

# involve

a journal of mathematics

Counting set classes with Burnside's lemma

Joshua Case, Lori Koban and Jordan LeGrand





# Counting set classes with Burnside's lemma

Joshua Case, Lori Koban and Jordan LeGrand

(Communicated by Kenneth S. Berenhaut)

Mathematical tools from combinatorics and abstract algebra have been used to study a variety of musical structures. One question asked by mathematicians and musicians is: how many  $d$ -note set classes exist in a  $c$ -note chromatic universe? In the music theory literature, this question is answered with the use of Pólya's enumeration theorem. We solve the problem using simpler techniques, including only Burnside's lemma and basic results from combinatorics and abstract algebra. We use interval arrays that are associated with pitch class sets as a tool for counting.

## 1. Introduction

For the past three decades, mathematical tools from combinatorics and abstract algebra have been used to study a variety of musical structures. The elements of a  $c$ -note chromatic universe are typically labeled  $0, 1, 2, \dots, c-1$  and are considered elements of  $Z_c$ , the group of integers modulo  $c$ . In the traditional 12-note chromatic universe,  $C$  is labeled 0. Following the language of [Clough and Myerson 1985], a  $d$ -note pitch class set in a  $c$ -note chromatic universe is a subset of  $\{0, 1, \dots, c-1\}$  of size  $d$ . As explained in [Reiner 1985; Hook 2007], two pitch class sets are considered equivalent if one can be obtained from the other either by rotation or reflection. A  $d$ -note set class contains all equivalent  $d$ -note pitch class sets. One question asked by musicians and music theorists is: how many  $d$ -note set classes exist in a  $c$ -note chromatic universe? Figure 1 shows a way to visualize the case where  $c = 12$  and  $d = 7$ .

Let  $n$  be a positive integer. The Euler  $\varphi$ -function,  $\varphi(n)$ , is the number of positive integers that are less than or equal to  $n$  that are also relatively prime to  $n$ .

**Theorem 1.1** [Reiner 1985; Hook 2007]. *The number of  $d$ -note set classes in a  $c$ -note chromatic universe is*

$$\frac{1}{2c}T(c, d) + \frac{1}{2}I(c, d), \quad (1-1)$$

*MSC2010:* 00A65, 05E18.

*Keywords:* set classes, pitch class sets, Burnside's lemma, group actions.

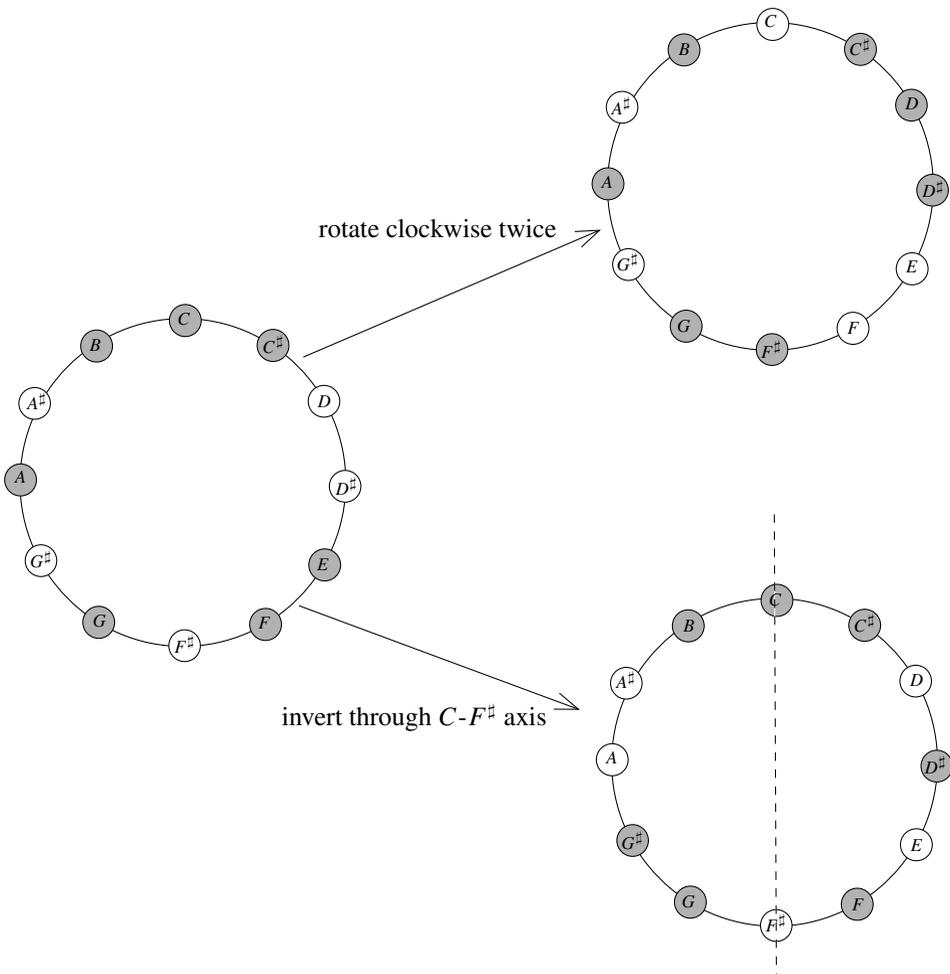
Case and LeGrand received financial support for this research from the University of Maine at Farmington's Wilson Scholar program.

where

$$T(c, d) = \sum_{j|\gcd(c,d)} \varphi(j) \binom{c/j}{d/j}$$

and

$$I(c, d) = \begin{cases} \binom{c/2-1}{\lfloor d/2 \rfloor} & \text{if } c \text{ is even and } d \text{ is odd,} \\ \binom{\lfloor c/2 \rfloor}{\lfloor d/2 \rfloor} & \text{otherwise.} \end{cases}$$



**Figure 1.** Visualizing a 7-note pitch class set in a 12-note chromatic universe. The three pitch class sets  $\{C, C^\sharp, E, F, G, A, B\}$ ,  $\{C^\sharp, D, D^\sharp, F^\sharp, G, A, B\}$ , and  $\{C, C^\sharp, D^\sharp, F, G, G^\sharp, B\}$  are equivalent and are therefore all part of the same set class.

In the music theory literature, Theorem 1.1 is proved using an advanced combinatorial theorem, namely Pólya's enumeration theorem (the final theorem stated in [Brualdi 2010]). Our contribution is that we make Theorem 1.1 more accessible by using only tools that would be seen in introductory classes in combinatorics and abstract algebra. The most advanced concept is Burnside's lemma, which appears in [Reiner 1985; Hook 2007] as a general tool for counting the number of equivalence classes generated by a group action, but is abandoned in the proof of Theorem 1.1 in favor of Pólya's result. In [Graham et al. 2008], the application of Burnside's lemma to our problem is discussed, but only specific examples, and not a general result, are reported. An additional contribution is that we use the structure of *interval arrays* (see Section 2), which were introduced in [Clough and Myerson 1985] and developed in [Fripertinger 1992], but have not been connected to this theorem.

### 2. Equivalent pitch class sets

The *dihedral group of order 2n*,  $D_{2n}$ , is the set of symmetries of a regular  $n$ -gon. There are  $n$  rotations and  $n$  reflections. Musically, rotations are known as transpositions and reflections are known as inversions.

Mathematically speaking, the number of  $d$ -note set classes in a  $c$ -note chromatic universe is the number of equivalence classes when  $D_{2c}$  acts on the set of  $d$ -note pitch class sets. In Figure 1, all 7-note pitch class sets that are equivalent to  $\{C, C^\sharp, E, F, G, A, B\}$  can be found by inverting and transposing the left-most figure in all 24 possible ways. Consult [Hook 2007] for more details about group actions in this context.

Let  $\{i_1, i_2, \dots, i_d\}$  be a  $d$ -note pitch class set. Without loss of generality, let  $i_1 < i_2 < \dots < i_d$ . The *interval array* associated with this  $d$ -note pitch class set is

$$\langle i_2 - i_1, i_3 - i_2, \dots, i_d - i_{d-1}, i_1 - i_d \rangle,$$

where all subtraction is done modulo  $d$  [Fripertinger 1992, Definition 2.5]. Note that  $\langle j_1, j_2, \dots, j_d \rangle$  is the interval array of a  $d$ -note pitch class set in a  $c$ -note chromatic universe if and only if  $j_1 + j_2 + \dots + j_d = c$  [Fripertinger 1992, Remark 2.4]. See Table 1.

Instead of counting the number of equivalence classes when  $D_{2c}$  acts on the set of  $d$ -note pitch class sets, we will count the number of equivalence classes when

7-note pitch class set	pitch class set in $Z_c$	interval array
$\{C, C^\sharp, E, F, G, A, B\}$	$\{0, 1, 4, 5, 7, 9, 11\}$	$\langle 1, 3, 1, 2, 2, 2, 1 \rangle$
$\{C^\sharp, D, D^\sharp, F^\sharp, G, A, B\}$	$\{1, 2, 3, 6, 7, 9, 11\}$	$\langle 1, 1, 3, 1, 2, 2, 2 \rangle$
$\{C, C^\sharp, D^\sharp, F, G, G^\sharp, B\}$	$\{0, 1, 3, 5, 7, 8, 11\}$	$\langle 1, 2, 2, 2, 1, 3, 1 \rangle$

**Table 1.** The interval arrays for the pitch class sets in Figure 1.

$D_{2d}$  acts on  $\{\langle j_1, j_2, \dots, j_d \rangle \mid j_1 + j_2 + \dots + j_d = c\}$ , the set of interval arrays. In Theorem 2.3 of the same work, Fripertinger proves that the number of equivalence classes is the same in both situations.

### 3. Algebraic and combinatorial tools

Below are the theorems from introductory combinatorics [Brualdi 2010] and abstract algebra [Dummit and Foote 2004] that we will apply.

**Theorem 3.1.** *Let  $n$  and  $k$  be positive integers. Then*

$$k \binom{n}{k} = n \binom{n-1}{k-1}.$$

**Theorem 3.2.** *The equation  $x_1 + x_2 + \dots + x_k = n$  has  $\binom{n-1}{k-1}$  positive-integral solutions.*

**Theorem 3.3** (hockey stick theorem). *If  $m$  and  $n$  are nonnegative integers, then*

$$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}.$$

**Theorem 3.4.** *Let  $j, k$ , and  $n$  be integers such that  $0 \leq j \leq k \leq n$ . Then*

$$\sum_{m=j}^{n-k+j} \binom{m}{j} \binom{n-m}{k-j} = \binom{n+1}{k+1}.$$

**Theorem 3.5.** *In a group, assume that element  $a$  has order  $d$ . Then*

$$\langle a^j \rangle = \langle a^{\gcd(d,j)} \rangle \quad \text{and} \quad |\langle a^j \rangle| = \frac{d}{\gcd(d,j)}.$$

**Theorem 3.6.** *If  $m$  is a positive divisor of  $d$ , then the number of elements of order  $m$  in a cyclic group of order  $d$  is  $\varphi(m)$ .*

**Theorem 3.7** (Burnside's lemma). *Let  $G$  be a group acting on a set  $S$ . The number of equivalence classes is*

$$\frac{1}{|G|} \sum_{g \in G} \text{Fix}(g),$$

where  $\text{Fix}(g)$  is the number of elements of  $S$  that are fixed by  $g$ .

### 4. The main theorem proved with Burnside's lemma

**Theorem 4.1.** *The number of  $d$ -note set classes in a  $c$ -note chromatic universe is*

$$\frac{1}{2d} T_B(c, d) + \frac{1}{2} I(c, d), \tag{4-1}$$

where

$$T_B(c, d) = \sum_{m|d \text{ and } d|cm} \varphi(d/m) \binom{cm/d-1}{m-1},$$

and  $I(c, d)$  is defined as in Theorem 1.1.

*Proof.* Instead of visualizing a regular  $c$ -gon and counting the number of equivalence classes when  $D_{2c}$  acts on the set of  $d$ -note pitch class sets, as is typically done, we visualize a regular  $d$ -gon and count the number of equivalence classes when  $D_{2d}$  acts on the set of interval arrays  $\{(j_1, j_2, \dots, j_d) \mid j_1 + j_2 + \dots + j_d = c\}$ . According to Burnside's lemma, we must count the number of interval arrays that are fixed by elements of  $D_{2d}$ .

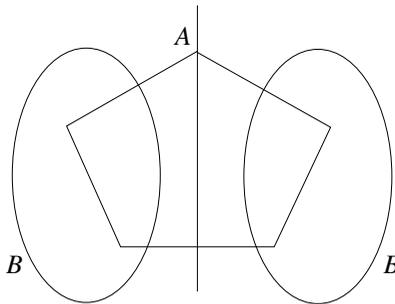
First, we consider the  $d$  inversions. Assume that  $c$  and  $d$  are both odd. We have a regular  $d$ -gon whose vertices are labeled  $j_1, j_2, \dots, j_d$ . Every possible axis of inversion passes through a single vertex. Let  $A$  be the value of that vertex, and let  $B = (c - A)/2$ . See Figure 2. Once the value of  $A$  is chosen, Theorem 3.2 says there are

$$\binom{\frac{c-A}{2} - 1}{\frac{d-1}{2} - 1}$$

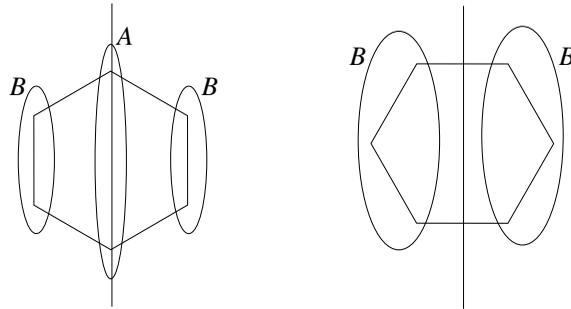
ways to assign values to the vertices that add up to  $B$ . Also note that  $A$  must be odd, and it ranges from 1 to  $c - (d - 1)$ . Thus the number of interval arrays fixed by this inversion is

$$\sum_{\substack{A=1 \\ A \text{ odd}}}^{c-(d-1)} \binom{\frac{c-A}{2} - 1}{\frac{d-1}{2} - 1},$$

which equals  $\binom{(c-1)/2}{(d-1)/2}$  by the hockey stick theorem. Since there are  $d$  inversions, the sum of the number of interval arrays fixed by an inversion is  $d \binom{\lfloor c/2 \rfloor}{\lfloor d/2 \rfloor}$ .



**Figure 2.** The inversion when  $d$  is odd.



**Figure 3.** Two inversions when  $d$  is even.

When  $c$  is even and  $d$  is odd, repeat the previous argument, except that  $A$  must be even and it ranges from 2 to  $c - (d - 1)$ . The hockey stick theorem yields

$$\binom{\frac{c-2}{2}}{\frac{d-1}{2}},$$

and the sum of the number of interval arrays fixed by an inversion is  $d \binom{c/2-1}{\lfloor d/2 \rfloor}$ .

Now assume that  $c$  and  $d$  are both even. When  $d$  is even, there are two types of inversions:  $d/2$  of each type in Figure 3. For an inversion through opposite edges, Theorem 3.2 says there are  $\binom{c/2-1}{d/2-1}$  ways to assign values to the  $d/2$  vertices that add up to  $B = c/2$ . For an inversion through a pair of vertices,  $A$  is chosen and then  $B = (c - A)/2$ . Note that  $A$  must be even and ranges from 2 to  $c - (d - 2)$ . The number of interval arrays fixed by this inversion is

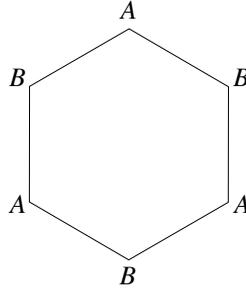
$$\begin{aligned} \sum_{\substack{A=2 \\ A \text{ even}}}^{c-(d-2)} \binom{A-1}{1} \binom{\frac{c-A}{2}-1}{\frac{d-2}{2}-1} &= \sum_{\substack{A=2 \\ A \text{ even}}}^{c-(d-2)} \binom{A}{1} \binom{\frac{c-A}{2}-1}{\frac{d-2}{2}-1} - \sum_{\substack{A=2 \\ A \text{ even}}}^{c-(d-2)} \binom{\frac{c-A}{2}-1}{\frac{d-2}{2}-1} \\ &= 2 \binom{\frac{c}{2}}{\frac{d}{2}} - \binom{\frac{c}{2}-1}{\frac{d}{2}-1}, \end{aligned}$$

where the first term simplifies by Theorem 3.4 and the second term simplifies by Theorem 3.3. The sum of the number of interval arrays fixed by the  $d$  inversions is

$$\frac{d}{2} \binom{\frac{c}{2}-1}{\frac{d}{2}-1} + \frac{d}{2} \left( 2 \binom{\frac{c}{2}}{\frac{d}{2}} - \binom{\frac{c}{2}-1}{\frac{d}{2}-1} \right) = d \binom{\frac{c}{2}}{\frac{d}{2}}.$$

The argument when  $c$  is odd and  $d$  is even is identical.

Second, we consider the  $d$  transpositions  $R^1, R^2, \dots, R^d$ , where  $R^1$  is a single transposition clockwise which generates the cyclic group of order  $d$ . Let  $m$  be a divisor of  $d$ . According to Theorem 3.5, each  $R^j$  with  $\gcd(d, j) = m$  generates the same subgroup, and this subgroup has order  $d/m$ . If an interval array can be fixed



**Figure 4.** If  $d = 6$ , rotating the hexagon  $120^\circ$  is acting on the interval arrays with  $R^2$ , an element of order 3. If an interval array is fixed, then the values  $A$  and  $B$  must each be repeated twice.

by a transposition of order  $d/m$ , it is necessary that  $(d/m) \mid c$  or, equivalently, that  $d \mid cm$ . Thus, if  $m \mid d$  and  $d \mid cm$ , the number of interval arrays fixed by an element of order  $d/m$  is the number of ordered partitions of

$$\frac{c}{d/m} = \frac{cm}{d}$$

into  $m$  parts. According to Theorem 3.2, this can be done  $\binom{cm/d-1}{m-1}$  ways. Moreover, Theorem 3.6 says that  $\varphi(d/m)$  transpositions have order  $d/m$ . Thus the sum of all  $\text{Fix}(R^j)$  is

$$\sum_{m \mid d \text{ and } d \mid cm} \varphi(d/m) \binom{cm/d-1}{m-1}.$$

See Figure 4 for an example. Applying Burnside's lemma completes the proof.  $\square$

**Theorem 4.2.** Expressions (1-1) and (4-1) are equal.

*Proof.* Since these expressions both count the number of  $d$ -note set classes in a  $c$ -note chromatic universe, they are equal. However, we provide a different proof, outside the context of music theory.

We must show that

$$\frac{1}{c} \sum_{j \mid \gcd(c,d)} \varphi(j) \binom{c/j}{d/j} = \frac{1}{d} \sum_{m \mid d \text{ and } d \mid cm} \varphi(d/m) \binom{cm/d-1}{m-1}. \tag{4-2}$$

We start with the right-hand side and reindex, letting  $j = d/m$ . Then

$$\begin{aligned} \frac{1}{d} \sum_{m \mid d \text{ and } d \mid cm} \varphi(d/m) \binom{cm/d-1}{m-1} &= \frac{1}{d} \sum_{d/j \mid d \text{ and } d \mid \frac{cd}{j}} \varphi(j) \binom{c/j-1}{d/j-1} \\ &= \frac{1}{d} \sum_{j \mid \gcd(c,d)} \varphi(j) \binom{c/j-1}{d/j-1}. \end{aligned}$$

The last equality is valid because

$$\{j : j \mid \gcd(c, d)\} = \{j : (d/j) \mid d \text{ and } d \mid (cd/j)\}.$$

The equality of (4-2) follows from termwise equality, as a result of Theorem 3.1.  $\square$

### References

- [Brualdi 2010] R. A. Brualdi, *Introductory combinatorics*, 5th ed., Pearson Prentice Hall, Upper Saddle River, NJ, 2010. MR 2012a:05001
- [Clough and Myerson 1985] J. Clough and G. Myerson, “Variety and multiplicity in diatonic systems”, *Journal of Music Theory* **29**:2 (1985), 249–270.
- [Dummit and Foote 2004] D. S. Dummit and R. M. Foote, *Abstract algebra*, 3rd ed., Wiley, Hoboken, NJ, 2004. MR 2007h:00003 Zbl 1037.00003
- [Friepertinger 1992] H. Friepertinger, “Enumeration in musical theory”, Beiträge zur elektronischen Musik 1, Hochschule für Musik und Darstellende Kunst, Graz, 1992, <http://iem.kug.ac.at/projects/workspace/projekte-bis-2008/publications/bem/bem1.html>.
- [Graham et al. 2008] J. Graham, A. Hack, and J. Wilson, “An application of Burnside’s theorem to music theory”, *The UMAP Journal* **29**:1 (2008), 45–57.
- [Hook 2007] J. Hook, “Why are there twenty-nine tetrachords? A tutorial on combinatorics and enumeration in music theory”, *Music Theory Online* **13**:4 (2007).
- [Reiner 1985] D. L. Reiner, “Enumeration in music theory”, *Amer. Math. Monthly* **92**:1 (1985), 51–54. MR 86c:05021 Zbl 0582.05005

Received: 2013-08-14    Revised: 2013-10-24    Accepted: 2013-12-23

joshua.case@maine.edu      *Mathematics Department, University of Maine at Farmington,  
228 South Street, Farmington, ME 04938, United States*

lori.koban@maine.edu      *Mathematics Department, University of Maine at Farmington,  
228 South Street, Farmington, ME 04938, United States*

jordan.legrand@maine.edu      *Mathematics Department, University of Maine at Farmington,  
228 South Street, Farmington, ME 04938, United States*

# involve

msp.org/involve

## EDITORS

### MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

### BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	La Trobe University, Australia P.Cerone@latrobe.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsteam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

## PRODUCTION

Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2015 is US \$140/year for the electronic version, and \$190/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2015 Mathematical Sciences Publishers

# involve

2015

vol. 8

no. 2

Enhancing multiple testing: two applications of the probability of correct selection statistic	181
ERIN IRWIN AND JASON WILSON	
On attractors and their basins	195
ALEXANDER ARBIETO AND DAVI OBATA	
Convergence of the maximum zeros of a class of Fibonacci-type polynomials	211
REBECCA GRIDER AND KRISTI KARBER	
Iteration digraphs of a linear function	221
HANNAH ROBERTS	
Numerical integration of rational bubble functions with multiple singularities	233
MICHAEL SCHNEIER	
Finite groups with some weakly $s$ -permutably embedded and weakly $s$ -supplemented subgroups	253
GUO ZHONG, XUANLONG MA, SHIXUN LIN, JIAYI XIA AND JIANXING JIN	
Ordering graphs in a normalized singular value measure	263
CHARLES R. JOHNSON, BRIAN LINS, VICTOR LUO AND SEAN MEEHAN	
More explicit formulas for Bernoulli and Euler numbers	275
FRANCESCA ROMANO	
Crossings of complex line segments	285
SAMULI LEPPÄNEN	
On the $\varepsilon$ -ascent chromatic index of complete graphs	295
JEAN A. BREYTENBACH AND C. M. (KIEKA) MYNHARDT	
Bisection envelopes	307
NOAH FECHTOR-PRADINES	
Degree 14 2-adic fields	329
CHAD AWTRY, NICOLE MILES, JONATHAN MILSTEAD, CHRISTOPHER SHILL AND ERIN STROSNIDER	
Counting set classes with Burnside's lemma	337
JOSHUA CASE, LORI KOBAN AND JORDAN LEGRAND	
Border rank of ternary trilinear forms and the $j$ -invariant	345
DEREK ALLUMS AND JOSEPH M. LANDSBERG	
On the least prime congruent to 1 modulo $n$	357
JACKSON S. MORROW	



1944-4176(2015)8:2;1-5