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# Embedding groups into distributive subsets of the monoid of binary operations

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Let $X$ be a set and $\mathrm{Bin}(X)$ the set of all binary operations on $X$. We say that $S \subset \mathrm{Bin}(X)$ is a distributive set of operations if all pairs of elements $*_\alpha, *_\beta \in S$ are right distributive, that is, $(a *_\alpha b) *_\beta c = (a *_\beta c) *_\alpha (b *_\beta c)$ (we allow $*_\alpha = *_\beta$).

The question of which groups can be realized as distributive sets was asked by J. Przytycki. The initial guess that embedding into $\mathrm{Bin}(X)$ for some $X$ holds for any $G$ was complicated by an observation that if $* \in S$ is idempotent ($a * a = a$), then $*$ commutes with every element of $S$. The first noncommutative subgroup of $\mathrm{Bin}(X)$ (the group $S_3$) was found in October 2011 by Y. Berman.

Here we show that any group can be embedded in $\mathrm{Bin}(X)$ for $X = G$ (as a set). We also discuss minimality of embeddings observing, in particular, that $X$ with six elements is the smallest set such that $\mathrm{Bin}(X)$ contains a nonabelian subgroup.



## 1. Introduction

Let $X$ be a set and $\mathrm{Bin}(X)$ the set of all distributive operations on $X$. We say that $S \subset \mathrm{Bin}(X)$ is a distributive set of operations if all pairs of elements $*_\alpha, *_\beta \in S$ are right distributive, that is, $(a *_\alpha b) *_\beta c = (a *_\beta c) *_\alpha (b *_\beta c)$ (we allow $*_\alpha = *_\beta$). It was observed in [Przytycki 2011] (see also [Romanowska and Smith 1985]) that $\mathrm{Bin}(X)$ is a monoid with composition $*_1 *_2$ given by $a *_1 *_2 b = (a *_1 b) *_2 b$ and the identity $*_0$ being the right trivial operation, that is, $a *_0 b = a$ for any $a, b \in X$.

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The submonoid of $\mathrm{Bin}(X)$ of all invertible elements in $\mathrm{Bin}(X)$ is a group denoted by $\mathrm{Bin}_{\mathrm{inv}}(X)$. If $* \in \mathrm{Bin}_{\mathrm{inv}}(X)$ then $*^{-1}$ is usually denoted by $\bar{*}$.

We say that a subset $S \subset \mathrm{Bin}(X)$ is a distributive set if all pairs of elements $*_\alpha, *_\beta \in S$ are right distributive, that is, $(a *_\alpha b) *_\beta c = (a *_\beta c) *_\alpha (b *_\beta c)$ (we allow $*_\alpha = *_\beta$). Additionally, $(X; S)$ is called a multishelf[1].

The following important basic lemma was proven in [Przytycki 2011]:

**Lemma 1.1.** (i) *If $S$ is a distributive set and $* \in S$ is invertible, then $S \cup \{\bar{*}\}$ is also a distributive set.*

(ii) *If $S$ is a distributive set and $M(S)$ is the monoid generated by $S$, then $M(S)$ is a distributive monoid.*

(iii) *If $S$ is a distributive set of invertible operations and $G(S)$ is the group generated by $S$, then $G(S)$ is a distributive group.*

The question of which groups can be realized as distributive sets was asked by J. Przytycki. Soon after the definition of a distributive submonoid of $\mathrm{Bin}(X)$ was given in [Przytycki 2011], Michal Jablonowski, a graduate student at Gdańsk University, noticed that any distributive monoid whose elements are idempotent operations is commutative.

**Proposition 1.2** [Przytycki 2011]. *Consider $*_\alpha, *_\beta \in \mathrm{Bin}(X)$ such that $*_\beta$ is idempotent ($a *_\beta a = a$) and distributive with respect to $*_\alpha$. Then $*_\alpha$ and $*_\beta$ commute. In particular*:

(i) *If $M$ is a distributive monoid and $*_\beta \in M$ is an idempotent operation, then $*_\beta$ is in the center of $M$.*

(ii) *A distributive monoid whose elements are idempotent operations is commutative.*

*Proof.* We have $(a *_\alpha b) *_\beta b \overset{\text{distrib}}{=} (a *_\beta b) *_\alpha (b *_\beta b) \overset{\text{idemp}}{=} (a *_\beta b) *_\alpha b$. □

A few months later, Agata Jastrzębska (also a graduate student at Gdańsk University) checked that any distributive group in $\mathrm{Bin}_{\mathrm{inv}}(X)$ for $|X| \leq 5$ is commutative.

The first noncommutative subgroup of $\mathrm{Bin}(X)$ (the group $S_3$) was found in October 2011 by Yosef Berman. Soon after, Berman and Carl Hammarsten constructed an embedding of a general dihedral group $D_{2\cdot n}$ in $\mathrm{Bin}(X)$ where $X$ has $2n$ elements. The embedding of Berman, $\phi : D_{2\cdot 3} \to \mathrm{Bin}(X)$, is given as follows: if $X = \{0, 1, 2, 3, 4, 5\}$ then the subgroup $D_{2\cdot 3} \subset \mathrm{Bin}(X)$ is generated by binary

[1] If $(X; *)$ is a magma and $*$ is a right self-distributive operation then $(X; *)$ is called a shelf, the term coined by Alissa Crans [2004].

operations $*_\tau$, which generates reflection, and $*_\sigma$, which generates a 3-cycle;

$$*_\tau = \begin{pmatrix} 1 & 1 & 3 & 5 & 5 & 3 \\ 0 & 0 & 4 & 2 & 2 & 4 \\ 3 & 3 & 5 & 1 & 1 & 5 \\ 2 & 2 & 0 & 4 & 4 & 0 \\ 5 & 5 & 1 & 3 & 3 & 1 \\ 4 & 4 & 2 & 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad *_\sigma = \begin{pmatrix} 2 & 4 & 2 & 4 & 2 & 4 \\ 5 & 3 & 5 & 3 & 5 & 3 \\ 4 & 0 & 4 & 0 & 4 & 0 \\ 1 & 5 & 1 & 5 & 1 & 5 \\ 0 & 2 & 0 & 2 & 0 & 2 \\ 3 & 1 & 3 & 1 & 3 & 1 \end{pmatrix},$$

where $i * j$ is placed in the $i$-th row and $j$-th column, and $D_{2\cdot 3} = \{\tau, \sigma \mid \tau\sigma\tau = \sigma^{-1}\}$.

## 2. Regular distributive embedding

We now show that any group $G$ can be embedded in $\mathrm{Bin}(X)$ for some $X$.

**Theorem 2.1** (Regular embedding). *Every group $G$ embeds in* $\mathrm{Bin}(G)$. *This embedding* (*monomorphism*), $\phi^{\mathrm{reg}} : G \to \mathrm{Bin}(G)$, *sends $g$ to $*_g$, where $a *_g b = ab^{-1}gb$.*

*Proof.* (i) We check that the set $\{*_g\}_{g\in G}$ is a distributive set. We have

$$(a *_{g_1} b) *_{g_2} c = (ab^{-1}g_1b) *_{g_2} c = ab^{-1}g_1bc^{-1}g_2c,$$

and

$$(a *_{g_2} c) *_{g_1} (b *_{g_2} c) = (ac^{-1}g_2c) *_{g_1} (bc^{-1}g_2c) = ab^{-1}g_1bc^{-1}g_2c,$$

as needed.

(ii) Now we check that the map $\phi^{\mathrm{reg}}$ is a monomorphism. The image of the identity $*_0$ is the identity in $\mathrm{Bin}(G)$. Furthermore, $a *_{g_1g_2} b = ab^{-1}g_1g_2b$ and $a *_{g_1} *_{g_2} b = (a *_{g_1} b) *_{g_2} b = ab^{-1}g_1bb^{-1}g_2b = ab^{-1}g_1g_2b$, as needed. We have proven that $\phi^{\mathrm{reg}}$ is a homomorphism. To show that $\phi^{\mathrm{reg}}$ is a monomorphism, we substitute $b = 1$ in the formula for $a *_g b$ to get $a *_g 1 = ag$; so different choices of $g$ give different binary operations in $\mathrm{Bin}(G)$. Notice that $\phi^{\mathrm{reg}}(g^{-1}) = \bar{*}_g$. □

We call our embedding *regular*, analogous to the regular representation of a group. We do not claim that the regular embedding is minimal, so finding minimal distributive embeddings is a very interesting problem in itself.

## 3. General conditions for a distributive embedding

We now discuss a method that can be used to embed groups into subsets of $\mathrm{Bin}_{\mathrm{inv}}(X)$ satisfying an arbitrary condition. We then use this method when the condition is right distributivity, which leads us to the regular distributive embedding of $G$ in $\mathrm{Bin}(G)$ and should be a natural tool to look for minimal embeddings. For the group $S_3$, we know, by Jastrzebska's calculations, that $X$ consisting of six elements is the minimal set such that $S_3$ embeds in $\mathrm{Bin}(X)$.

We start from the following basic observation:

**Lemma 3.1.** *There is an isomorphism between* $\mathrm{Bin}_{\mathrm{inv}}(X)$ *and* $S_{|X|}^{|X|}$, *where* $|X|$ *is the cardinality of* $|X|$ *and* $S_{|X|}$ *is the group of permutations on set* $X$ *( i.e., bijections of the set* $X$*). The isomorphism* $\alpha : \mathrm{Bin}_{\mathrm{inv}}(X) \to S_X^{|X|} = \Pi_{y \in X} S_X^y$ *is described as follows*: $\alpha(*)(y) : X \to X$ *is the bijection where* $(\alpha(*)(y))(x) = x * y$. *In other words*, $\alpha(*)(y)$ *is the bijection corresponding to the* $y$*-coordinate of* $S_X^{|X|}$.

Using the map $\alpha$, we can translate conditions on a set of binary operations in $\mathrm{Bin}(X)$ into a group-theoretic condition on (coordinates of) elements of $S_X^{|X|}$. With some work, we can use this to find an embedding of a group into $\mathrm{Bin}(X)$. This is possible since the group axioms require that such an embedding must sit inside $\mathrm{Bin}_{\mathrm{inv}}(X)$. Let us consider distributive, invertible sets $\mathcal{S}$ of binary operations in $\mathrm{Bin}_{\mathrm{inv}}(X)$. These are subsets $\mathcal{S} \subseteq \mathrm{Bin}_{\mathrm{inv}}(X)$ that satisfy

$$(x *_i y) *_j z = (x *_j z) *_i (y *_j z) \quad \text{for all } *_i, *_j \in S \text{ and } x, y, z \in X.$$

Let $\sigma_{i,y} = p_y \alpha(*_i)$, where $p_y : S_X^{|X|} \to S_X$ is projection onto the $y$-th coordinate. Then translating the distributivity condition via $\alpha$,

$$\sigma_{j,z}(x *_i y) = \sigma_{i,(y *_j z)}(x *_j z)$$

or

$$\sigma_{j,z}(\sigma_{i,y}(x)) = \sigma_{i,\sigma_{j,z}(y)}(\sigma_{j,z}(x)),$$

which leads to

$$\sigma_{i,\sigma_{j,z}(y)} = \sigma_{j,z} \sigma_{i,y} \sigma_{j,z}^{-1}.$$

Now the problem of embedding a group into $\mathrm{Bin}_{\mathrm{inv}}(X)$ is reduced to finding subsets of $S_{|X|}^{|X|}$ satisfying the condition above that are isomorphic to the group. We can then use tools of group theory (e.g., representation theory) to solve the problem. This process can be attempted for subsets of $\mathrm{Bin}_{\mathrm{inv}}(X)$ satisfying any condition and leads to the embedding defined in the previous section for distributive subsets.

## 4. Future directions; multiterm homology

Przytycki [2011] defined multiterm homology for any distributive set. This provided motivation to have many examples of distributive sets. The regular embedding of a group (Theorem 2.1) provides an interesting family of distributive sets ripe for the study of their homology (compare with [Crans et al. 2014; Przytycki 2011; 2012; Przytycki and Putyra 2013; Przytycki and Sikora 2014]). As a nontrivial example, we propose computing $n$-term distributive homology related to the regular embedding of the cyclic group $Z_n$. Another problem related to Theorem 2.1 is determining which monoids are distributive submonoids of $\mathrm{Bin}(X)$.

A key motivation is to use multiterm distributive homology in knot theory. This possibility arises from the relation of the third Reidemeister move with right distributivity (and eventually the Yang–Baxter operator) and the important work of Carter, Kamada, and Saito [2001] and other researchers on applications of quandle homology to knot theory.

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