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An extension of Young's segregation game

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In *Individual strategy and social structure* (2001), Young demonstrated that the stochastically stable configurations of his *segregation game* are precisely those that are segregated. This paper extends the work of Young to configurations involving three types of individuals. We show that the stochastically stable configurations in this more general setting are again precisely those that are segregated.

Schelling [1971] investigated self-organizing systems consisting of two groups of individuals, two of whom could trade locations at each discrete time interval to improve at least one's contentment level without diminishing the other's. He identified the equilibria of these systems under various conditions. Most of the time, these equilibria were more segregated in the sense that the individual members of each of the groups tended to gather in larger clusters rather than be uniformly mixed. Young [2001] used a Markov chain model to identify the stochastically stable equilibria of these self-organizing systems with two groups of individuals.

By an *equilibrium* we mean a state in which no pair of individuals exist who would prefer to trade positions. These equilibria are stable in sense that once one is reached, there will be no further change in the system.

However, if we allow for the possibility of error, that is, trades of pairs of individuals which do not benefit at least one of the two, without harming the other, it is possible to move from some equilibria to others. Those equilibria which remain stable in this more general context are called *stochastically stable* equilibria. They are precisely the segregated equilibria, those with all of the individuals of a group gathered into a single cluster.

After seeing this behavior modeled in a classroom activity, a student asked the faculty author of this paper whether the same phenomena happened if there were more than two types of individuals. Responding to that question, in [Burek et al. 2009] we showed that there are both segregated (all members of each group living next to each other in a single cluster) and non-segregated equilibria in such a model,

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consistent with the work of Schelling. In this paper, we will show that the segregated equilibria are the only stochastically stable equilibria, consistent with the work of Young.

A real world example of this type of self-organizing behavior was provided in 2004 when Bill Bishop received national attention when he made the following claim and coined the neologism the *big sort*: the phenomenon that Americans have been sorting themselves into increasingly homogeneous political communities according to city and even neighborhood. He published his argument in [Bishop 2009] using demographical data to justify his claims. Therefore, in recognition of Bishop's work, we will refer to our three groups of people as Republicans, Democrats, and Libertarians.

Terminology

Let R, D, and L represent a individual that is a Republican, Democrat, and Libertarian, respectively. A *configuration* is an linear arrangement of individuals members that contains at least four members from each party,with an explanation for this restriction being given later. In general, let r, d, and l represent the total number of individuals in each of the Republican, Democrat, and Libertarian parties, respectively. We assume that our configurations are circular in the sense that the first and last individuals are assumed to be neighbors of each other; this allows us to not worry about end conditions. For instance, in the following configuration, the leftmost R is considered to be next to the rightmost L:

$$RDLLLLLLLLLLLRRRRRRDRDLDDDLLLLRRRRRRRRRRL.$$

We consider the positions of the Republicans, Democrats, and Libertarians to be ordered in this configuration. Thus, the configuration above is distinct from the one obtained by shifting each individual nine positions to the right, displayed here:

$$RRRRRRRRRRLRDLLLLLLLLLLLRRRRRRDRDLDDDLLLL.$$

To avoid unnecessary repetition, we use exponential notation and define a cluster of Y^m to be a string of m Y's in a row, where $2 \le m \le q$ where q denotes the total number of members in Y's party. Thus, the first configuration displayed above can be somewhat more compactly conveyed as

$$RDL^{11}R^6DRDLD^3L^4R^{10}L.$$

While the positions are distinct, the individuals themselves are not distinguished beyond their party affiliation.

Given any configuration, we need to determine an individual's contentment level. Measuring contentment was straightforward in [Young 2001] since Young only considered two types of individuals: either you're next to at least one individual like yourself (and are content) or you are not (and are therefore not content). Introducing a third group adds a layer of complexity in the form of bias: which individuals (aside from those of your own party) do you prefer to be next to, which are you neutral towards, and which do you prefer not to be next to at all? In [Burek et al. 2009], we describe seven different scenarios with varying levels of bias. In this paper, our focus is on individuals who have no aversion towards individuals of either of the other two parties, but do have a preference for neighbors of their own party.

We can describe this low level of bias as follows, since we do not need to specify the utility functions for our purposes. Let X, Y, and Z be arbitrary individuals, not necessarily of distinct parties. Given an individual Y in a configuration, consider the ordered triple consisting of Y and its immediate neighbors to the left and the right, X and Z, respectively. Y has the highest contentment if both X and Z are of the same party as Y. Y has a somewhat lower contentment level if exactly one of X and Z is of the same party as Y. Finally Y has the lowest contentment level if neither X nor Z is of the same party as Y. For example, in the configuration

$$RDL^{11}R^{6}DRDLD^{3}L^{4}R^{10}L,$$

the first D individual has the lowest contentment level and the second to last D has the highest contentment level. More than three levels of contentment would be possible were we to allow higher levels of bias, as described in [Burek et al. 2009].

Two individuals in a configuration are willing to trade positions if at least one of the individual's contentment level increases as a result of this trade, and the other individual's contentment level does not decrease as a result of the trade. We call this a *favorable trade*.

Notice that when two individuals trade positions, it moves us from the original configuration s to a new configuration s'. When we move forward to a new time period, a pair of individuals are randomly chosen from among those pairs for whom a favorable trade exists and these two individuals trade positions. Eventually, no favorable trades remain and the system reaches an *equilibrium configuration*. Some of these are *segregated* equilibrium configurations, in the form

$$R^r D^d L^l \quad \text{or} \quad R^r L^l D^d.$$

Segregated equilibrium configurations could start with Democrats or Libertarians as well. In particular, note that the configuratons

$$L^l R^r D^d \quad \text{and} \quad D^d L^l R^r$$

can be obtained from the first segregated equilibribium configuration above by shifted positions to the right, but they can not be obtained from the second segregated equilibribium above. Thus there are two fundamental classes of segregated equilibria, those of the form *RDL* and those of the form *RLD*.

Other equilibria are *non-segregated*. In a non-segregated equilibrium, the members of at least one party are separated into two disjoint clusters, each of which contains at least two members. Some examples (with $r = 6$, $d = 10$, and $l = 8$) are

$$D^8R^3L^2D^2L^4R^3L^2, \quad R^3D^3L^8D^5R^5, \quad \text{and} \quad L^2D^{10}L^6R^6.$$

If there are only three members of a party, then they must be in a single cluster in every equilibria. Thus if there are only three members of each of the three parties, there are no non-segregated equilibria. Thus to ensure that we have non-segregated equilibria, we require that there be at least four individuals in each party.

We denote the set of all equilibrium configurations by E, the set of those equilibria that are segregated by E^S and those equilibria that are non-segregated by E^{NS}. Thus, $E = E^S \cup E^{NS}$.

In our discussion so far, we have only allowed favorable trades to occur. To investigate stochastically stable configurations, we need to allow the possibility of non-favorable trades to occur as well. We define three types of such trades. Let a, b, and c denote positive real numbers such that $0 < a < b < c$. A *type a perturbation* occurs when two individuals trade with one individual's contentment level rising and the other's falling, or when two individuals trade with neither individual's contentment level changing. A *type b perturbation* occurs when two individuals trade positions such that one individual's contentment level decreases, but the other individual's contentment level remains constant. Finally, a *type c perturbation* occurs when two individuals trade positions such that both of the individuals' contentment levels go down.

Markov chain model

Both the basic situation and the perturbed situation can be modeled as a Markov chain. In this section, we describe those models, identify their key properties, their relationship, and give the key theorem that we will use in our analysis. The reader interested in more detailed discussion of Markov chains should consult [Ghahramani 2005] or [Norris 1998] for an introduction to the subject, or [Ross 2000] for a more rigorous treatment.

We model the basic situation as a Markov chain, P, by letting the set of states, S, be the various configurations, s, of our neighborhoods. For each s, set $p_{s,s'} = 0$ for any state s' such that there is no favorable trade which moves s to s'. For all other s', $p_{s,s'} = k/n$, where n is the number of favorable trades in s and k is the number of favorable trades which move s to s'. As favorable trades occur, the system can be thought to randomly evolve over time, with at most one trade occurring during each time period.

If we allow the possibility of non-favorable trades as well, we can obtain a second Markov chain, P^ϵ, in which a type a perturbation occurs with probability ϵ^a, where $1 > \epsilon > 0$. A type b perturbation occurs with probability ϵ^b, and a type c perturbation occurs with probability ϵ^c. Favorable trades occur with equal probabilities which sum to $1 - \sum pr(x)$, where x ranges across all of the non-favorable trades. Thus P^ϵ has the same state space as P, and

$$p_{s,s'} = \sum pr(y) + \sum pr(x),$$

where y ranges across all favorable trades moving s to s' and x ranges across all non-favorable trades doing the same.

We can say the following about P and P^ϵ:

(1) The absorbing states of P are precisely the equilibrium states.

(2) P^ϵ is irreducible.

(3) P^ϵ has a unique stationary distribution, μ^ϵ.

(4) P^ϵ satisfies $\lim_{\epsilon \to 0} p^\epsilon_{s,s'} = p_{s,s'}$, and there exists a unique $r(s, s') > 0$ such that whenever $p^\epsilon_{s,s'} > 0$ for some $\epsilon > 0$,

$$0 < \lim_{\epsilon \to 0} \frac{p^\epsilon_{s,s'}}{\epsilon^{r(s,s')}} < \infty.$$

(5) P^ϵ is regular perturbed.

Briefly, these five items are justified as follows. In any non-equilibrium configuration, there are a finite number of favorable trades; as time advances and these trades happen, they are eventually depleted resulting in a configuration that is at equilibrium and is an absorbing state of P. Because all trades (favorable and non-favorable) have positive probability in P^ϵ, there exists a positive probability that of moving from any configuration to any other configuration in the future. Hence P^ϵ is irreducible. Further, since P^ϵ has a finite state space, it has a unique stationary distribution. The first limit in item four follows from our definition of $p^\epsilon_{s,s'}$. The second limit follows from our assignments of probabilities to the various non-favorable trades. Finally, item five follows from items two and four.

In general, $r(s, s')$ is called the resistance to moving from state s to state s', and is the minimum, taken over all sequences of trades that begin in state s and end in state s', of the sum of the resistances on the individual trades in the sequence. The values a, b, and c are the resistance to the corresponding types of non-favorable trades. A favorable trade has resistance 0.

We now construct a graph theoretic model to compute the stochastically stable states of P^ϵ. Recall that the only absorbing states of P are the equilibrium states in S. Denote these by $E = \{z_1, z_2, z_3, \ldots\}$. Construct a weighted complete directed graph whose vertices are the elements of E and whose edges have weights equal to the resistances $r(z_i, z_j)$. A *z-tree* is a set of $|E| - 1$ directed edges such that, from every vertex different from $z \in E$, there is a unique directed path in the tree to z. The resistance of a z-tree is the sum of the resistances on the edges that compose it. The *stochastic potential* of the state z is the minimum resistance over all z-trees.

Figure 1 illustrates one such tree. In this illustration, z is an *RLD* segregated equilibria, and each *RLD* and *RDL* vertex represents a one position shift from its parent vertex. The *ns* vertices represent generic non-segregated equilibria. The choice of edge weights, a, b, and $a + b$, will be explained after Theorem 1.

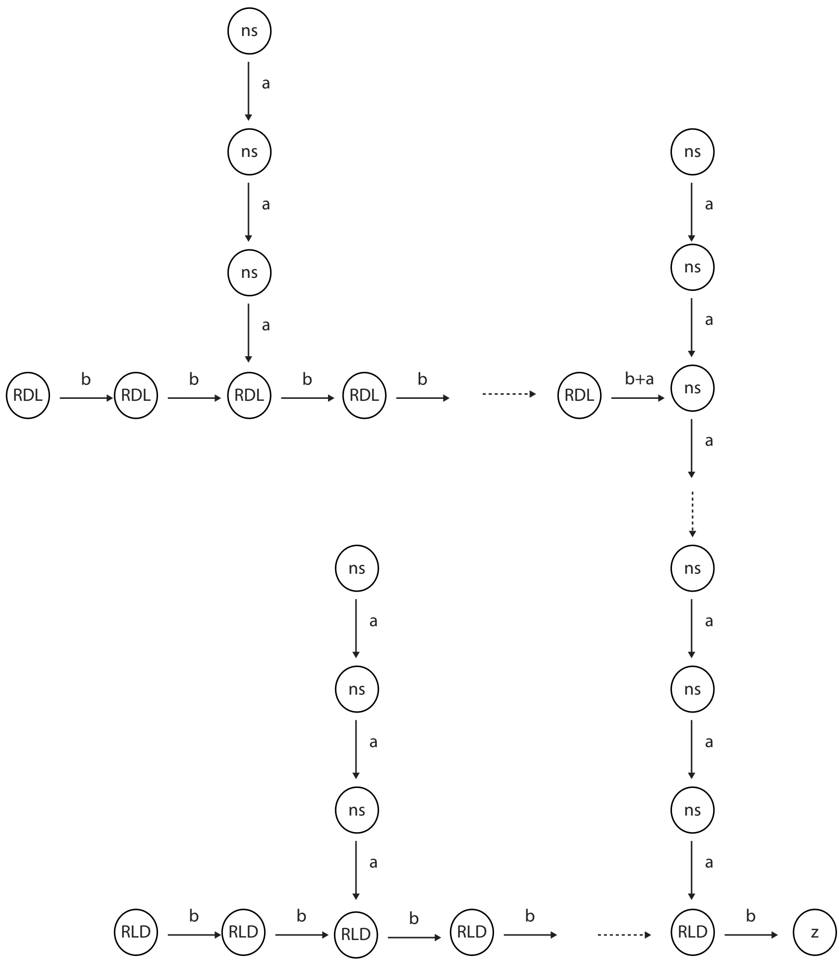

Figure 1. A z-tree for a *RLD* segregated equilibrium.

The stochastically stable states are those states that occur with positive probability in the long run while the probility of error, ϵ, is small but non-vanishing. That is, the state $s \in E$ is stochastically stable for the Markov chain P^ϵ if

$$\lim_{\epsilon \to 0} \mu_s^\epsilon > 0,$$

where μ^ϵ is the unique stationary distribution of P^ϵ. Young's theorem provides a method for determining these states, which is the goal of this paper.

Young's theorem. *Let P^ϵ be a regular perturbed Markov chain and let μ^ϵ be the unique stationary distribution of P^ϵ for each $\epsilon > 0$. Then the stochastically stable states are precisely those states that are absorbing states of P having minimum stochastic potential* [Young 1993].

Main result

In this section, we construct z-trees for both segregated and non-segregated equilibria and demonstrate that the former have minimal stochastic potential. We begin by proving three lemmas which will develop our argument.

Lemma 1. *Given a non-segregated equilibrium, the resistance to moving to another equilibrium by making a trade which moves an individual from one cluster to another cluster of like individuals is a.*

Proof. Given a non-segregated equilibribium state, suppose that one party, say the Rs, has at least two clusters. Then at least two of the R clusters have neighbor clusters of the same type, say L. Otherwise, there are exactly two R clusters, one with two D clusters as neighbors and the other with two L cluster neighbors. (The pattern is $D - R - D - L - R - L$.) In this case, we change our perspective to the two D clusters, which have a common R cluster as a neighbor.

There are three patterns possible for the two R clusters and their L cluster neighbors:

$$L^{l_1-1}\boldsymbol{L}R^{r_1}\dots L^{l_2}\boldsymbol{R}R^{r_2-1},$$
$$L^{l_1-1}\boldsymbol{L}R^{r_1}\dots R^{r_2-1}\boldsymbol{R}L^{l_2},$$
$$R^{r_1-1}\boldsymbol{R}L^{l_1}\dots L^{l_2-1}\boldsymbol{L}R^{r_2}.$$

In each case, trading the bold faced individuals shifts one individual from one cluster to another and results in a new equilibrium state. Each of these trades has resistance a. □

Lemma 2. *Given any segregated equilibrium, the minimum resistance to shifting to another segregated equilibrium is b.*

Proof. Consider the segregated equilibrium

$$R^{r-1}\boldsymbol{R}D^{d}L^{l-1}\boldsymbol{L}.$$

We trade the boldfaced individuals, with resistance b, to get

$$R^{r-1}\boldsymbol{L}D^{d}L^{l-1}\boldsymbol{R}.$$

However, this configuration is not an equilibrium. Therefore, we need to make a favorable trade, which has resistance 0, to return to equilibrium. The two individuals involved in this trade are indicated in bold:

$$R^{r-1}\boldsymbol{L}D^{d-1}\boldsymbol{D}L^{l-1}R.$$

Trading these two individuals results in the segregated configuration:

$$R^{r-1}\boldsymbol{D}D^{d-1}\boldsymbol{L}L^{l-1}R.$$

Note that the new configuration is the the original equilibrium configuration shifted one position to the left.

To obtain a smaller resistance, either one of the individuals trading had an increase in their contentment level while the second had a decrease, or neither of the individuals trading had any change in contentment level. However, when we begin with a segregated equilibrium, both cases imply that any individual who trades must trade with another individual of the same party, and that results in the same equilibrium after the trade as before. Thus it is not possible to shift from one segregated equilibrium to another with a resistance less than b. □

Lemma 3. *Given any equilibrium, the minimum resistance to creating a new cluster is* $b+a$.

Proof. Without loss of generality, consider an equilibrium containing the sequence

$$\dots L^{l_1-1}\boldsymbol{L}D^{d_1-1}\boldsymbol{D}R^{r_1}\dots.$$

In a trade between the bold L and the bold D, L's contentment level would drop, while D's contentment level stays the same, resulting in a trade with resistance b. The resulting configuration,

$$\dots L^{l_1-1}\boldsymbol{D}D^{d_1-1}\boldsymbol{L}R^{r_1}\dots$$

is not in equilibrium, so a trade between the second right-most L with the (new) right-most D, with resistance a, results in an equilibrium with an additional cluster of consisting of two L's. This is the smallest resistance possible, since creating a new cluster requires isolating an individual and consequently lowering their contentment level, a type b perturbation. To return to an equilibrium state with this new cluster, another trade must occur in order to have a second individual join the first. At best this is a type a perturbation. □

With these three lemmas in hand, we are ready to compute the minimum stochastic potential for the z-trees. In the proof of Lemma 2, we assumed that the segregated configuration has the ordering *RDL* of clusters. The alternative ordering is *RLD*. Clearly the Lemma applies to this ordering as well. However, in both Theorem 1 and Theorem 2 below, we do need to treat the two orderings separately,so we let E^S_{RDL} and E^S_{RLD} denote the two sets of segregated equilibriums, respectively. We begin with $z \in E^S$.

Theorem 1. *For each $z \in E^S$, its stochastic potential is*

$$a \cdot |E^{NS}| + b \cdot (|E^S_{RDL}| - 1) + b \cdot (|E^S_{RLD}| - 1) + (b + a) = a \cdot |E^{NS}| + b \cdot (|E^S| - 1) + a.$$

Proof. We will assume that z is an *RLD* type of segregated equilibrium. Each non-segregated equilibrium has an outbound edge to another equilibrium in which one of the clusters has one fewer individuals. By Lemma 1, this edge has resistance (weight) a. All but two of the segregated equilibriums have an outbound edge to another segregated equilibriums, which rotates the positions of the individuals by one position. By Lemma 2, each of these edges has resistance b. The first exception to the previous statement is the root equilibrium, z, which has no outbound edge associated with it. The second exception is the *RDL* equilibrium at which a new cluster is generated in order to begin the transition to an *RLD* equilibrium. By Lemma 3, this particular equilibrium has an outbound edge that has resistance $b + a$. Summing the resistances on the various edges gives the result. □

Figure 1 illustrates the proof for a typical z-tree, when z is a segregated equilibrium. The target *RLD* equilibrium is in the lower right corner, and the transitional *RDL* equilibrium has a resistance of $b + a$. In this illustration, each segregated equilibrium is rotated until it reaches z, or until the transitional configuration is reached. Each non-segregated state progressively moves to states with smaller and/or fewer clusters, eventually becoming segregated.

Next, we compute the minimum stochastic potential for an arbitrary z-tree where z is in E^{NS}. Notice that in Theorem 1, we were able to calculate the minimum stochastic resistance precisely. In the following theorem, we are only able to determine a lower bound. This is because it is may be required to create many new clusters, with the creation of each of these clusters increasing the sum given in the theorem. Fortunately, the result is sufficient for our purposes.

Theorem 2. *For each $z \in E^{NS}$, its stochastic potential is at least*

$$\begin{aligned} a \cdot (|E^{NS}| - 1) + b \cdot (|E^S_{RDL}| - 1) + (b + a) + b \cdot (|E^S_{RLD}| - 1) + (b + a) \\ = a \cdot |E^{NS}| + b \cdot (|E^S| - 1) + a + b. \end{aligned}$$

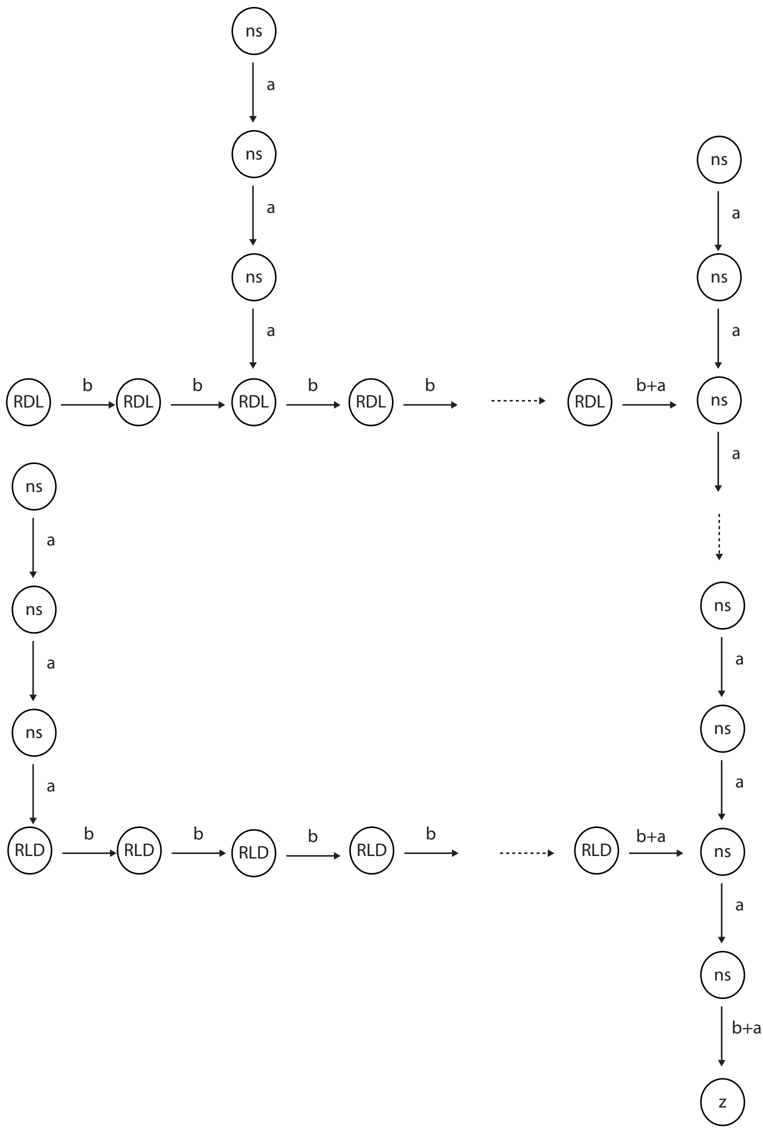

Figure 2. Minimal z-tree for a non-segregated equilibrium.

Proof. We will assume that z has only four clusters, the minimum possible in a non-segregated equilibriums. Each non-segregated equilibrium, other than z, has an outbound edge to another equilibrium with in which one of the clusters has one fewer individuals. By Lemma 1, this edge has resistance a. All but two of the segregated equilibriums has an outbound edge to another segregated equilibriums rotating the positions of the individuals by one position. By Lemma 2, each of these edges has resistance b. The two exceptions to the previous are the *RLD* equilibrium and the *RDL* equilibrium at which new clusters are created; by Lemma 3, these two equilibriums have outbound edges with resistance $b + a$. Summing the resistance on the various edges gives the result. □

Figure 2 illustrates the proof for a z-tree in which z is a non-segregated equilibrium. Again, the target non-segregated equilibrium is in the lower right corner, and the transitional *RLD* and *RDL* equilibriums have resistance $b + a$.

Since the sum in Theorem 1 is smaller than the sum in Theorem 2, we are able state our main result.

Theorem 3. *In segregation games with three types of individuals and the lowest level of bias, the stochastically stable equilibriums are precisely those that are segregated.*

Open questions

The model described in this paper assumes that no individuals have a bias against members of one of the other groups. In [Burek et al. 2009], we outline six other scenarios describing varying biases that are available among three groups. For example, would we get the same results in a scenario where Republicans and Democrats each prefer to live near Libertarians over each other, but Libertarians hold no such bias? What if Democrats prefer Republicans, Republicans prefer Libertarians, and Libertarians prefer Republicans? Demonstrating stochastic results similar to those presented in this paper would extend our model.

Furthermore, it would be interesting to extend the analysis in this paper to a 2-dimensional perspective. Doing so would allow for a more realistic geo-political interpretation of the results, such as that suggested by Bishop's work.

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