such that $c_2(v_\ell v_\ell) \in \{m+3, \ldots, 3m+1\}$ and thus (Sa) holds. If $f = u_{2m} v_{2m}$ and $i \neq t$, then $v_i, v_{2m}, u_{2m}, v_t$ is a $c_2$-ascent of length three and (Sa) holds. All other cases are dealt with as in Case 3 of the proof of Theorem 3.

Combining Theorems 1 and 8 and Corollary 4 determines $\chi_\varepsilon(K_n)$ for all $n \equiv 2 \pmod{10}$, $n \geq 6$.

Corollary 9. For all $n \geq 6$ and $n \equiv 2 \pmod{10}$, we have $\chi_\varepsilon(K_n) = n + 1$.

4. Conclusion

In Theorem 1 we proved a lower bound for $\chi_\varepsilon(K_n)$, and in Corollary 4 we improved the previously known general upper bound for $\chi_\varepsilon(K_n)$ from $2n - 3$ to $\lfloor (3n - 3)/2 \rfloor$. Corollary 7 improves this bound for $n \equiv 0 \pmod{4}$ and allows us to bound $\chi_\varepsilon(K_{4m})$ by $4m \leq \chi_\varepsilon(K_n) \leq 4m + 1$. Finally, Corollary 9 determines $\chi_\varepsilon(K_n)$ for all $n \equiv 2 \pmod{4}$, $n \geq 6$. Based on the results for even $n$ and the values $\chi_\varepsilon(K_5) = 7$ and $\chi_\varepsilon(K_7) = 9$, we formulate the following conjecture.

Conjecture 10. For all $n \geq 4$, we have $\chi_\varepsilon(K_n) = \chi'(K_n) + 2$.

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