involve

a journal of mathematics

Bisection envelopes

Noah Fechtor-Pradines

msp

2015 vol. 8, no. 2

INVOLVE 8:2 (2015)
dx.doi.org/10.2140/involve.2015.8.307

Bisection envelopes

Noah Fechtor-Pradines

(Communicated by Frank Morgan)

We study the envelope of the family of lines which bisect the interior region of a simple, closed curve in the plane. We determine this bisection envelope for polygons and show that polygons with no parallel pairs of sides are characterized by their bisection envelope. We show that the bisection envelope always has at least three and an odd number of cusps. We investigate the winding numbers of bisection envelopes, and use this to show that there are an infinite number of curves with any given bisection envelope and show how to generate them. We obtain results on the intersections of bisecting lines. Finally, we give a relationship between the internal area of a curve and that of its bisection envelope.

1. Introduction and overview

We study the envelope of the family of lines that bisect the interior region of a given simple, closed curve in the plane. This concept, which we call the bisection envelope, was explored in [Fusco and Pratelli 2011]; however, here we apply it to a more general class of curves. Fusco and Pratelli only used the bisection envelope in relation to Zindler sets — convex sets whose bisecting chords have fixed length: they used as a tool to rewrite the problem of minimizing the area of a Zindler set with fixed bisecting chord length.

Specifically, let $\mathcal{S}$ be a simple compact curve which is piecewise of class C^1 with a finite number of pieces. Let $\mathcal{L}$ be the set of lines l_θ that have direction θ and bisect the interior of $\mathcal{S}$. The bisection envelope of $\mathcal{S}$ is the envelope of the lines in $\mathcal{L}$. For curves $\mathcal{S}$ that are bisection convex (see Definition 2.2), we show that the bisection envelope is the midpoint locus of bisecting chords. Furthermore, we show that for curves that are strictly bisection convex (see Definition 2.3) we can parametrize the bisection envelope by a function f such that $f(\theta)$ lies on l_θ, and find the derivative of f, defined at all but a finite number of points. Where

MSC2010: 26B15, 51M25.
Keywords: bisection envelope, area, winding number, envelope, geometry, bisection.
At the time of the writing, the author was a student at the British International School of Boston. He is now an undergraduate at Harvard University.

this derivative exists we show it is of the form $v_\theta(\cos\theta, \sin\theta)$, and give conditions on $\mathcal{S}$ such that for a scalar v_θ,

$$f(\theta) = f(0) + \int_0^\theta v_t(\cos t, \sin t)\, dt.$$

We show that zeros of $f'(\theta)$ (which are also zeros of v_θ) each corresponds to a bisecting chord at whose endpoints the tangents to $\mathcal{S}$ are parallel. We also show a relation between sign changes of v_θ and the appearance of cusps on the bisection envelope. These results are summarized in Theorem 1.

In Section 3, we examine the bisection envelopes of polygons, showing that they are the union of sections of hyperbolas. Furthermore, for each hyperbola, there exist two sides of $\mathcal{S}$ which are segments of its asymptotes. We also show Theorem 2, which states that polygons with no mutually tangent sides are uniquely defined by their bisection envelopes.

Section 4 addresses curves with identical bisection envelopes. We show how to generate a curve $\mathcal{S}'$ from the bisection envelope $\mathcal{B}$ of a strictly bisection convex curve $\mathcal{S}$ satisfying certain criteria by letting $\mathcal{S}'$ be the image of a function g, defined as

$$g(\theta) = f(\theta) + r(\theta)(\cos\theta, \sin\theta),$$

where $r(\theta)$ is a radius function that can be changed to produce different $\mathcal{S}'$. The main result of Section 4 is Theorem 3, which states that if the generated $\mathcal{S}'$ does not intersect $\mathcal{B}$, then $\mathcal{B}$ is indeed the bisection envelope of both $\mathcal{S}$ and $\mathcal{S}'$.

To prove Theorem 3, we first prove Theorem 4, which concerns the winding numbers of bisection envelopes. Specifically, let m_P be the number of lines through a point P tangent to $\mathcal{B}$. We show that

$$m_P = -2w(P) + 1,$$

where $w(P)$ is the winding number of $\mathcal{B}$ about P with θ increasing from 0 to π.

In Section 5, we examine the interior areas of $\mathcal{S}'$ and $\mathcal{B}$. The interior area of $\mathcal{B}$ is usually not well-defined, as it can be self-intersecting, therefore we define the interior area of a curve Γ by the integral

$$\mathscr{A}(\Gamma) = \frac{1}{2}\oint_\Gamma x\, dy - y\, dx.$$

From this definition, we use the construction in Section 4 to break apart $\mathscr{A}(\mathcal{S}')$ to give Theorem 5, which states that

$$\mathscr{A}(\mathcal{S}') = \int_0^{2\pi} \frac{r^2(\theta)}{2}\, d\theta + 2\mathscr{A}(\mathcal{B}).$$

We also show that $\mathscr{A}(\mathcal{B})$ is never positive and use this to show that certain curves with maximal interior area are rotationally symmetric (see Corollary 5.3). We conclude by computing the internal area of the bisection envelope of an equilateral triangle, and thus deduce a constant universal to all triangles: $\frac{3}{4}\ln 2 - \frac{1}{2}$, the ratio of the area of a triangle to the area of its bisection envelope.

2. Basic properties

For the entirety of this paper, it is assumed that $\mathcal{S}$ is a curve in $\mathbb{R}^2$ which is compact, continuous, simple, and piecewise of class C^1 with a finite number of pieces.

We now define the bisection envelope.

Definition 2.1. Given such a curve $\mathcal{S}$, define $\mathcal{L}$ to be the family of lines that bisect the interior area of $\mathcal{S}$. Each $l_\theta \in \mathcal{L}$ is the bisecting line in direction θ. Define the *bisection envelope* $\mathcal{B}$ of $\mathcal{S}$ to be the envelope of $\mathcal{L}$; that is,

$$\mathcal{B} = \left\{ P \mid P = \lim_{\epsilon \to 0} l_\theta \cap l_{\theta+\epsilon}\ ,\ 0 \le \theta < \pi \right\}.$$

We now restrict the class of curves $\mathcal{S}$ to be studied.

Definition 2.2. Define $\mathcal{S}$ and $\mathcal{L}$ as above. We say that $\mathcal{S}$ is *bisection convex* if for all θ, l_θ intersects $\mathcal{S}$ in exactly two points. Alternatively, for every point A on $\mathcal{S}$, there exists a unique point B also on $\mathcal{S}$ such that the line AB bisects the interior area of $\mathcal{S}$.

We also create a tighter restriction.

Definition 2.3. Define $\mathcal{S}$ and $\mathcal{L}$ as before. We say that $\mathcal{S}$ is *strictly bisection convex* if it is bisection convex and for all θ, l_θ is not tangent to $\mathcal{S}$. At any point where there are two tangents to $\mathcal{S}$ — one from each side — the l_θ through that point is distinct from both tangents.

Henceforth, unless otherwise stated, *it is assumed that $\mathcal{S}$ is strictly bisection convex.*

Define $A(\theta)$ and $B(\theta)$ to be the endpoints of the bisecting chord in direction θ, with $B(\theta) = A(\theta + \pi)$. We distinguish between $A(\theta)$ and $B(\theta)$ by demanding that for each point $Q \neq A(\theta), B(\theta)$ on the bisecting chord, the vector $A(\theta) - Q$ points in positive direction θ and the vector $B(\theta) - Q$ points in positive direction $\theta + \pi$.

Proposition 2.4. *Assume that $\mathcal{S}$ is bisection convex. Then $A(\theta)$ varies continuously with θ.*

Proof. First, we note that any two bisecting chords must intersect in the interior of $\mathcal{S}$, for if they did not, the interior of $\mathcal{S}$ would be split into three regions, one of which would have zero area, which does not make sense.

From this, we have $\lim_{\epsilon\to 0} l_{\theta+\epsilon} = l_\theta$, as the limit of the intersection point $l_{\theta+\epsilon} \cap l_\theta$ is bounded. This also implies that the limit as $\epsilon \to 0$ of the distance from $A(\theta+\epsilon)$ to the intersection point $l_{\theta+\epsilon} \cap l_\theta$ is bounded. Therefore, the limit as $\epsilon \to 0$ of the perpendicular distance from $A(\theta+\epsilon)$ to l_θ is zero.

We have that $\lim_{\epsilon\to 0} A(\theta+\epsilon)$ must be a point P on l_θ which intersects $\mathcal{S}$, where for every other point Q on the bisecting chord with direction θ, the vector $P-Q$ points in positive direction θ. There is only one such point, $A(\theta)$; therefore,

$$\lim_{\epsilon\to 0} A(\theta+\epsilon) = A(\theta),$$

and $A(\theta)$ varies continuously with θ. □

From this, $B(\theta)$ also varies continuously with θ. We now determine the bisection envelope of bisection convex curves.

Proposition 2.5. *Let $\mathcal{S}$ be bisection convex. Fix θ and let $A = A(\theta)$ and $B = B(\theta)$. Then,*

$$\lim_{\epsilon\to 0} l_\theta \cap l_{\theta+\epsilon} = \frac{A+B}{2}. \tag{2-1}$$

Proof. Let $A(\theta+\epsilon) = A_\epsilon$ and $B(\theta+\epsilon) = B_\epsilon$. Let $l_\theta \cap l_{\theta+\epsilon} = O_\epsilon$, and let $\lim_{\epsilon\to 0} l_\theta \cap l_{\theta+\epsilon} = O$; see Figure 1. Define $a(\epsilon) = d(A_\epsilon, O_\epsilon)$, $b(\epsilon) = d(B_\epsilon, O_\epsilon)$, and extend to let $a(0) = d(A, O)$ and $b(0) = d(O, B)$.

Since $l_\theta, l_{\theta+\epsilon}$ are bisecting line segments,

$$\mathcal{A}(AO_\epsilon A_\epsilon) = \mathcal{A}(BO_\epsilon B_\epsilon), \tag{2-2}$$

where $AO_\epsilon A_\epsilon$ and $BO_\epsilon B_\epsilon$ are not triangles, but rather the regions enclosed by $\mathcal{S}$, l_θ, and $l_{\theta+\epsilon}$.

For fixed ϵ, we have the inequality

$$\tfrac{1}{2}\epsilon m^2 \le \mathcal{A}(AO_\epsilon A_\epsilon) \le \tfrac{1}{2}\epsilon M^2,$$

where m and M are the minimum and maximum values of $d(A_\delta, O_\epsilon)$ for $0 \le \delta \le \epsilon$.

As $m \le a(\epsilon) \le M$,

$$\tfrac{1}{2}\epsilon m^2 \le \tfrac{1}{2}\epsilon a^2(\epsilon) \le \tfrac{1}{2}\epsilon M^2.$$

The previous two inequalities have the same bounds, therefore

$$\left|\mathcal{A}(AO_\epsilon A_\epsilon) - \tfrac{1}{2}\epsilon a^2(\epsilon)\right| \le \tfrac{1}{2}\epsilon(M^2 - m^2). \tag{2-3}$$

From the continuity of $\mathcal{S}$, we have

$$\lim_{\epsilon\to 0} \frac{\tfrac{1}{2}\epsilon(M^2 - m^2)}{\epsilon} = \tfrac{1}{2}(a^2(0) - a^2(0)) = 0.$$

Combining this with (2-3) and using an identical argument for $\mathcal{A}(BO_\epsilon B_\epsilon)$, we have

$$\begin{aligned} \left|\mathcal{A}(AO_\epsilon A_\epsilon) - \tfrac{1}{2}\epsilon a^2(\epsilon)\right| &= o(\epsilon), \\ \left|\mathcal{A}(BO_\epsilon B_\epsilon) - \tfrac{1}{2}\epsilon b^2(\epsilon)\right| &= o(\epsilon). \end{aligned} \tag{2-4}$$

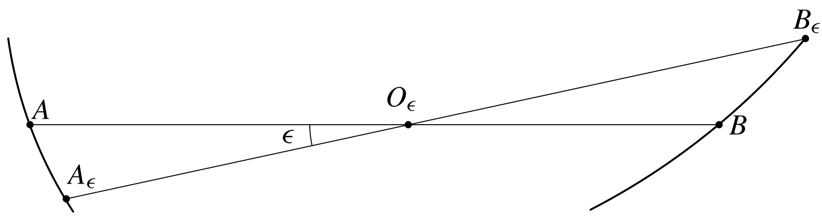

Figure 1. The situation considered in the proof of Proposition 2.5.

By the triangle inequality and (2-2), we have that

$$\begin{aligned}\left|\tfrac{1}{2}\epsilon a^2(\epsilon)-\tfrac{1}{2}\epsilon b^2(\epsilon)\right| &\le \left|\tfrac{1}{2}\epsilon a^2(\epsilon)-\mathscr{A}(AO_\epsilon A_\epsilon)\right|+\left|\mathscr{A}(AO_\epsilon A_\epsilon)-\tfrac{1}{2}\epsilon b^2(\epsilon)\right|\\ &=\left|\mathscr{A}(AO_\epsilon A_\epsilon)-\tfrac{1}{2}\epsilon a^2(\epsilon)\right|+\left|\mathscr{A}(BO_\epsilon B_\epsilon)-\tfrac{1}{2}\epsilon b^2(\epsilon)\right|.\end{aligned}$$

It follows from this and (2-4) that

$$\begin{aligned}\left|\tfrac{1}{2}\epsilon a^2(\epsilon)-\tfrac{1}{2}\epsilon b^2(\epsilon)\right| &= o(\epsilon),\\ \left|\tfrac{1}{2}a^2(0)-\tfrac{1}{2}b^2(0)\right| &= 0,\\ a(0) &= b(0).\end{aligned} \tag{2-5}$$

Therefore O is the midpoint of A and B. □

Hence, $\mathcal{B}$ is the locus of midpoints of the intersections of each $l_\theta \in \mathcal{L}$ with $\mathcal{S}$.

Define a function $f:\mathbb{R}\to\mathbb{R}^2$, with $f(\theta+\pi)=f(\theta)$, such that $f(\theta)$ signifies the point on $\mathcal{B}$ that is the midpoint of the bisecting chord of $\mathcal{S}$ with direction θ. The image of this function is $\mathcal{B}$. We are interested in the derivative of this function, where it exists.

Proposition 2.6. *Let $\mathcal{S}$ be strictly bisection convex. Fix θ such that $\mathcal{S}$ is of class C^1 at the endpoints $A(\theta)$, $B(\theta)$ of the bisecting chord with direction θ. Then $f'(\theta)$ is defined, and if $f'(\theta)$ is nonzero, then l_θ is tangent to $\mathcal{B}$ at $f(\theta)$.*

Proof. It suffices to derive $f'(\theta)$ and show that it is either zero or a nonzero vector pointing in direction θ.

Without loss of generality, let the axes be redefined such that direction θ is along the x-axis.

Define A, B, A_ϵ, B_ϵ, O_ϵ as in the proof of Proposition 2.5. Let

$$M=\frac{A+B}{2} \quad\text{and}\quad M_\epsilon=\frac{A_\epsilon+B_\epsilon}{2}.$$

Let $r=d(A,M)=d(M,B)$, $r(\epsilon)=d(A_\epsilon,M_\epsilon)=d(M_\epsilon,B_\epsilon)$, $\lambda(\epsilon)=d(O_\epsilon,M_\epsilon)$. Let $\alpha(\epsilon)=m\angle A_\epsilon A O_\epsilon$ and $\beta(\epsilon)=m\angle B_\epsilon B O_\epsilon$.

Let $a_h(\epsilon)$ and $a_v(\epsilon)$ be the horizontal and vertical components of $\overrightarrow{AA_\epsilon}$, positive in directions θ and $\theta+\pi/2$ respectively. Define $b_h(\epsilon)$ and $b_v(\epsilon)$ similarly; see Figure 2.

By definition,

$$f'(\theta) = \lim_{\epsilon\to 0} \frac{\overrightarrow{MM_\epsilon}}{\epsilon} = \lim_{\epsilon\to 0} \frac{\overrightarrow{AA_\epsilon} + \overrightarrow{BB_\epsilon}}{2\epsilon} = \lim_{\epsilon\to 0}\left(\frac{a_h(\epsilon)+b_h(\epsilon)}{2\epsilon}, \frac{a_v(\epsilon)+b_v(\epsilon)}{2\epsilon}\right). \tag{2-6}$$

By inspection,

$$a_v(\epsilon) = -(r(\epsilon)-\lambda(\epsilon))\sin\epsilon \quad \text{and} \quad b_v(\epsilon) = (r+\lambda(\epsilon))\sin\epsilon.$$

Thus

$$\lim_{\epsilon\to 0} \frac{a_v(\epsilon)+b_v(\epsilon)}{\epsilon} = \lim_{\epsilon\to 0}(r - r(\epsilon) + 2\lambda(\epsilon))\frac{\sin\epsilon}{\epsilon} = 0, \tag{2-7}$$

as $\lim_{\epsilon\to 0} r(\epsilon) = r$ and $\lim_{\epsilon\to 0} M_\epsilon = \lim_{\epsilon\to 0} O_\epsilon = M$, which follow from definition and Proposition 2.5.

As $a_h(\epsilon) = -a_v(\epsilon)\cot(\alpha(\epsilon))$ and $b_h(\epsilon) = -b_v(\epsilon)\cot(\beta(\epsilon))$, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0} \frac{a_h(\epsilon)+b_h(\epsilon)}{\epsilon} \\
&= \lim_{\epsilon\to 0}\big(r(\epsilon)\cot(\alpha(\epsilon)) - r\cot(\beta(\epsilon)) - \lambda(\epsilon)\cot(\beta(\epsilon)) - \lambda(\epsilon)\cot(\alpha(\epsilon))\big)\frac{\sin\epsilon}{\epsilon} \\
&= \lim_{\epsilon\to 0}\big(r(\cot(\alpha(\epsilon)) - \cot(\beta(\epsilon))) - \cot(\alpha(\epsilon))(r-r(\epsilon)) - \lambda(\epsilon)\cot(\beta(\epsilon)) \\
&\qquad\qquad - \lambda(\epsilon)\cot(\alpha(\epsilon))\big)\frac{\sin\epsilon}{\epsilon} \\
&= r(\cot\alpha - \cot\beta), \quad \text{where } \alpha = \lim_{\epsilon\to 0}\alpha(\epsilon),\ \beta = \lim_{\epsilon\to 0}\beta(\epsilon).
\end{aligned} \tag{2-8}$$

This follows from the same limits stated earlier, as $\mathcal{S}$ is strictly bisection convex and thus neither α nor β are 0 or π. Note that α and β are not necessarily defined — the limits only exist if $\mathcal{S}$ is of class C^1 locally at A and B, and thus α and β are not defined for only a finite number of values of θ. Where they are defined, we can combine (2-6), (2-7), and (2-8), giving

$$f'(\theta) = \left(\frac{r(\cot\alpha - \cot\beta)}{2}, 0\right), \tag{2-9}$$

and so $f'(\theta)$ is defined. Since $f'(\theta)$ has y-component 0, it points in direction θ if it is nonzero. □

Directly from (2-9), we have:

Corollary 2.7. *We have $f'(\theta) = 0$ if and only if the tangents to $\mathcal{S}$ at the endpoints of the bisecting chord with direction θ are parallel, that is, when $\alpha = \beta$.*

Proposition 2.6 can be further extended to cover more points on $\mathcal{B}$.

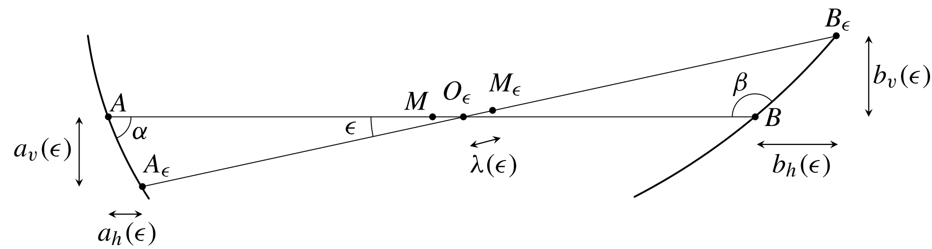

Figure 2. The situation considered in the proof of Proposition 2.6.

Proposition 2.8. *If f' is zero or undefined at a finite number of points, then for all θ, l_θ is tangent to $\mathcal{B}$ at $f(\theta)$.*

Proof. Define t_θ to be the tangent to $\mathcal{B}$ at $f(\theta)$.

If there are only a finite number of points for which f' is zero or undefined, then there are only a finite number of values of θ for which Proposition 2.6 does not hold. Thus, around any of these values θ_0, there exists a neighborhood for which Proposition 2.6 does hold. For small ϵ, $\theta_0 + \epsilon$ will lie in this neighborhood. Also, f is continuous, so the lines $l_\theta \in \mathcal{L}$ vary continuously with θ, and it is clear that

$$t_{\theta_0} = \lim_{\epsilon \to 0} t_{\theta_0 + \epsilon} = \lim_{\epsilon \to 0} l_{\theta_0 + \epsilon} = l_{\theta_0}. \qquad \square$$

From the derivation in Proposition 2.6, it is true that wherever f' is defined, it points in direction θ; thus, each defined $f'(\theta)$ is a scalar multiple of $(\cos\theta, \sin\theta)$.

Also from Proposition 2.6, we have:

Proposition 2.9. *Wherever $f'(\theta)$ is defined, f' is continuous at θ.*

Proof. From (2-9) we have that, where $f'(\theta)$ is defined, it is continuous if r, $\cot\alpha$, and $\cot\beta$ vary continuously with θ.

We have that r is half of the distance between the points $A(\theta)$ and $B(\theta)$, which vary continuously by Proposition 2.4, and therefore varies continuously for any θ.

From the fact that $\mathcal{S}$ is strictly bisection convex, the angle α must remain between 0 and π; therefore, $\cot\alpha$ varies continuously if α varies continuously. The angle α is defined as the difference in direction of the bisecting line and the direction of the tangent to $\mathcal{S}$ at $A(\theta)$. The direction of the bisecting line is θ, so it varies continuously. Where $f(\theta)$ is defined, $\mathcal{S}$ is of class C^1 locally at $A(\theta)$, and as $A(\theta)$ is a continuous parametrization of $\mathcal{S}$, the tangents to $\mathcal{S}$ around $A(\theta)$ vary continuously with θ. Thus α varies continuously with θ.

An identical argument can be used to show that β varies continuously with θ, and the result follows. $\square$

From this, we have that f' is undefined in at most a finite number of places over any period of length 2π, and it is only at these points that it is discontinuous.

Definition 2.10. Define $v_\theta := f'(\theta) \cdot (\cos\theta, \sin\theta)$, where f' is defined. Then v_θ has the following properties:

(1) $|v_\theta| = |f'(\theta)|$.

(2) $v_{\theta+\pi} = -v_\theta$.

(3) $f'(\theta) = v_\theta(\cos\theta, \sin\theta)$.

(4) $\int_{\theta_0}^{\theta_0+\pi} v_\theta(\cos\theta, \sin\theta)\, d\theta = (0, 0)$.

These follow directly from the definition of v_θ and from Proposition 2.6. Also note that the integral shown is defined, as the number of discontinuities of v_θ over the interval is the same as the number of discontinuities of f', thus finite, and the set of discontinuity points has measure 0.

Proposition 2.11. *If v_θ is not identically zero, then over any interval $[\theta_0, \theta_0+\pi]$ where $v_{\theta_0} \neq 0$, v_θ changes sign an odd number of times, and at least thrice.*

Proof. As $v_{\theta_0+\pi} = -v_{\theta_0}$, we know that v_θ must change sign at least once in the interval and must change an odd number of times.

Assume that only one sign change occurs over the interval $[\theta_0, \theta_0+\pi]$. Then there exists a value θ_1 (not necessarily unique) with $\theta_0 < \theta_1 < \theta_0+\pi$ such that over the interval $[\theta_0, \theta_1]$, $v_\theta \le 0$ and over the interval $[\theta_1, \theta_0+\pi]$, $v_\theta \ge 0$, or vice versa. Either way, this ensures that v_θ does not change sign over the interval $[\theta_1, \theta_1+\pi]$.

Consider the component of $f'(\theta)$ in direction $\theta_1 + \pi/2$. We observe that

$$\begin{aligned} 0 &= \int_{\theta_1}^{\theta_1+\pi} f'(\theta) \cdot (\cos(\theta_1+\pi/2), \sin(\theta_1+\pi/2))\, d\theta \\ &= \int_{\theta_1}^{\theta_1+\pi} v_\theta(\cos\theta, \sin\theta) \cdot (-\sin(\theta_1), \cos(\theta_1))\, d\theta \\ &= \int_{\theta_1}^{\theta_1+\pi} v_\theta \sin(\theta-\theta_1)\, d\theta. \end{aligned} \tag{2-10}$$

Neither v_θ nor $\sin(\theta-\theta_1)$ change sign between the bounds of the integral; thus, their product does not change sign (and is not identically zero by assumption), and (2-10) cannot be equal to 0, a contradiction.

This implies there is more than one sign change in any such interval $[\theta_0, \theta_0+\pi]$, so there are at least three, the next odd number. □

Remark 2.12. The notion of sign changes of v_θ has a geometric manifestation. For every point or interval where v_θ changes sign, a cusp or corner, respectively, appears on $\mathcal{B}$. If v_θ is zero at a finite number of points, then corners do not occur, and we have one cusp per sign change in an interval of length π. With these conditions, we extend Proposition 2.11 to $\mathcal{B}$ geometrically — if $\mathcal{B}$ is not a point and has no corners, then it has an odd number of cusps, and at least three cusps.

Note that this collection of results becomes much cleaner if we assume $\mathcal{S}$ to be entirely of class C^1.

Theorem 1. *If $\mathcal{S}$ is strictly bisection convex and of class C^1, then there exist $n \geq 3$ lines l_θ that bisect the interior area of $\mathcal{S}$ such that the tangents to $\mathcal{S}$ at $A(\theta)$ and $B(\theta)$ are parallel. If n is finite, then there exist m cusps on the bisection envelope $\mathcal{B}$ of $\mathcal{S}$, with $n \geq m \geq 3$ and m odd.*

Proof. From our assumptions and Propositions 2.6 and 2.9, f' is defined everywhere and is continuous; therefore, from the definition of v_θ, we know that v_θ is continuous. Therefore, if we let m be the number of sign changes of v_θ and n be the number of zeros, we have $n \geq m$. A zero of v_θ is a zero of $f(\theta)$, and thus by Corollary 2.7, there are n lines l_θ such that the tangents to $\mathcal{S}$ at $A(\theta)$ and $B(\theta)$ are parallel. If n is finite, then v_θ is not identically zero, so by Proposition 2.11, m is odd and at least 3. With n finite, no corners exist on $\mathcal{B}$, so from Remark 2.12, we have that m is the number of cusps on $\mathcal{B}$. □

3. Bisection envelopes of polygons

From Proposition 2.5, we know that the bisection envelope of a bisection convex curve is the midpoint locus of the bisecting chords of its interior area. We apply this fact to the computation of the bisection envelope of a bisection convex polygon.

Let $A(\theta)$, $B(\theta)$ be the endpoints of the bisecting chord with direction θ, with $A(\theta+\pi) = B(\theta) = A(\theta-\pi)$. If $\mathcal{S}$ is a polygon, we can split the interval $[0, \pi)$ into a finite number of subintervals $[0, \theta_1), [\theta_1, \theta_2), \dots, [\theta_n, \pi)$ such that on each subinterval, the locus of each of $A(\theta)$ and $B(\theta)$ is a line segment.

Proposition 3.1. *The locus of points $M(\theta) = (A(\theta) + B(\theta))/2$ over any of the intervals $[\theta_i, \theta_{i+1})$ is either a section of a hyperbola or a point.*

Proof. Let all points $A(\theta)$ lie on line k_1 and all points $B(\theta)$ lie on line k_2. If k_1 and k_2 are parallel, it follows from Corollary 2.7 that the locus of $M(\theta)$ is a point. Otherwise, k_1 and k_2 meet at a point Q. Let $a(\theta) = d(A(\theta), Q)$ and $b(\theta) = d(B(\theta), Q)$.

If we construct the triangles $\triangle A(\theta) Q B(\theta)$, they each have area $\frac{1}{2}a(\theta)b(\theta)\sin\gamma$, where γ is the angle between k_1 and k_2, a constant; see Figure 3. Furthermore, the chords $\overline{A(\theta)B(\theta)}$ are area preserving on $\mathcal{S}$; therefore, the triangles have constant area, or

$$\tfrac{1}{2}a(\theta)b(\theta)\sin\gamma = c a(\theta)b(\theta) = \frac{2c}{\sin\gamma} = c', \tag{3-1}$$

for some constant c'.

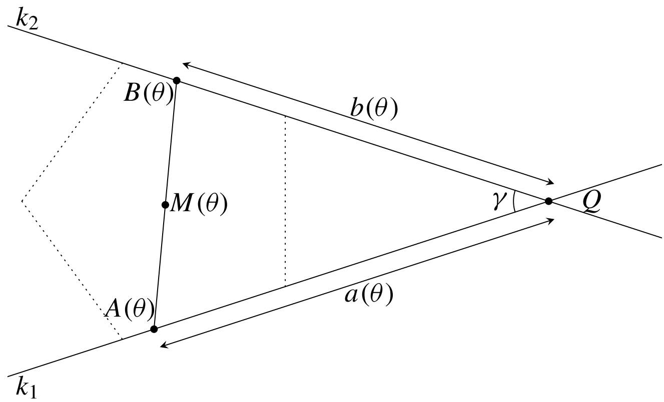

Figure 3. The situation considered in the proof of Proposition 3.1

Thus there exist distinct unit vectors w_1, w_2 parallel to k_1, k_2 respectively such that

$$M(\theta) = Q + \frac{a(\theta)w_1 + b(\theta)w_2}{2} = Q + \frac{a(\theta)w_1 + (c'/a(\theta))w_2}{2}. \tag{3-2}$$

We see that $M(\theta)$ is a linear transformation of the set of points

$$\left(a(\theta), \frac{c'}{a(\theta)}\right),$$

which represents a section of a hyperbola. Note that the image of a hyperbola under a linear transformation is itself a hyperbola. □

Proposition 3.2. *On any such interval* $[\theta_i, \theta_i + 1)$, *if the locus of* $M(\theta)$ *is a section of a hyperbola, the asymptotes of the hyperbola are the two lines* k_1 *and* k_2, *where* k_1 *and* k_2 *contain all* $A(\theta)$ *and* $B(\theta)$, *respectively.*

The proof of Proposition 3.2 is left to the reader.

Proposition 3.3. *The bisection envelope* $\mathcal{B}$ *of a polygon* $\mathcal{S}$ *is the union of a finite number of sections of hyperbolas. Let the set of all asymptotes of these hyperbolas be* H, *and let the set of all lines that contain the sides of* $\mathcal{S}$ *be* G. *Then* $H \subseteq G$, *with equality if no two lines in* G *are parallel.*

This follows from the previous two propositions.

This makes the calculation of a bisection envelope of a polygon significantly easier — one must only find the bisecting lines through the vertices and their midpoints; this then strictly defines each of the hyperbolas on each section $[\theta_i, \theta_{i+1})$.

Example 3.4. The bisection envelope of an equilateral triangle $\triangle ABC$ of side length two centered on the origin with $A = (0, 2/\sqrt{3})$, $B = (1, -1/\sqrt{3})$, and $C = (-1, -1/\sqrt{3})$ can be found as follows.

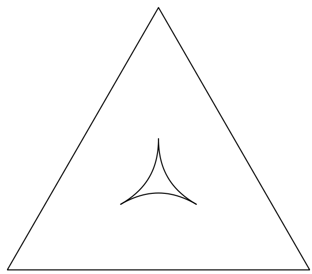

Figure 4. The bisection envelope of an equilateral triangle found in Example 3.4.

Let A', B', C' be on the triangle such that the chord AA' is bisecting, and so forth. The bisection envelope is split into 3 sections: a section of a hyperbola from $(A+A')/2$ to $(B+B')/2$ with asymptotes AC and BC, and two other congruent hyperbolic sections; see Figure 4.

Specifically, we have

$$A' = \left(0, -\frac{1}{\sqrt{3}}\right), \quad B' = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right), \quad C' = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right).$$

Therefore

$$\frac{A+A'}{2} = \left(0, \frac{1}{2\sqrt{3}}\right), \quad \frac{B+B'}{2} = \left(\frac{1}{4}, -\frac{1}{4\sqrt{3}}\right), \quad \frac{C+C'}{2} = \left(-\frac{1}{4}, -\frac{1}{4\sqrt{3}}\right). \tag{3-3}$$

The three hyperbolas, from A to B, B to C, and C to A respectively, are

$$\left(\left(y - \frac{2}{\sqrt{3}}\right) + \sqrt{3}\,x\right)\left(y + \frac{1}{\sqrt{3}}\right) = c_1, \tag{3-4}$$

$$\left(\left(y - \frac{2}{\sqrt{3}}\right) - \sqrt{3}\,x\right)\left(\left(y - \frac{2}{\sqrt{3}}\right) + \sqrt{3}\,x\right) = c_2, \tag{3-5}$$

$$\left(y + \frac{1}{\sqrt{3}}\right)\left(\left(y - \frac{2}{\sqrt{3}}\right) - \sqrt{3}\,x\right) = c_3. \tag{3-6}$$

By plugging in (3-3) above, we can find

$$c_1 = -\tfrac{3}{4}, \quad c_2 = \tfrac{3}{2}, \quad c_3 = -\tfrac{3}{4}.$$

This defines the bisection envelope fully.

Theorem 2. *A polygon with no mutually parallel sides is uniquely defined by its bisection envelope.*

Proof. From observations in Proposition 3.1, the assumptions in the theorem give us that the bisection envelope of this polygon does not contain any static points. This is to say, over each of the intervals $[\theta_i, \theta_{i+1})$, $M(\theta)$ is not a point but a section of a hyperbola, and therefore, there exists a bijection between the points on the interval $[\theta_i, \theta_{i+1})$ and the points on the locus of the restriction of $M(\theta)$ to that range.

From Proposition 3.2, we know the two lines k_1, k_2, upon which $A(\theta)$, $B(\theta)$ must lie. $A(\theta)$ and $B(\theta)$ must each lie on the line in direction θ through $M(\theta)$, a line distinct from k_1 and k_2, so the points $A(\theta)$, $B(\theta)$ are strictly determined over the interval $[\theta_i, \theta_{i+1})$. This can be done for every such interval, and the union of all such intervals is $[0, \pi)$; thus we achieve uniqueness for the loci of $A(\theta)$, $B(\theta)$ over all θ, giving the result. □

4. Backwards construction

The natural question arises: are there multiple curves with the same bisection envelope? Given a bisection envelope $\mathcal{B}$, can we generate all suitable curves with $\mathcal{B}$ as their bisection envelope?

First we ask, what curves can be bisection envelopes? Suppose that $\mathcal{B}$ is a bisection envelope associated to some strictly bisection convex curve $\mathcal{S}$ which is piecewise of class C^1 with a finite number of pieces. Its bisecting lines are $\mathcal{L} = \{l_\theta\}$, as explained earlier.

Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(\theta) = \lim_{\epsilon\to 0} l_\theta \cap l_{\theta+\epsilon}$. From Proposition 2.5, we know this is the midpoint of the bisecting chord in direction θ, described by the function $M(\theta)$ presented in Proposition 3.1. Then we have:

Proposition 4.1. *The function f is continuous.*

Proof. This follows immediately from the definition $M(\theta) := (A(\theta) + B(\theta))/2$, as we have from Proposition 2.4 that $A(\theta)$ and $B(\theta)$ vary continuously along $\mathcal{S}$. □

Since $\mathcal{S}$ has tangents which vary continuously everywhere except a finite number of points, by Proposition 2.6, f' is defined everywhere but a finite number of points, and where it is defined, it is of the form $v_\theta(\cos\theta, \sin\theta)$ for a scalar v_θ. Therefore, it is possible to define f as the Lebesgue integral of f', giving

$$f(\theta) := f(0) + \int_0^\theta v_t(\cos t, \sin t)\, dt. \tag{4-1}$$

The value of $f(0)$ is unimportant — it can just be set to the origin.

Now we generate a curve $\mathcal{S}'$ from f and a radius function $r : \mathbb{R} \to \mathbb{R}$, with $r(\theta + \pi) = r(\theta)$ and $r(\theta) > 0$. We define the function r to be continuous and piecewise of class C^1 with a finite number of pieces.

Define $\mathcal{S}'$ to be the image of the function

$$g(\theta) := f(\theta) + r(\theta)(\cos\theta, \sin\theta). \tag{4-2}$$

We have then that $\mathcal{S}'$ is continuous, compact, and piecewise of class C^1 with a finite number of pieces; however, we do not have that it is simple. It is clear that

$$g(\theta+2\pi)=g(\theta) \quad \text{and} \quad \frac{g(\theta)+g(\theta+\pi)}{2}=f(\theta).$$

Thus the chords $\overline{g(\theta)g(\theta+\pi)}$ are area-preserving if $\mathcal{S}'$ has a well-defined interior and if the chords lie strictly within this interior except at their endpoints, that is, if $\mathcal{S}'$ is simple and bisection convex. The remainder of Section 4 is concerned with the proof of Theorem 3.

Theorem 3. *Let f, g be defined as above.*

Let $\mathcal{S}'$ be the image of g and $\mathcal{B}$ be the image of f. If $\mathcal{S}'\cap\mathcal{B}=\varnothing$, then $\mathcal{B}$ is the bisection envelope of $\mathcal{S}'$.

To prove Theorem 3, we use a consequence of the following result.

Theorem 4. *Let f be defined as above with image $\mathcal{B}$. Let $\mathcal{L}$ be the set of lines l_θ through $f(\theta)$ in direction θ for all θ.*

Given a point $P\in\mathbb{R}^2\setminus\mathcal{B}$, let m_P be the number of lines in $\mathcal{L}$ for which P lies on l_θ, and let $w(P)$ be the winding number of f around P with θ increasing over an interval of π. Then

$$m_P=-2w(P)+1. \tag{4-3}$$

The proof of Theorem 4 begins by looking at the winding number of a simpler function.

Lemma 4.2. *Define the function*

$$f_P(\theta)=(f(\theta)-P)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

If $f(\theta)\neq P$ for all θ then, over the interval $0\le\theta<2\pi$, let n_P be the number of values of θ for which $f_P(\theta)$ lies on the x-axis, and let w_P be the winding number of $f_P(\theta)$ about the origin. Then

$$w_P=-\tfrac{1}{2}n_P. \tag{4-4}$$

Proof. We have

$$\begin{aligned} f_P'(\theta)&=f'(\theta)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}+(f(\theta)-P)\begin{pmatrix}-\sin\theta & -\cos\theta\\ \cos\theta & -\sin\theta\end{pmatrix}\\ &=v_\theta(1,0)+f_P(\theta)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\\ &=(v_\theta+y,-x), \quad \text{where } f_P(\theta)=(x,y). \end{aligned} \tag{4-5}$$

Note that if $x>0$, $y'<0$, and vice versa.

Now consider $f_P(\theta)$ over the half-open interval $[0, 2\pi)$. We have $f_P(\theta+\pi) = f_P(\theta)$, so the image of f_P is a closed loop, and $f_P(\theta)$ is never equal to (0, 0), so it has a winding number about the origin.

Let $\theta_1 < \theta_2 < \cdots < \theta_{n_P}$ be the values of θ for which $f_P(\theta)$ lies on the x-axis. Let $f_P(\theta_1) = (x_1, 0)$ and so on, with $x_i \neq 0$ by assumption. Then

$$\begin{aligned} x_i = g - f_P'(\theta_i)\cdot(0,1) &= -\lim_{h\to 0^+} \frac{f(\theta_i+h)\cdot(0,1) - f(\theta_i)\cdot(0,1)}{h} \\ &= g - \lim_{h\to 0^+} \frac{f(\theta_i+h)\cdot(0,1)}{h}. \end{aligned}$$

Similarly,

$$x_{i+1} = \lim_{\lambda\to 0^+} \frac{f(\theta_i-\lambda)\cdot(0,1)}{\lambda}.$$

But in the domain (θ_i, θ_{i+1}) we have that $f(\theta)\cdot(0,1)$ is continuous and, by our choices of θ_i, nonzero, so it has constant sign. Therefore, for all h, λ sufficiently small and greater than zero,

$$\operatorname{sign}\bigl(f(\theta_i+h)\cdot(0,1)\bigr) = \operatorname{sign}\bigl(f(\theta_{i+1}-\lambda)\cdot(0,1)\bigr).$$

Thus

$$\operatorname{sign} x_i = -\operatorname{sign}\frac{f(\theta_i+h)\cdot(0,1)}{h} = -\operatorname{sign}\frac{f(\theta_{i+1}-\lambda)\cdot(0,1)}{\lambda} = -\operatorname{sign} x_{i+1}.$$

Therefore the x_i alternate signs. This also implies n_P is even and x_i is positive for $n_P/2$ values.

The winding number of a curve Γ about a point P can be calculated descriptively by fixing a ray R from P in any direction and counting the number of intersections of Γ with R. For each intersection where the derivative is counterclockwise about P, we add 1, and where the derivative is clockwise, we subtract 1. The final total is the winding number. Note that if the derivative is along the ray or zero at any intersections, a more subtle approach is required, but this is not the case here.

If we fix the ray from the origin along the x-axis in positive direction for f_P, we see from (4-5) that at each intersection the derivative is counterclockwise about the origin; therefore $w_P = -\frac{1}{2}n_P$. □

Now we show the relation between the winding numbers of $f(\theta)$ about P and $f_P(\theta)$ about the origin.

Lemma 4.3. *Let the winding number of $f(\theta)$ about P over the interval $[0, \pi)$ be $w(P)$ and the winding number of $f_P(\theta)$ about the origin over the interval $[0, 2\pi)$ be w_P. Then*

$$w_P = 2w(P) - 1. \tag{4-6}$$

Proof. An alternative method of determining the winding number of a function relies on the calculation of an integral; several forms exist, although this proof uses the form

$$\frac{1}{2\pi}\int_a^b \frac{f'(x)\cdot\Big((f(x)-P)\big(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\big)\Big)}{|f(x)-P|^2}\,dx \tag{4-7}$$

for a function $f(x)$ about P on the interval (a, b). Now we calculate

$$\begin{aligned}
w_P &= \frac{1}{2\pi}\int_0^{2\pi} \frac{f_P'(\theta)\cdot\Big((f_P(\theta)-0)\big(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\big)\Big)}{|f_P(\theta)-0|^2}\,d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\Big(f'(\theta)\big(\begin{smallmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{smallmatrix}\big)+(f(\theta)-P)\big(\begin{smallmatrix}-\sin\theta&-\cos\theta\\\cos\theta&-\sin\theta\end{smallmatrix}\big)\Big)}{|f(\theta)-P|^2}\\
&\qquad\qquad\cdot\Big((f(\theta)-P)\big(\begin{smallmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{smallmatrix}\big)\big(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\big)\Big)\,d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\Big(f'(\theta)\big(\begin{smallmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{smallmatrix}\big)\Big)\cdot\Big((f(\theta)-P)\big(\begin{smallmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{smallmatrix}\big)\big(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\big)\Big)}{|f(\theta)-P|^2}\,d\theta\\
&\quad+\frac{1}{2\pi}\int_0^{2\pi} \frac{\Big((f(\theta)-P)\big(\begin{smallmatrix}-\sin\theta&-\cos\theta\\\cos\theta&-\sin\theta\end{smallmatrix}\big)\Big)\cdot\Big((f(\theta)-P)\big(\begin{smallmatrix}\sin\theta&\cos\theta\\-\cos\theta&\sin\theta\end{smallmatrix}\big)\Big)}{|f(\theta)-P|^2}\,d\theta.
\end{aligned}$$

For the first half of this sum we note that $\big(\begin{smallmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{smallmatrix}\big)$ and $\big(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\big)$ commute, and recall that a dot product is unaffected by an isometry applied to both multiplicands. Furthermore, note that f is periodic in π, so this integral can be split into two identical parts. For the second half of the sum, recall that $v\cdot(-v)=-|v|^2$. This allows us to simplify to

$$\begin{aligned}
w_P &= 2\left(\frac{1}{2\pi}\int_0^{\pi} \frac{f'(\theta)\cdot\Big((f(\theta)-P)\big(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\big)\Big)}{|f(\theta)-P|^2}\,d\theta\right)+\frac{1}{2\pi}\int_0^{2\pi}-1\,d\theta\\
&= 2w(P)-1.
\end{aligned}$$

□

The results of the two preceding lemmas can be combined to achieve Theorem 4.

Proof of Theorem 4. When P is on l_θ, $f_P(\theta)$ lies on the x-axis, but

$$f_P(\theta+\pi)=-f_P(\theta)\neq(0,0)\quad\text{and}\quad l_{\theta+\pi}=l_\theta,$$

so the number m_P of distinct lines l_θ containing P is equal to half the number of times $f_P(\theta)$ lies on the x-axis in the interval $[0, 2\pi)$. Using Lemmas 4.2 and 4.3,

$$m_P=n_P/2=-w_P=-2w(P)+1.$$

□

Corollary 4.4. *Every point P in the exterior of $\mathcal{B}$ lies on precisely one bisecting line l_θ.*

Proof. Since P is on the exterior of $\mathcal{B}$, we have $w(P) = 0$, and the result follows from Theorem 4. □

Remark 4.5. This also implies that no bisection envelope can have strictly positive winding number about any point, or the value m_P would be negative and have no meaning. Intuitively, this could be observed from f_P, which may not wind counterclockwise about the origin.

Theorem 4 can be used to show the first step in proving Theorem 3.

Lemma 4.6. *If $\mathcal{S}'$ lies on the exterior of $\mathcal{B}$, then it is not self-intersecting and each l_θ intersects $\mathcal{S}'$ exactly twice, at $g(\theta)$ and $g(\theta+\pi)$.*

Proof. If either of these conditions are false, there exist two lines $l_{\theta_1}, l_{\theta_2}$ that intersect at some point on $\mathcal{S}'$, say at P. But $\mathcal{S}'$, and thus P, lies on the exterior of $\mathcal{B}$; thus $w(P) = 0$. By Theorem 4, this leads to the contradiction

$$2 \le m_P = 2(-0) + 1 = 1.$$ □

Lemma 4.7. *Given two continuous, compact curves $C_1, C_2 \in \mathbb{R}^2$, if C_2 lies fully in the interior of C_1, then for each $P \in \mathbb{R}^2$, there exists a point $P_1 \in C_1$ such that for all $P_2 \in C_2$,*

$$d(P_1, P) > d(P_2, P).$$

Proof. If P_2 lies on the interior of C_1, then there is a ball around P_2 that lies on the interior of C_1. The ray starting at P passing through P_2 extends to points past P_2 but still in the interior of C_1. Since C_1 is bounded, eventually this ray must intersect C_1 at a point Q, and $d(Q, P) > d(P_2, P)$.

Let P_1 be a point on C_1 such that $d(P_1, P)$ is maximal (this can be done as C_1 is compact). Then

$$d(P_1, P) \ge d(Q, P) > d(P_2, P)$$

for all P_2. This can be done for every point P. □

Lemma 4.8. *$\mathcal{S}'$ cannot lie fully in the interior of $\mathcal{B}$.*

Proof. From the definition of g, for a point P_1 on $\mathcal{B}$, there exist points $P_2 = P_1 + a$ and $P_2' = P_1 - a$ on $\mathcal{S}'$ for some nonzero vector a (r is defined to be greater than zero); then P_2, P_1, and P_2' are collinear in that order.

It follows that given any reference point P, P_1 cannot be the furthest of these points from P; thus by the contrapositive of Lemma 4.7, $\mathcal{S}'$ is not fully in the exterior of $\mathcal{B}$. □

Proof of Theorem 3. If $\mathcal{S}' \cap \mathcal{B} = \varnothing$, then $\mathcal{S}'$ lies fully in the exterior of $\mathcal{B}$ — it cannot lie in the interior by Lemma 4.8. By Lemma 4.6, $\mathcal{S}'$ must not be self-intersecting, so it has a well-defined interior, and each line l_θ touches $\mathcal{S}'$ at exactly two points. Thus the chords $\overline{g_\theta g_{\theta+\pi}}$ are fully contained in the interior of $\mathcal{S}'$. By Proposition 2.5, they are area preserving, and $\overline{g_\theta g_{\theta+\pi}} = \overline{g_{\theta+\pi} g_{\theta+2\pi}}$, so they are bisecting lines of the interior of $\mathcal{S}'$. From the definitions, $f(\theta)$ is the midpoint of $g(\theta)$ and $g(\theta+\pi)$, so again by Proposition 2.5, $\mathcal{B}$ is the bisection envelope of $\mathcal{S}'$. □

Remark 4.9. Note that $\mathcal{S}$ is strictly bisection convex; therefore by Proposition 2.6, there are no points on $\mathcal{B}$ where the limit of $|f'(\theta)|$ is infinite. However, $\mathcal{B}$ is also the bisection envelope of $\mathcal{S}'$, so $\mathcal{S}'$ is also strictly bisection convex.

Remark 4.10. If the radius function r is sufficiently large, $\mathcal{S}'$ cannot intersect $\mathcal{B}$. This implies that for any strictly bisection convex $\mathcal{S}$, there are an infinite number of other strictly bisection convex curves $\mathcal{S}'$ that share its bisection envelope, each generated by a different r.

5. Relations between areas

Using the construction from the previous section, we now determine the interior area of $\mathcal{S}'$ as the sum of two integrals, one involving $r(\theta)$ and another that gives the interior area of $f(\theta)$. Note that we assume f and g are differentiable almost everywhere throughout this section.

We define (and denote) interior area of a closed, continuous curve purely based upon the line integral

$$\mathscr{A}(\Gamma) = \frac{1}{2} \oint_\Gamma x\, dy - y\, dx \tag{5-1}$$

irrespective of whether the curve has a well-defined interior. Note that whenever the curve Γ is simple, that is, when discussion of area makes sense, this area function gives its exact area, positive or negative depending on the direction we integrate about Γ. Also note that this integral functions equivalently to the double integral

$$\iint_{\mathbb{R}^2 \setminus \Gamma} w(\Gamma, P)\, dx\, dy, \tag{5-2}$$

where $P = (x, y)$ and $w(\Gamma, P)$ is the winding number of Γ about P.

Theorem 5.

$$\mathscr{A}(\mathcal{S}') = \int_0^{2\pi} \frac{r^2(\theta)}{2}\, d\theta + 2\mathscr{A}(\mathcal{B}). \tag{5-3}$$

Proof. We recall that $\mathcal{S}'$ is parametrized by

$$g(\theta) = f(0) + \int_0^\theta v_t(\cos t, \sin t)\, dt + r(\theta)(\cos\theta, \sin\theta).$$

Since $f(0)$ is arbitrary, we take it to be zero.

Next we take the derivative and separate the x and y components, giving

$$g'(\theta) = \big(v_\theta \cos\theta + r'(\theta)\cos\theta - r(\theta)\sin\theta,\, v_\theta \sin\theta + r'(\theta)\sin\theta + r(\theta)\cos\theta\big).$$

We expand and simplify $\mathscr{A}(\mathcal{S}')$ using standard trigonometric identities.

$$\begin{aligned}
\mathscr{A}(\mathcal{S}') &= \frac{1}{2}\oint_{\mathcal{S}} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right)d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\Bigg(\bigg(\int_0^\theta v_t\cos t\,dt + r(\theta)\cos\theta\bigg)\big(v_\theta\sin\theta + r'(\theta)\sin\theta + r(\theta)\cos\theta\big) \\
&\qquad - \bigg(\int_0^\theta v_t\sin t\,dt + r(\theta)\sin\theta\bigg)\big(v_\theta\cos\theta + r'(\theta)\cos\theta - r(\theta)\sin\theta\big)\Bigg)\,d\theta \\
&= \int_0^{2\pi}\frac{r^2(\theta)}{2}\,d\theta + \frac{1}{2}\int_0^{2\pi}\int_0^\theta v_t v_\theta(\sin(\theta - t))\,dt\,d\theta \\
&\quad + \frac{1}{2}\int_0^{2\pi}\bigg(r'(\theta)\int_0^\theta v_t\sin(\theta - t)\,dt + r(\theta)\int_0^\theta v_t\cos(\theta - t)\,dt\bigg)\,d\theta. \qquad (5\text{-}4)
\end{aligned}$$

Observe that, from the points in Definition 2.10,

$$\begin{aligned}
\int_0^{\theta+\pi} v_t\sin((\theta+\pi) - t)\,dt &= \int_\pi^{\theta+\pi} -v_{t+\pi}\sin((\theta+\pi) - t)\,dt \\
&= -\int_0^\theta v_t\sin(\theta - t)\,dt. \qquad (5\text{-}5)
\end{aligned}$$

Similarly,

$$\int_0^{\theta+\pi} v_t\cos((\theta+\pi) - t)\,dt = -\int_0^\theta v_t\cos(\theta - t)\,dt. \qquad (5\text{-}6)$$

By splitting the integrals and replacing variables, the final line of (5-4) can be rewritten to give

$$\begin{aligned}
&\frac{1}{2}\int_0^\pi r'(\theta)\int_0^\theta v_t\sin(\theta - t)\,dt\,d\theta + \frac{1}{2}\int_0^\pi r'(\theta+\pi)\int_0^{\theta+\pi} v_t\sin(\theta+\pi - t)\,dt\,d\theta \\
&+ \frac{1}{2}\int_0^\pi r(\theta)\int_0^\theta v_t\cos(\theta - t)\,dt\,d\theta + \frac{1}{2}\int_0^\pi r(\theta+\pi)\int_0^{\theta+\pi} v_t\cos(\theta+\pi - t)\,dt\,d\theta.
\end{aligned}$$

As $r(\theta+\pi) = r(\theta)$, this can further be written as

$$\begin{aligned}
&\frac{1}{2}\int_0^\pi r'(\theta)\bigg(\int_0^\theta v_t\sin(\theta - t)\,dt + \int_0^{\theta+\pi} v_t\sin(\theta+\pi - t)\,dt\bigg)\,d\theta \\
&\qquad + \frac{1}{2}\int_0^\pi r(\theta)\bigg(\int_0^\theta v_t\cos(\theta - t)\,dt + \int_0^{\theta+\pi} v_t\cos(\theta+\pi - t)\,dt\bigg)\,d\theta.
\end{aligned}$$

However, from (5-5) and (5-6) this entire expression amounts to zero. From (5-4), we are left with

$$\mathcal{A}(\mathcal{S}') = \int_0^{2\pi} \frac{r^2(\theta)}{2}\,d\theta + \frac{1}{2}\int_0^{2\pi}\int_0^{\theta} v_t v_\theta(\sin(\theta - t))\,dt\,d\theta. \tag{5-7}$$

In a similar fashion to the above, the second term can be rewritten as

$$\frac{1}{2}\int_0^{\pi} v_\theta\bigg(\int_0^{\theta} v_t \sin(\theta - t)\,dt - \int_0^{\theta+\pi} v_t \sin(\theta + \pi - t)\,dt\bigg)\,d\theta.$$

Note the change in the negative sign, as $v_{\theta+\pi} = -v_\theta$. By (5-5) this is equal to

$$2\bigg(\frac{1}{2}\int_0^{\pi}\int_0^{\theta} v_t v_\theta \sin(\theta - t)\,dt\,d\theta\bigg). \tag{5-8}$$

Now applying (5-1) to $\mathcal{B}$, we recall that $\mathcal{B}$ is parametrized by

$$f(\theta) = f(0) + \int_0^{\theta} v_t(\cos t, \sin t)\,dt,$$

with derivative

$$f'(\theta) = (v_\theta \cos\theta,\, v_\theta \sin\theta).$$

Thus

$$\begin{aligned}\mathcal{A}(\mathcal{B}) &= \frac{1}{2}\oint_{\mathcal{B}} x\,dy - y\,dx = \frac{1}{2}\int_0^{\pi}\bigg(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\bigg)\,d\theta \\ &= \frac{1}{2}\int_0^{\pi}\bigg(\int_0^{\theta} v_t \cos t\, v_\theta \sin\theta\,dt - \int_0^{\theta} v_t \sin t\, v_\theta \cos\theta\,dt\bigg)\,d\theta \\ &= \frac{1}{2}\int_0^{\pi}\int_0^{\theta} v_t v_\theta \sin(\theta - t)\,dt\,d\theta. \end{aligned}\tag{5-9}$$

Combining (5-7), (5-8), and (5-9), it is finally achieved that

$$\mathcal{A}(\mathcal{S}') = \int_0^{2\pi} \frac{r^2(\theta)}{2}\,d\theta + 2\mathcal{A}(\mathcal{B}). \qquad \square$$

This formula may be useful in determining the area of a bisection envelope where the integral (5-9) is much more difficult than finding $r(\theta)$ then calculating (5-3).

A property of $\mathcal{B}$ described in Remark 4.5 allows us to bound $\mathcal{A}(\mathcal{B})$.

Proposition 5.1. $$\mathcal{A}(\mathcal{B}) \le 0.$$

Proof. Remark 4.5 notes that, for all $P \notin \mathcal{B}$,

$$w(\mathcal{B}, P) \le 0.$$

Thus from (5-2),

$$\mathscr{A}(\mathcal{B}) = \iint_{\mathbb{R}^2\setminus\mathcal{B}} w(\mathcal{B}, P)\, dx\, dy \leq 0. \qquad \square$$

Proposition 5.2. *Let $\mathcal{S}'$ be piecewise of class C^1 with a finite number of pieces. If $\mathscr{A}(\mathcal{B}) = 0$, then $\mathcal{B}$ is a point.*

Proof. If $\mathscr{A}(\mathcal{B}) = 0$, then from the reasoning in Proposition 5.1, $w(\mathcal{B}, P) = 0$ for all P not on $\mathcal{B}$.

Consider three bisecting lines $l_{\theta_1}, l_{\theta_2}, l_{\theta_3}$ with mutual intersections A, B, C. Assume the three points are distinct. From continuity, we have that all points P in the interior of $\triangle ABC$ lie on at least three lines l_θ. By Theorem 3, this implies that for all such P, $w(P) \leq -1$, and therefore P must be on B. Hence, B is a space-filling curve on some subset of $\mathbb{R}^2$ that contains $\triangle ABC$. However, f is of class C^1 at all but a finite number of points, so it cannot be a space-filling curve.

It follows that any three bisecting lines are concurrent, and thus, all bisecting lines are concurrent, and $\mathcal{B}$ is a point. $\square$

Corollary 5.3. *Of all bisection convex curves $\mathcal{S}'$ piecewise of class C^1 with a finite number of pieces such that*

$$\int_0^{2\pi} \frac{r^2(\theta)}{2}\, d\theta = k$$

for some fixed k, those with maximal interior area have $180°$ *rotational symmetry.*

Proof. From Theorem 5 and Proposition 5.1, these curves clearly have maximal interior area when $\mathscr{A}(\mathcal{B}) = 0$. By Proposition 5.2, this is only possible if $\mathcal{B}$ is a point, say P. From the definition of g, $\mathcal{S}'$ has $180°$ rotational symmetry about P. $\square$

Remark 5.4. The proof of Corollary 5.3 shows that if we drop the restriction that $\mathcal{S}'$ is piecewise of class C^1 with a finite number of pieces and rather assume it is only piecewise of class C^1, then the bisection envelope consists of all the points of intersection between bisecting lines and this envelope might be space-filling. We are unable to rule out the possibility of a space-filling bisection envelope and leave it as an open question: can f be differentiable almost everywhere and space-filling?

Lastly, we use Theorem 5 to find the internal area of the bisection envelope of an equilateral triangle calculated in Example 3.4.

Example 5.5. The bisection envelope of a triangle is not self-intersecting; therefore its interior area is well-defined and is recognized to be $-\mathscr{A}(\mathcal{B})$. Rearranging Theorem 5, we have

$$-\mathscr{A}(\mathcal{B}) = \frac{\int_0^{2\pi} r^2(\theta)/2\, d\theta - \mathscr{A}(\mathcal{S}')}{2}.$$

Now $\mathscr{A}(\mathcal{S}')$ is the area of an equilateral triangle with side length 2 or $\sqrt{3}$. Also, by symmetry, r has period $\pi/3$, and therefore we rewrite

$$-\mathscr{A}(\mathcal{B}) = 3\int_0^{\pi/3} \frac{r^2(\theta)}{2}\,d\theta - \frac{\sqrt{3}}{2}. \tag{5-10}$$

Rotation of the triangle has no effect on area, and thus we rotate so that the three medians have directions $0, \pi/3, 2\pi/3$ with A, B, C being the vertices that lie on the respective medians.

Let $A(\theta)$, $B(\theta)$ be the intersection points of l_θ with the triangle, where $A(0) = A$, $B(\pi/3) = B$. Let $a(\theta) = d(A(\theta), C)$ and $b(\theta) = d(B(\theta), C)$. Since the $A(\theta)B(\theta)$ are bisecting chords, we have $\frac{1}{2}a(\theta)b(\theta)\sin(\pi/3) = \sqrt{3}/2$, which implies

$$a(\theta)b(\theta) = 2. \tag{5-11}$$

We now apply the sine and cosine laws to get $2r(\theta)\sin\left(\frac{\pi}{2} - \theta\right) = a(\theta)\sin\frac{\pi}{3}$ on the one hand, which yields

$$a(\theta) = \frac{4}{\sqrt{3}} r(\theta)\cos\theta, \tag{5-12}$$

and on the other hand

$$4r^2(\theta) = a^2(\theta) + b^2(\theta) - 2a(\theta)b(\theta)\cos\frac{\pi}{3}. \tag{5-13}$$

Combining (5-11), (5-12), and (5-13) we have

$$(2 - \tfrac{8}{3}\cos^2\theta)r^4(\theta) + r^2(\theta) - \frac{3}{8\cos^2\theta} = 0. \tag{5-14}$$

Thus we find

$$\frac{r^2(\theta)}{2} = \frac{1 \pm \sqrt{3}\tan\theta}{\frac{32}{3}\cos^2\theta - 8}. \tag{5-15}$$

We choose the $\pm$ to be a $-$, otherwise as $\theta \to \pi/6$, we have that $r^2(\theta)$ goes to infinity. This function is integrable by standard methods by a change of variable to $u = \cot\theta$ and then through use of partial fractions. We calculate

$$\int_0^{\pi/3} \frac{r^2(\theta)}{2}\,d\theta = \tfrac{1}{8}\sqrt{3}\ln(1 + \sqrt{3}\tan\theta)\,\Big|_0^{\pi/3} = \frac{\sqrt{3}}{4}\ln 2. \tag{5-16}$$

This can now be inserted back into (5-10), giving the result

$$-\mathscr{A}(\mathcal{B}) = \frac{3\sqrt{3}}{4}\ln 2 - \frac{\sqrt{3}}{2} \approx 0.03440. \tag{5-17}$$

Remark 5.6. As ratios of areas and ratios of lengths along a line are unaffected by linear transformations, the bisection envelope of a curve will remain unchanged under a linear transformation. As any triangle is the image of any other triangle under some linear transformation, it follows that the ratio $\mathscr{A}(\mathcal{B}) : \mathscr{A}(\mathcal{S}')$ is a constant when $\mathcal{S}'$ is a triangle. Therefore, for all triangles $\mathcal{S}'$,

$$\frac{\mathscr{A}(\mathcal{B})}{\mathscr{A}(\mathcal{S}')} = \tfrac{3}{4}\ln 2 - \tfrac{1}{2} \approx 0.01986. \tag{5-18}$$

In other words, every triangle has a bisection envelope with area roughly a fiftieth of its area.

Acknowledgements

I would like to thank C. Kenneth Fan for all of the time and effort he has put in making this paper a possibility. From offering initial ideas and working through proofs, to reviewing and editing a final product, he has been a valuable collaborator and mentor. I would also like to thank the referee their insightful comments.

References

[Fusco and Pratelli 2011] N. Fusco and A. Pratelli, "On a conjecture by Auerbach", *J. Eur. Math. Soc. (JEMS)* **13**:6 (2011), 1633–1676. MR 2012h:52021 Zbl 1227.49047

Received: 2013-07-30 Revised: 2013-10-23 Accepted: 2013-11-12

noahfp@gmail.com *Harvard University, 1405 Harvard Yard Mail Center, Cambridge, MA 02138, United States*

involve

msp.org/involve

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2015 is US $140/year for the electronic version, and $190/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2015 vol. 8 no. 2