

# involve

a journal of mathematics

Two-parameter taxicab trigonometric functions

Kelly Delp and Michael Filipski



# Two-parameter taxicab trigonometric functions

Kelly Delp and Michael Filipiski

(Communicated by Frank Morgan)

In this paper, we review some of the fundamental properties of the  $\ell^1$ , or taxicab, metric on  $\mathbb{R}^2$ . We define and give explicit formulas for two-parameter sine and cosine functions for this metric space. We also determine the maximum of these functions, which is greater than 1.

## 1. Introduction

The  $\ell^1$  metric on  $\mathbb{R}^2$ , the so-called taxicab metric, is often one of the first non-Euclidean metrics a mathematics student encounters. For any points  $p = (p_1, p_2)$  and  $q = (q_1, q_2)$  in  $\mathbb{R}^2$ , the metric is given by the formula

$$d_T(p, q) = |p_1 - q_1| + |p_2 - q_2|.$$

The  $\ell^1$  metric is just one metric in a class of metrics defined on  $\mathbb{R}^2$  known as *Minkowski metrics*; see [Álvarez Paiva and Thompson 2005] for an introduction to these metric spaces. Let  $\Omega$  be a closed, bounded convex set in  $\mathbb{R}^2$  which contains and is symmetric about the origin. The set  $\Omega$  defines a norm on  $\mathbb{R}^2$ , where  $\Omega$  is the unit disk. Given a norm  $\|\cdot\|$ , one can define a metric on  $\mathbb{R}^2$  by  $d(p, q) = \|p - q\|$ . Examples of Minkowski metrics include the  $\ell^p$  metrics, the  $\ell^\infty$  or max metric, and metrics with a unit disk that is a regular  $2n$ -gon.

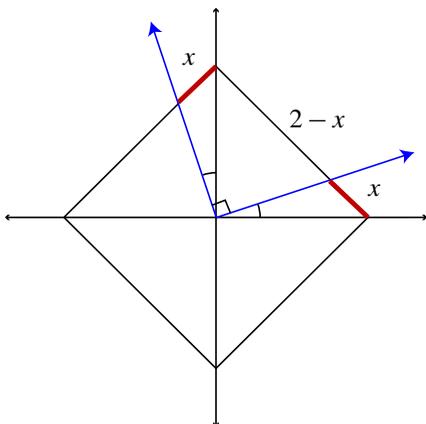
Length minimizing paths in the taxicab plane are not necessarily unique, so we use the vector space properties of  $\mathbb{R}^2$  and define *lines* to be the sets of points of the form  $L = \{t\mathbf{v} + \mathbf{b} \mid t \in \mathbb{R}\}$  for some fixed  $\mathbf{v}$  and  $\mathbf{b}$ . We can similarly define *line segments*, *triangles*, *rays*, and *angles* (pairs of rays sharing an initial point). We define the length of a line segment  $\overline{AB}$  to be the distance between the endpoints,  $d_T(A, B)$ .

Given a metric  $d$  on a set  $X$ , a circle  $C$  of radius  $r$  is the set of all points  $p \in X$  equidistant from a given point called the center. A circle in the taxicab metric is a square with diagonals parallel to the  $x$ - and  $y$ -axes. In Euclidean space there is an intrinsic notion of angle measure, radian measure, which is determined by the

---

MSC2010: 51F99, 52A21.

Keywords: taxicab trigonometry, Minkowski geometry.



**Figure 1.** Euclidean right angles have taxicab angle measure of 2.

length of a particular circle arc. We can similarly define an intrinsic angle measure in the taxicab plane, called *t-radians*.

**Definition 1.** Let  $C$  be a circle with radius  $r$  and center  $P$ . Given an angle with vertex  $P$ , let  $s$  be the length of the subtended arc. The t-radian measure,  $\theta$ , of a taxicab angle is given by

$$\theta = \frac{s}{r}.$$

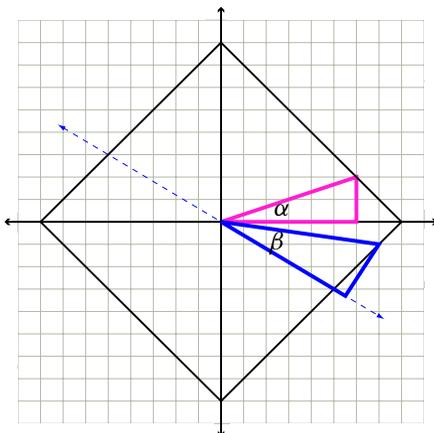
It is this notion of angle measure which was used in these previous works [Akça and Kaya 1997; Brisbin and Artola 1985; Thompson and Dray 2000] on taxicab trigonometry. Another well-studied angle measure in a Minkowski metric uses the *area* of the sector of the circle, rather than arc length, to define the angle measure. (Due to a theorem of Haar, any area measure  $\mu$  is proportional to Lebesgue measure; see [Álvarez Paiva and Thompson 2004] for a discussion of areas in normed spaces.) By Theorem 1 in [Düvelmeyer 2005], these two notions are equivalent (up to scale) because the taxicab circle is an example of an equiframed curve. See [Düvelmeyer 2005] for the definition of equiframed curve.

Note that an  $\ell^1$  circle has 8 t-radians, which means in this metric, 4 is the analogue of  $\pi$ . Some of the properties from Euclidean geometry have analogous statements which are true in the taxicab plane. We will use the following propositions.

**Proposition 2** [Thompson and Dray 2000, Theorem 4.2]. *The angle sum of a taxicab triangle is 4 t-radians.*

We define a taxicab right angle to be an angle with measure 2 t-radians, which, as in Euclidean geometry, is an angle which has measure equal to its supplement.

**Proposition 3** [Thompson and Dray 2000, Lemma 2.5]. *A Euclidean right angle has taxicab angle measure of 2 t-radians, and the converse is also true.*



**Figure 2.** An  $\ell^1$  circle with two right-angled triangles.

Figure 1 gives a sketch of a proof of Proposition 3.

Proposition 3 implies that the vectors  $\mathbf{x}$  and  $\mathbf{y}$  form a right angle in the taxicab plane if and only if they are orthogonal in the Euclidean sense. The study of different notions of orthogonality in Minkowski spaces is an active area of research. Two important orthogonality types in Minkowski spaces are Birkhoff orthogonality ( $\mathbf{x} \perp \mathbf{y}$  if and only if  $\|\mathbf{x} - \alpha\mathbf{y}\| \geq \|\mathbf{x}\|$  for all  $\alpha$ ) and James (or isosceles) orthogonality ( $\mathbf{x} \perp \mathbf{y}$  if and only if  $\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$ ). In the taxicab plane, Birkhoff orthogonality is not symmetric and James orthogonality is not invariant under scalar multiplication, which implies neither notion is equivalent to the definition of right angle that we use above; see the recent survey [Alonso et al. 2012] for an explanation of these facts and extensive discussion of orthogonality in normed linear spaces.

Not all angles in the taxicab geometry behave as nicely as right angles. In Figure 2, the Euclidean angles  $\alpha$  and  $\beta$  of the two triangles depicted are not equal, but the taxicab angle measure of both is  $\frac{1}{2}$ .

A taxicab right triangle is in *standard position* if the base of the triangle is parallel to the  $x$ -axis (see  $\alpha$ -triangle in Figure 2). For triangles in standard position, we can define the taxicab sine and cosine functions as we do in Euclidean geometry with the  $\cos \theta$  and  $\sin \theta$  equal to the  $x$ - and  $y$ -coordinates on the unit circle. Indeed, the piecewise linear formulas for these functions are given in [Thompson and Dray 2000; Akça and Kaya 1997] and with slightly different formulas in [Brisbin and Artola 1985]. However, if we define sine and cosine as ratios of sides of right triangles, considering only triangles in standard position will not give all possible values. To illustrate this, we refer again to Figure 2.

Both triangles are right triangles with hypotenuse (the side opposite the 2-radian angle) of length 1. Also, since  $\alpha$  and  $\beta$  both have angle measure  $\frac{1}{2}$ , the other

nonright angle is  $4 - 2 - \frac{1}{2} = \frac{3}{2}$ . In the  $\alpha$ -triangle, we compute the cosine of  $\alpha$  by taking the ratio of the lengths of the adjacent side and the hypotenuse, which is  $\frac{3}{4}$ . However, looking at the  $\beta$ -triangle, we see the vertex of the right angle falls outside of the unit circle, which implies that the length of the side adjacent to  $\beta$ , and therefore the cosine of  $\beta$ , is *greater* than 1.

A natural question arises: what is the maximum value of the cosine of an angle in the taxicab plane? In this paper, we define and give explicit formulas for two-parameter sine and cosine functions, describing the possible side ratios of right triangles in the taxicab plane. Using these formulas, we show the maximum value to be  $1/2 + 1/\sqrt{2}$ , which is greater than 1. Thus we obtain a quantitative measure of a difference between the Euclidean and taxicab plane.

We would like to thank the referee for pointing out many references on the geometry of Minkowski metric spaces, including [Thompson 1996]. In Chapter 8 of this text, Thompson defines two-parameter sine and cosine functions for general Minkowski spaces. For Thompson's function, the Minkowski cosine of two vectors is zero if and only if the vectors  $x_1$  and  $x_2$  are Birkhoff orthogonal. This property does not hold for our definition of cosine, so our functions are not a special case of those defined by Thompson, even up to scale. Using the sine function, Thompson defines an  $\alpha$  which measures how far the Minkowski space is from Euclidean space, leaving us with a question: is this  $\alpha$  related to the value we obtain for the maximum of our taxicab sine function?

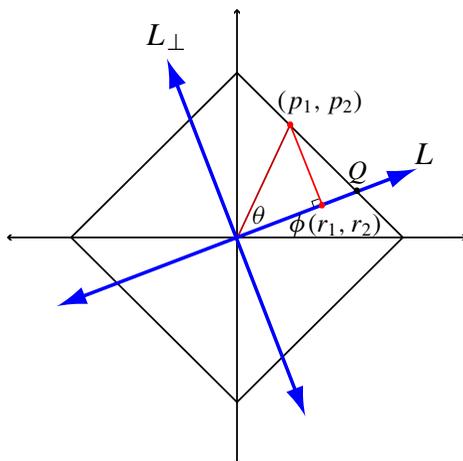
## 2. A two-parameter sine and cosine function

**Definition 4.** Given two metric spaces  $(X, d_1)$  and  $(Y, d_2)$ , a bijection  $f : X \rightarrow Y$  is an *isometry* if for any two points  $p, q \in X$ ,

$$d_1(p, q) = d_2(f(p), f(q)).$$

Given a metric space  $X$ , the set of all isometries  $\phi : X \rightarrow X$  forms a group, and the set of isometries that fix a point forms a subgroup of this group. An important subgroup is the set of isometries which fix the origin, which, by the Mazur–Ulam theorem (see [Thompson 1996, Chapter 3]), are linear. Using this fact and the fact that isometries map circles to circles with the same radius, one can see that the group of isometries that fix the origin of  $(\mathbb{R}^2, d_T)$  is the group of symmetries of a square, also called the dihedral group  $D_4$ . This includes the set of rotations (by  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$ , and  $270^\circ$ ) and reflections across the  $x$ -axis,  $y$ -axis and the lines passing through the origin with slope  $\pm 1$ . The full group of isometries is the semidirect product  $D_4 \rtimes \mathbb{R}^2$ , which is proved in [Schattschneider 1984]. This group is generated by translations and isometries that fix the origin.

Two triangles  $T_1, T_2$  in the taxicab plane are *congruent* if there is a taxicab isometry  $\phi$  such that  $\phi(T_1) = T_2$ . Note that due to the rigidity of the isometry group,



**Figure 3.** Defining sine and cosine.

there is no taxicab isometry taking the  $\alpha$ -triangle in Figure 2 to the  $\beta$ -triangle, so there is no angle-side-angle theorem in taxicab geometry. We will define the taxicab sine and cosine functions to have two angle parameters; one parameter is the usual  $\theta$ -angle parameter measured from a fixed axis, and the other  $\phi$ -parameter will denote the “direction” of the triangle in the plane (see Figure 3).

Before giving the definition, we describe a notion of orthogonal projection in the taxicab plane. Let  $L$  be a line and  $P$  be a point. If  $P$  is on  $L$ , the *orthogonal projection of  $P$  onto  $L$*  is  $P$ . If  $P$  is not on  $L$ , the orthogonal projection is a unique point  $R$  on  $L$  for which the line segment  $\overline{OPR}$  makes a Euclidean right angle with  $L$ ; Proposition 3 implies that this point  $R$  is also the unique point on  $L$  which makes a taxicab right angle. The following definition, which is convenient for later proofs, may seem somewhat unnatural; we refer the reader to Propositions 6 and 7 which justify that this definition gives the desired “signed ratio” of side lengths.

**Definition 5.** Let  $L$  be the line through the origin  $O$  which makes reference angle  $\phi$  with the  $x$ -axis, where  $0 \leq \phi < 2$ , and let  $P = (p_1, p_2)$  be a point on the unit circle so that  $\overline{OP}$  makes angle  $\theta$  with  $L$ . Let  $R = (r_1, r_2)$  be the orthogonal projection of  $P$  onto  $L$ . We define the taxicab cosine and sine of angle  $\theta$  at reference angle  $\phi$  as

$$\text{tcos}_\phi \theta = r_1 + r_2, \quad \text{tsin}_\phi \theta = (r_1 - p_1) + (p_2 - r_2).$$

Given a right triangle  $T$  with hypotenuse of length 1, there is a taxicab isometry which maps  $T$  to a triangle of the form  $\triangle PRO$  given in Definition 5, so  $T$  is congruent to  $\triangle PRO$ .

Let  $L_\perp$  be the perpendicular to  $L$  which also passes through the origin. The lines  $L$  and  $L_\perp$  divide the plane into four quadrants, which we number I, II, III, IV in the usual way.

**Proposition 6.** *The value of  $t\cos_\phi \theta$  is positive for  $\theta$  in  $L$ - $L_\perp$  quadrants I and IV, and negative for  $\theta$  in quadrants II and III. Similarly  $t\sin_\phi \theta$  is positive for  $\theta$  in quadrants I and II, and negative for  $\theta$  in quadrants III and IV.*

*Proof.* Let  $P = (p_1, p_2)$  and  $R = (r_1, r_2)$  be as given in [Definition 5](#). When  $\theta$  is in quadrants I and IV, as defined by  $L$  and  $L_\perp$ , the coordinate  $r_1$  is positive and  $r_2$  is nonnegative (when  $\phi = 0$ , the line  $L$  is the x-axis and  $r_2 = 0$ ). Therefore  $t\cos_\phi \theta$ , which is the sum of these coordinates, is positive. Similarly, when  $\theta$  is in quadrants II and III,  $r_1$  is negative and  $r_2$  is nonpositive; hence  $t\cos_\phi \theta$  is negative.

Recall that  $t\sin_\phi \theta = (r_1 - p_1) + (p_2 - r_2)$ . For a fixed  $\phi$ , the coordinates of  $P$  and  $R$  are continuous real-valued functions of  $\theta$ , and therefore the functions  $r_1 - p_1$  and  $p_2 - r_2$  are also continuous functions. When  $0 < \phi < 2$ , each of these functions is zero if and only if  $\theta = 4n$  for some integer  $n$ . This follows from the fact that the slope of  $L$  is positive, which implies that the line through  $P$  and  $R$  has negative slope; so  $p_1 = r_1$  or  $p_2 = r_2$  if and only if  $P = R$ . Therefore the sign of each of these functions,  $r_1 - p_1$  and  $p_2 - r_2$ , is constant for  $\theta$  in quadrants I and II. Picking a specific angle such as  $\theta = 2$  allows us to verify that both are positive, and therefore  $t\sin_\phi \theta$  is positive. Choosing an angle in the range  $4 < \theta < 8$  shows that both of these functions are negative, and therefore  $t\sin_\phi \theta$  is also negative when  $\theta$  is in quadrants III and IV.

When  $\phi = 0$ , we have that  $r_2 = 0$  and  $r_1 = p_1$ ; then  $t\sin_\phi \theta = p_2$ , and the result follows.  $\square$

**Proposition 7.** *In the right triangle made by  $P$ ,  $R$  and the origin  $O$ ,  $|t\cos_\phi \theta|$  gives the length of the side adjacent to  $\theta$ , and  $|t\sin_\phi \theta|$  gives the length of the opposite side.*

*Proof.* Fix an angle  $0 \leq \phi < 2$ . The length of the adjacent side is the distance from  $R$  to the origin, which is  $|r_1| + |r_2|$ . When  $\theta$  is in quadrants I and IV (defined by  $L$  and  $L_\perp$ ), both  $r_1$  and  $r_2$  are nonnegative, so

$$|r_1| + |r_2| = r_1 + r_2 = |t\cos_\phi \theta|.$$

When  $\theta$  lies in quadrants II and III, both  $r_1$  and  $r_2$  are nonpositive, so

$$|r_1| + |r_2| = -r_1 - r_2 = -(r_1 + r_2) = |t\cos_\phi \theta|.$$

The length of the side opposite of  $\theta$  in triangle  $OPR$  is given by the distance between  $P$  and  $R$ , which is  $|p_1 - r_1| + |p_2 - r_2|$ . Arguing as in [Proposition 6](#), when  $\theta$  is in quadrants I and II, we have

$$|p_1 - r_1| + |p_2 - r_2| = (r_1 - p_1) + (p_2 - r_2) = |t\sin_\phi \theta|,$$

and when  $\theta$  is in quadrants III and IV,

$$\begin{aligned} |p_1 - r_1| + |p_2 - r_2| &= -(r_1 - p_1) - (p_2 - r_2) \\ &= -((r_1 - p_1) + (p_2 - r_2)) = |\operatorname{tsin}_\phi \theta|. \quad \square \end{aligned}$$

**Proposition 8.** *The following identities hold.*

$$\operatorname{tsin}_\phi(\theta - 4) = -\operatorname{tsin}_\phi \theta \quad \text{and} \quad \operatorname{tcos}_\phi(\theta - 4) = -\operatorname{tcos}_\phi \theta.$$

*Proof.* Let  $P$  and  $R$  be the points given in [Definition 5](#) corresponding to  $\theta$ , and  $P'$  and  $R'$  the points corresponding to  $\theta - 4$ . By [Proposition 3](#), taxicab angles of measure 2 are Euclidean right angles, which means  $P$  and  $P'$  are antipodal points on the unit circle and  $P' = -P$ . The map  $(x, y) \rightarrow (-x, -y)$  is an isometry of the taxicab plane which maps  $P$  to  $P'$ . Angles are defined by the metric, and therefore isometries preserve angle measure. It follows from the definition of  $R$  that  $R' = -R$ . Therefore,

$$\operatorname{tcos}_\phi(\theta - 4) = -r_1 - r_2 = -(r_1 + r_2) = -\operatorname{tcos}_\phi \theta$$

and

$$\begin{aligned} \operatorname{tsin}_\phi(\theta - 4) &= (-r_1 + p_1) + (-p_2 + r_2) \\ &= -[(r_1 - p_1) + (p_2 - r_2)] = -\operatorname{tsin}_\phi \theta. \quad \square \end{aligned}$$

### 3. Explicit formulas for sine and cosine functions

**Theorem 9.** *Let  $\phi$  be a taxicab reference angle such that  $0 \leq \phi < 2$  and let  $\theta$  be a taxicab angle measured relative to  $\phi$ . Let*

$$\alpha = \frac{1}{\phi^2 - 2\phi + 2},$$

*which is well-defined for all  $\phi$  since  $\phi^2 - 2\phi + 2 > 0$ . The sine and cosine of  $\theta$  with reference angle  $\phi$  are given by*

$$\operatorname{tsin}_\phi \theta = \begin{cases} \alpha \theta & \text{if } -\phi \leq \theta \leq 2 - \phi, \\ 1 + \alpha(\theta - 2)(\phi - 1) & \text{if } 2 - \phi \leq \theta \leq 4 - \phi, \\ \alpha(4 - \theta) & \text{if } 4 - \phi \leq \theta \leq 6 - \phi, \\ -1 + \alpha(6 - \theta)(\phi - 1) & \text{if } 6 - \phi \leq \theta \leq 8 - \phi, \end{cases}$$

and

$$\operatorname{tcos}_\phi \theta = \begin{cases} 1 + \alpha \theta(\phi - 1) & \text{if } -\phi \leq \theta \leq 2 - \phi, \\ \alpha(2 - \theta) & \text{if } 2 - \phi \leq \theta \leq 4 - \phi, \\ -1 + \alpha(4 - \theta)(\phi - 1) & \text{if } 4 - \phi \leq \theta \leq 6 - \phi, \\ \alpha(\theta - 6) & \text{if } 6 - \phi \leq \theta \leq 8 - \phi. \end{cases}$$

**Lemma 10.** *Let  $L$  be a line through the origin that makes angle  $\phi$  with the  $x$ -axis, where  $0 \leq \phi < 2$ . The point of intersection between  $L$  and the unit taxicab circle is*

$$Q = \left( \frac{2-\phi}{2}, \frac{\phi}{2} \right).$$

*Proof.* Let  $Q = (q_1, q_2)$ . Since  $Q$  lies on the unit circle and  $0 \leq \phi < 2$ , both coordinates are positive and

$$q_1 + q_2 = 1. \tag{1}$$

Since the radius of the unit circle is 1, the definition of angle implies that  $\phi$  is the distance between  $Q$  and  $(1, 0)$ . This distance is given by

$$|q_1 - 1| + |q_2 - 0| = 1 - q_1 + q_2 = \phi. \tag{2}$$

We solve the system of linear equations consisting of (1) and (2) for  $q_2$  by adding the two equations to get

$$q_2 = \frac{\phi}{2};$$

substituting  $q_2$  into (1) gives us  $q_1 = 1 - \phi/2$ , which is the desired result.  $\square$

**3.1. Proof of Theorem 9 for  $-\phi \leq \theta \leq 2 - \phi$ .** Let  $0 \leq \phi < 2$  and  $-\phi \leq \theta \leq 2 - \phi$ . We will determine the coordinates of  $P$  and  $R$ , given in Definition 5, as functions of  $\phi$  and  $\theta$ . Lemma 10 implies that the  $\phi$ -axis (line  $L$  in Figure 3) intersects the circle at

$$Q = \left( \frac{2-\phi}{2}, \frac{\phi}{2} \right).$$

Since the  $\phi$ -axis passes through the origin, we find that the equation is

$$L(x) = \frac{\phi}{2-\phi} x. \tag{3}$$

Next, we determine the coordinates of  $P$ , the point of intersection between the circle and the  $(\theta + \phi)$ -ray. Applying Lemma 10 again with angle  $\theta + \phi$  gives coordinates

$$P = \left( \frac{2-\phi-\theta}{2}, \frac{\phi+\theta}{2} \right).$$

Proposition 3 implies that Euclidean right angles are taxicab right angles. Therefore, to find the point  $R$  we determine the equation of the line perpendicular (in the usual Euclidean sense) to the  $\phi$ -axis,  $L_P$ , through point  $P$ . Since the  $\phi$ -axis has slope  $\phi/(2-\phi)$ ,  $L_P$  has slope  $(\phi-2)/\phi$ . Since we know the coordinates of

$P = (p_1, p_2)$  and the slope, we can determine the equation for  $L_P$ , which is

$$\begin{aligned} L_P(x) &= \left( \frac{\phi - 2}{\phi} \right) (x - p_1) + p_2 \\ &= \frac{(\phi - 2)x}{\phi} + \frac{(\phi - 2)(\theta + \phi - 2) + \phi(\theta + \phi)}{2\phi}. \end{aligned} \quad (4)$$

The point  $R$  is the intersection between the  $\phi$ -axis and  $L_P$ . Setting equations (3) and (4) equal to each other and solving for the  $x$ -coordinate of  $R$  yields

$$r_1 = \frac{2 - \phi}{2} + \frac{(2 - \phi)(\phi\theta - \theta)}{2(\phi^2 - 2\phi + 2)}.$$

Plugging  $r_1$  into  $L$  (or  $L_P$ ) gives the  $y$ -coordinate of  $R$ ,

$$r_2 = \frac{\phi}{2} + \frac{\phi^2\theta - \phi\theta}{2(\phi^2 - 2\phi + 2)}.$$

Thus, the coordinates of  $R$  are

$$R = \left( \frac{2 - \phi}{2} + \frac{(2 - \phi)(\phi\theta - \theta)}{2(\phi^2 - 2\phi + 2)}, \frac{\phi}{2} + \frac{\phi^2\theta - \phi\theta}{2(\phi^2 - 2\phi + 2)} \right).$$

The result now follows by using the coordinates of  $R$  and  $P$  to compute  $\text{tsin}_\phi \theta$  and  $\text{tcos}_\phi \theta$  by the formulas given in [Definition 5](#).  $\square$

**3.2. Proof for  $2 - \phi \leq \theta \leq 4 - \phi$ .** We again find the coordinates of  $P$  and  $R$  to compute  $\text{tsin}_\phi \theta$  and  $\text{tcos}_\phi \theta$ . When  $2 < \theta + \phi < 4$ , the point  $P$  is in the second quadrant (as defined by the  $x$ - and  $y$ -axes). Let  $\theta_1$  be the portion of  $\theta$  measured from the  $y$ -axis, so  $\theta_1 = \phi + \theta - 2$ .

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the map defined by  $(x, y) \mapsto (y, -x)$ . This map is an order 4 isometry of the  $\ell^1$  metric. Note that  $f(0, 1) = (1, 0)$  and  $f(P)$  is in the first quadrant. Since angle measure is defined by the metric, angle measure is preserved by isometries. We can therefore apply [Lemma 10](#) to  $f(P)$  to obtain the coordinates

$$f(P) = \left( \frac{2 - \theta_1}{2}, \frac{\theta_1}{2} \right).$$

To obtain the coordinates for  $P$  we apply the inverse map:

$$P = f^{-1} \left( \frac{2 - \theta_1}{2}, \frac{\theta_1}{2} \right) = \left( -\frac{\theta_1}{2}, \frac{2 - \theta_1}{2} \right) = \left( \frac{2 - \phi - \theta}{2}, \frac{4 - \phi - \theta}{2} \right).$$

To finish the proof for this interval, we use the same procedure as in the proof for the first interval; that is, we find the equation of the line perpendicular to the  $\phi$ -axis through  $P$  to determine the coordinates of the point  $R$ . The line through  $P$

perpendicular to  $L(x)$  is

$$\begin{aligned} L_P(x) &= \left( \frac{\phi - 2}{\phi} \right) (x - p_1) + p_2 \\ &= \frac{(\phi - 2)x}{\phi} + \frac{(\phi - 2)(\theta + \phi - 2) + \phi(4 - \theta - \phi)}{2\phi}. \end{aligned} \quad (5)$$

To find  $r_1$ , we set equations (3) and (5) equal to one another and solve for  $x$ , which gives

$$r_1 = \frac{(\phi - 2)(\theta - 2)}{2(\phi^2 - 2\phi + 2)}.$$

Plugging  $r_1$  into  $L(x)$  (Equation (3)) gives

$$r_2 = \frac{-\phi(\theta - 2)}{2(\phi^2 - 2\phi + 2)}.$$

The sine and cosine functions can now be computed from the formulas given in Definition 5.  $\square$

**3.3. Proof for  $4 - \phi \leq \theta \leq 8 - \phi$ .** We will use the symmetry of the functions to establish the formulas for the third and fourth intervals. Let  $\theta$  be in the given interval, and  $\theta^* = \theta - 4$ . Then  $-\phi \leq \theta^* \leq 4 - \phi$ . We have determined formulas for  $\text{tsin}_\phi(\theta^*)$  and  $\text{tcos}_\phi(\theta^*)$  in this interval, so applying Proposition 8 gives formulas for angle  $\theta$  in the remaining two intervals.  $\square$

It should be noted that our formulas are a generalization of those formulas in [Thompson and Dray 2000; Akça and Kaya 1997]; if  $\phi = 0$ , then  $\theta$  is in standard position and we obtain identical formulas.

## 4. Properties of the functions

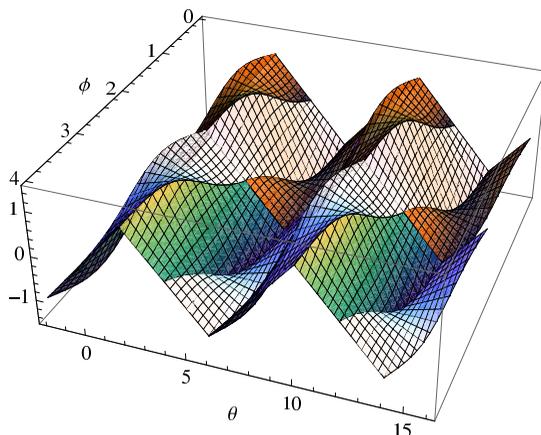
**4.1. Periodic extensions and graphs.** In Definition 5, the generalized sine and cosine functions were defined for all real numbers  $\theta$  and for values of  $\phi$  such that  $0 \leq \phi < 2$ . It is evident from the definition that the  $\theta$ -period of these functions is 8, so for any integer  $k$ ,

$$\text{tcos}_\phi(\theta + 8k) = \text{tcos}_\phi \theta \quad \text{and} \quad \text{tsin}_\phi(\theta + 8k) = \text{tsin}_\phi \theta.$$

There is a natural  $\phi$ -extension of these functions; since rotation by right angles gives isometries of the  $\ell^1$  metric, we extend the  $\phi$ -domain of the generalized sine and cosine functions to be  $\phi$ -periodic with period 2. Therefore, for any integer  $s$ ,

$$\text{tcos}_{\phi+2s} \theta = \text{tcos}_\phi \theta \quad \text{and} \quad \text{tsin}_{\phi+2s} \theta = \text{tsin}_\phi \theta.$$

It should be noted that the formulas for  $\text{tsin}_\phi \theta$  and  $\text{tcos}_\phi \theta$  given by  $P$  and  $R$  from Definition 5 are only valid for values of  $\phi$  in the first quadrant. Since Theorem 9



**Figure 4.** Graph of the generalized sine function.

gives explicit formulas for entire  $\phi$  and  $\theta$  periods, we may use this theorem and the two periodic properties stated above to give values for  $\text{tsin}_\phi \theta$  and  $\text{tcos}_\phi \theta$  for any  $(\phi, \theta) \in \mathbb{R} \times \mathbb{R}$ . Figure 4 contains a graph of  $\text{tsin}_\phi \theta$  for two periods of  $\phi$  and two periods of  $\theta$ .

Table 1 shows a family of cross-sections. Referring to the formulas in Theorem 9, we see that for a fixed  $\phi$  these functions are piecewise linear. We invite the interested reader to verify that these functions are constant when  $\theta = 2n$  for some integer  $n$ .

Recall that in the Euclidean metric,  $\sin(\theta + \pi/2) = \cos \theta$ . The cross-sections for the sine and cosine functions when  $\phi$  is fixed suggest a similar identity.

**Proposition 11.**  $\text{tsin}_\phi(\theta + 2) = \text{tcos}_\phi \theta$ .

*Proof.* While this identity follows from the symmetry of the space, Theorem 9 gives explicit formulas for  $\text{tsin}_\phi \theta$  and  $\text{tcos}_\phi \theta$ , so we need only check the formulas to verify this identity. Assume that  $0 \leq \phi < 2$  and  $-\phi \leq \theta \leq 2 - \phi$ , which implies  $2 - \phi \leq \theta + 2 \leq 4 - \phi$ . For angles in the interval  $[2 - \phi, 4 - \phi]$ ,

$$\text{tsin}_\phi \theta = 1 + \alpha(\theta - 2)(\phi - 1).$$

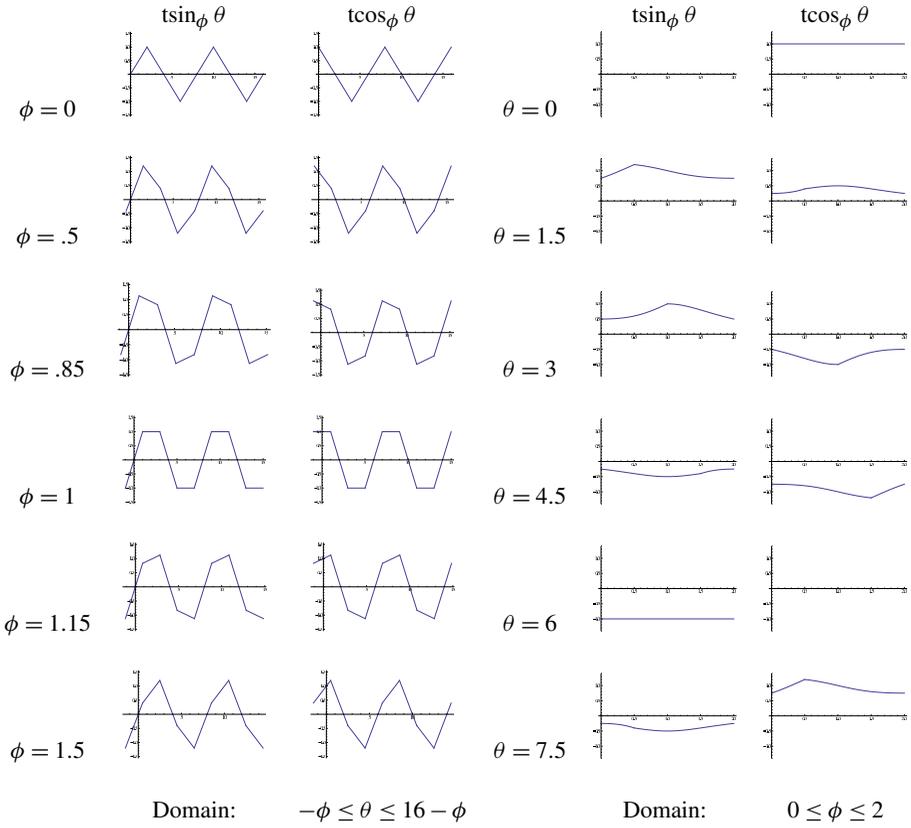
Therefore,

$$\text{tsin}_\phi(\theta + 2) = 1 + \alpha((\theta + 2) - 2)(\phi - 1) = 1 + \alpha \theta(\phi - 1),$$

which is equal to  $\text{tcos}_\phi \theta$  when  $-\phi \leq \theta \leq 2 - \phi$ . The other intervals can be verified similarly. □

#### 4.2. Maximum and minimum values.

**Theorem 12.** The maximum value of  $\text{tsin}_\phi \theta$  and  $\text{tcos}_\phi \theta$  is  $1/2 + 1/\sqrt{2}$ ; the minimum value is  $-(1/2 + 1/\sqrt{2})$ .



**Table 1.** Cross sections.

*Proof.* By Proposition 11, the maximum of the sine function is equal to the maximum of the cosine function. Also, by Proposition 8, the minimum of the sine function is equal to the negative of the maximum. Therefore it is sufficient to verify the maximum of the sine function.

The sine function has a  $\theta$ -period of 8 and a  $\phi$ -period of 2. However, the maximum of the sine function must occur when sine is positive, and hence  $\theta$  must be in the interval  $[0, 4]$  by Proposition 6. It is therefore sufficient to find the maximum of  $\text{tsin}_\phi \theta$  on the region defined by  $0 \leq \phi \leq 2$  and  $0 \leq \theta \leq 4$ . We will use standard techniques from multivariable calculus to maximize this function.

As  $\text{tsin}_\phi \theta$  is piecewise defined, we will consider the intervals

$$[0, 2 - \phi], \quad [2 - \phi, 4 - \phi], \quad \text{and} \quad [4 - \phi, 4].$$

Recall that

$$\alpha = \frac{1}{\phi^2 - 2\phi + 2} = \frac{1}{(\phi - 1)^2 + 1},$$

which is positive for all  $\phi$ . When  $\theta$  is in the interval  $[0, 2 - \phi]$ , we have  $\text{tsin}_\phi \theta = \alpha\theta$ , and  $\theta$  in  $[4 - \phi, 4]$  implies  $\text{tsin}_\phi \theta = \alpha(4 - \theta)$ . The partial derivatives with respect to  $\theta$  of these functions are  $\alpha$  and  $-\alpha$ ; therefore,  $\text{tsin}_\phi \theta$  is increasing with respect to  $\theta$  on  $[0, 2 - \phi]$  and decreasing in  $\theta$  on  $[4 - \phi, 4]$ . This implies the absolute maximum of  $\text{tsin}_\phi \theta$  occurs when  $\theta$  is in the middle interval.

When  $2 - \phi \leq \theta \leq 4 - \phi$ ,

$$\text{tsin}_\phi \theta = 1 + \frac{(\theta - 2)(\phi - 1)}{\phi^2 - 2\phi + 2}.$$

The partial derivatives are

$$\begin{aligned} \frac{\partial}{\partial \phi} \left[ 1 + \frac{(\theta - 2)(\phi - 1)}{\phi^2 - 2\phi + 2} \right] &= \frac{(2\phi - \phi^2)(\theta - 2)}{(\phi^2 - 2\phi + 2)^2}, \\ \frac{\partial}{\partial \theta} \left[ 1 + \frac{(\theta - 2)(\phi - 1)}{\phi^2 - 2\phi + 2} \right] &= \frac{\phi - 1}{\phi^2 - 2\phi + 2}. \end{aligned}$$

These are both zero only when  $(\phi, \theta) = (1, 2)$ . In this case,  $\text{tsin}_1(2) = 1$ . We now check the boundary conditions.

When  $\phi = 0$ , we have  $2 \leq \theta \leq 4$  and  $\text{tsin}_\phi \theta = 2 + (-\theta/2)$ , which has a maximum of 1. Note that  $\text{tsin}_\phi \theta$  has the same maximum when  $\phi = 2$  because of the  $\phi$ -periodic property previously stated.

When  $\theta = 2 - \phi$ , we have

$$g(\phi) = \text{tsin}_\phi(2 - \phi) = 1 - \frac{(\phi - 1)\phi}{\phi^2 - 2\phi + 2}.$$

The derivative of this function is

$$g'(\phi) = \frac{\phi^2 - 4\phi + 2}{(\phi^2 - 2\phi + 2)^2}.$$

This function is zero when  $\phi = 2 \pm \sqrt{2}$ . Only one of these values,  $\phi = 2 - \sqrt{2}$ , is in the region under consideration. For this value of  $\phi$ , we have  $\theta = \sqrt{2}$  and we see the value of the sine function is

$$\text{tsin}_{2-\sqrt{2}} \sqrt{2} = 1/2 + 1/\sqrt{2}.$$

When  $\theta = 4 - \phi$ , we have

$$h(\phi) = \text{tsin}_\phi(4 - \phi) = 1 - \frac{(\phi - 2)(\phi - 1)}{\phi^2 - 2\phi + 2}.$$

The derivative of this function is

$$h'(\phi) = \frac{2 - \phi^2}{(\phi^2 - 2\phi + 2)^2}.$$

For values of  $\phi$  in the interval  $[0, 2]$ , this derivative is zero when  $\phi = \sqrt{2}$ . Then  $\theta = 4 - \sqrt{2}$ , and

$$\text{tsin}_{\sqrt{2}}(4 - \sqrt{2}) = \frac{1}{2} + \frac{1}{\sqrt{2}}.$$

We can therefore conclude for values in the region  $0 \leq \phi \leq 2$  and  $0 \leq \theta \leq 4$ , the function  $\text{tsin}_{\phi} \theta$  achieves its absolute maximum,  $1/2 + 1/\sqrt{2}$ , in two locations:  $(2 - \sqrt{2}, \sqrt{2})$  and  $(\sqrt{2}, 4 - \sqrt{2})$ .  $\square$

**Corollary 13.** *The hypotenuse of a right triangle in taxicab space is not always the longest side of the triangle.*

## References

- [Akça and Kaya 1997] Z. Akça and R. Kaya, “On the taxicab trigonometry”, *J. Inst. Math. Comput. Sci. Math. Ser.* **10**:3 (1997), 151–159. [MR 99c:51022](#) [Zbl 0926.51026](#)
- [Alonso et al. 2012] J. Alonso, H. Martini, and S. Wu, “On Birkhoff orthogonality and isosceles orthogonality in normed linear spaces”, *Aequationes Math.* **83**:1-2 (2012), 153–189. [MR 2012m:46001](#) [Zbl 1241.46006](#)
- [Álvarez Paiva and Thompson 2004] J. C. Álvarez Paiva and A. C. Thompson, “Volumes on normed and Finsler spaces”, pp. 1–48 in *A sampler of Riemann–Finsler geometry*, Math. Sci. Res. Inst. Publ. **50**, Cambridge Univ. Press, 2004. [MR 2006c:53079](#) [Zbl 1078.53072](#)
- [Álvarez Paiva and Thompson 2005] J. C. Álvarez Paiva and A. Thompson, “On the perimeter and area of the unit disc”, *Amer. Math. Monthly* **112**:2 (2005), 141–154. [MR 2005i:51012](#) [Zbl 1084.52007](#)
- [Brisbin and Artola 1985] R. Brisbin and P. Artola, “Taxicab trigonometry”, *Pi Mu Epsilon Journal* **8**:2 (1985), 89–95.
- [Düvelmeyer 2005] N. Düvelmeyer, “Angle measures and bisectors in Minkowski planes”, *Canad. Math. Bull.* **48**:4 (2005), 523–534. [MR 2006g:52008](#) [Zbl 1093.52002](#)
- [Schattschneider 1984] D. J. Schattschneider, “The taxicab group”, *Amer. Math. Monthly* **91**:7 (1984), 423–428. [MR 86b:51027](#) [Zbl 0564.51005](#)
- [Thompson 1996] A. C. Thompson, *Minkowski geometry*, Encyclopedia of Mathematics and its Applications **63**, Cambridge University Press, Cambridge, 1996. [MR 97f:52001](#) [Zbl 0868.52001](#)
- [Thompson and Dray 2000] K. Thompson and T. Dray, “Taxicab angles and trigonometry”, *Pi Mu Epsilon Journal* **11**:2 (2000), 87–96.

Received: 2013-06-24

Revised: 2013-11-06

Accepted: 2013-11-07

[kelly.delp@gmail.com](mailto:kelly.delp@gmail.com)

*Department of Mathematics, Ithaca College,  
201 Muller Center, Ithaca, NY 14850, United States*

[mgfilips@buffalo.edu](mailto:mgfilips@buffalo.edu)

*Department of Mathematics, University at Buffalo,  
244 Mathematics Building, Buffalo, NY 14260, United States*

## EDITORS

### MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, [berenhks@wfu.edu](mailto:berenhks@wfu.edu)

### BOARD OF EDITORS

|                      |   |                        |  |
|----------------------|---|------------------------|--|
| Colin Adams          | Williams College, USA<br><a href="mailto:colin.c.adams@williams.edu">colin.c.adams@williams.edu</a>               | David Larson           | Texas A&M University, USA<br><a href="mailto:larson@math.tamu.edu">larson@math.tamu.edu</a>                          |
| John V. Baxley       | Wake Forest University, NC, USA<br><a href="mailto:baxley@wfu.edu">baxley@wfu.edu</a>                             | Suzanne Lenhart        | University of Tennessee, USA<br><a href="mailto:lenhart@math.utk.edu">lenhart@math.utk.edu</a>                       |
| Arthur T. Benjamin   | Harvey Mudd College, USA<br><a href="mailto:benjamin@hmc.edu">benjamin@hmc.edu</a>                                | Chi-Kwong Li           | College of William and Mary, USA<br><a href="mailto:ckli@math.wm.edu">ckli@math.wm.edu</a>                           |
| Martin Bohner        | Missouri U of Science and Technology, USA<br><a href="mailto:bohner@mst.edu">bohner@mst.edu</a>                   | Robert B. Lund         | Clemson University, USA<br><a href="mailto:lund@clemson.edu">lund@clemson.edu</a>                                    |
| Nigel Boston         | University of Wisconsin, USA<br><a href="mailto:boston@math.wisc.edu">boston@math.wisc.edu</a>                    | Gaven J. Martin        | Massey University, New Zealand<br><a href="mailto:g.j.martin@massey.ac.nz">g.j.martin@massey.ac.nz</a>               |
| Amarjit S. Budhiraja | U of North Carolina, Chapel Hill, USA<br><a href="mailto:budhiraj@email.unc.edu">budhiraj@email.unc.edu</a>       | Mary Meyer             | Colorado State University, USA<br><a href="mailto:meyer@stat.colostate.edu">meyer@stat.colostate.edu</a>             |
| Pietro Cerone        | La Trobe University, Australia<br><a href="mailto:P.Cerone@latrobe.edu.au">P.Cerone@latrobe.edu.au</a>            | Emil Minchev           | Ruse, Bulgaria<br><a href="mailto:eminchev@hotmail.com">eminchev@hotmail.com</a>                                     |
| Scott Chapman        | Sam Houston State University, USA<br><a href="mailto:scott.chapman@shsu.edu">scott.chapman@shsu.edu</a>           | Frank Morgan           | Williams College, USA<br><a href="mailto:frank.morgan@williams.edu">frank.morgan@williams.edu</a>                    |
| Joshua N. Cooper     | University of South Carolina, USA<br><a href="mailto:cooper@math.sc.edu">cooper@math.sc.edu</a>                   | Mohammad Sal Moslehian | Ferdowsi University of Mashhad, Iran<br><a href="mailto:moslehian@ferdowsi.um.ac.ir">moslehian@ferdowsi.um.ac.ir</a> |
| Jem N. Corcoran      | University of Colorado, USA<br><a href="mailto:corcoran@colorado.edu">corcoran@colorado.edu</a>                   | Zuhair Nashed          | University of Central Florida, USA<br><a href="mailto:znashed@mail.ucf.edu">znashed@mail.ucf.edu</a>                 |
| Toka Diagana         | Howard University, USA<br><a href="mailto:tdiagana@howard.edu">tdiagana@howard.edu</a>                            | Ken Ono                | Emory University, USA<br><a href="mailto:ono@mathcs.emory.edu">ono@mathcs.emory.edu</a>                              |
| Michael Dorff        | Brigham Young University, USA<br><a href="mailto:mdorff@math.byu.edu">mdorff@math.byu.edu</a>                     | Timothy E. O'Brien     | Loyola University Chicago, USA<br><a href="mailto:tbriell@luc.edu">tbriell@luc.edu</a>                               |
| Sever S. Dragomir    | Victoria University, Australia<br><a href="mailto:sever@matilda.vu.edu.au">sever@matilda.vu.edu.au</a>            | Joseph O'Rourke        | Smith College, USA<br><a href="mailto:orourke@cs.smith.edu">orourke@cs.smith.edu</a>                                 |
| Behrouz Emamizadeh   | The Petroleum Institute, UAE<br><a href="mailto:bemamizadeh@pi.ac.ae">bemamizadeh@pi.ac.ae</a>                    | Yuval Peres            | Microsoft Research, USA<br><a href="mailto:peres@microsoft.com">peres@microsoft.com</a>                              |
| Joel Foisy           | SUNY Potsdam<br><a href="mailto:foisyjs@potsdam.edu">foisyjs@potsdam.edu</a>                                      | Y.-F. S. Pétermann     | Université de Genève, Switzerland<br><a href="mailto:petermann@math.unige.ch">petermann@math.unige.ch</a>            |
| Errin W. Fulp        | Wake Forest University, USA<br><a href="mailto:fulp@wfu.edu">fulp@wfu.edu</a>                                     | Robert J. Plemmons     | Wake Forest University, USA<br><a href="mailto:rplemmons@wfu.edu">rplemmons@wfu.edu</a>                              |
| Joseph Gallian       | University of Minnesota Duluth, USA<br><a href="mailto:kgallian@d.umn.edu">kgallian@d.umn.edu</a>                 | Carl B. Pomerance      | Dartmouth College, USA<br><a href="mailto:carl.pomerance@dartmouth.edu">carl.pomerance@dartmouth.edu</a>             |
| Stephan R. Garcia    | Pomona College, USA<br><a href="mailto:stephan.garcia@pomona.edu">stephan.garcia@pomona.edu</a>                   | Vadim Ponomarenko      | San Diego State University, USA<br><a href="mailto:vadim@sciences.sdsu.edu">vadim@sciences.sdsu.edu</a>              |
| Anant Godbole        | East Tennessee State University, USA<br><a href="mailto:godbole@etsu.edu">godbole@etsu.edu</a>                    | Bjorn Poonen           | UC Berkeley, USA<br><a href="mailto:poonen@math.berkeley.edu">poonen@math.berkeley.edu</a>                           |
| Ron Gould            | Emory University, USA<br><a href="mailto:rg@mathcs.emory.edu">rg@mathcs.emory.edu</a>                             | James Propp            | U Mass Lowell, USA<br><a href="mailto:jpropp@cs.uml.edu">jpropp@cs.uml.edu</a>                                       |
| Andrew Granville     | Université Montréal, Canada<br><a href="mailto:andrew.andrew@dms.umontreal.ca">andrew.andrew@dms.umontreal.ca</a> | József H. Przytycki    | George Washington University, USA<br><a href="mailto:przytyck@gwu.edu">przytyck@gwu.edu</a>                          |
| Jerrold Griggs       | University of South Carolina, USA<br><a href="mailto:griggs@math.sc.edu">griggs@math.sc.edu</a>                   | Richard Rebarber       | University of Nebraska, USA<br><a href="mailto:rrebarbe@math.unl.edu">rrebarbe@math.unl.edu</a>                      |
| Sat Gupta            | U of North Carolina, Greensboro, USA<br><a href="mailto:sgupta@uncg.edu">sgupta@uncg.edu</a>                      | Robert W. Robinson     | University of Georgia, USA<br><a href="mailto:rwr@cs.uga.edu">rwr@cs.uga.edu</a>                                     |
| Jim Haglund          | University of Pennsylvania, USA<br><a href="mailto:jhaglund@math.upenn.edu">jhaglund@math.upenn.edu</a>           | Filip Saidak           | U of North Carolina, Greensboro, USA<br><a href="mailto:f_saidak@uncg.edu">f_saidak@uncg.edu</a>                     |
| Johnny Henderson     | Baylor University, USA<br><a href="mailto:johnny_henderson@baylor.edu">johnny_henderson@baylor.edu</a>            | James A. Sellers       | Penn State University, USA<br><a href="mailto:sellersj@math.psu.edu">sellersj@math.psu.edu</a>                       |
| Jim Hoste            | Pitzer College<br><a href="mailto:jhoste@pitzer.edu">jhoste@pitzer.edu</a>  | Andrew J. Sterge       | Honorary Editor<br><a href="mailto:andy@ajsterge.com">andy@ajsterge.com</a>  |
| Natalia Hritonenko   | Prairie View A&M University, USA<br><a href="mailto:nhritonenko@pvamu.edu">nhritonenko@pvamu.edu</a>              | Ann Trenk              | Wellesley College, USA<br><a href="mailto:atrenk@wellesley.edu">atrenk@wellesley.edu</a>                             |
| Glenn H. Hurlbert    | Arizona State University, USA<br><a href="mailto:hurlbert@asu.edu">hurlbert@asu.edu</a>                           | Ravi Vakil             | Stanford University, USA<br><a href="mailto:vakil@math.stanford.edu">vakil@math.stanford.edu</a>                     |
| Charles R. Johnson   | College of William and Mary, USA<br><a href="mailto:crjohnso@math.wm.edu">crjohnso@math.wm.edu</a>                | Antonia Vecchio        | Consiglio Nazionale delle Ricerche, Italy<br><a href="mailto:antonia.vecchio@cnr.it">antonia.vecchio@cnr.it</a>      |
| K. B. Kulasekera     | Clemson University, USA<br><a href="mailto:kk@ces.clemson.edu">kk@ces.clemson.edu</a>                             | Ram U. Verma           | University of Toledo, USA<br><a href="mailto:verma99@msn.com">verma99@msn.com</a>                                    |
| Gerry Ladas          | University of Rhode Island, USA<br><a href="mailto:gladas@math.uri.edu">gladas@math.uri.edu</a>                   | John C. Wierman        | Johns Hopkins University, USA<br><a href="mailto:wierman@jhu.edu">wierman@jhu.edu</a>                                |
|                      |   | Michael E. Zieve       | University of Michigan, USA<br><a href="mailto:zieve@umich.edu">zieve@umich.edu</a>                                  |

## PRODUCTION

Silvio Levy, Scientific Editor

See inside back cover or [msp.org/involve](http://msp.org/involve) for submission instructions. The subscription price for 2015 is US \$140/year for the electronic version, and \$190/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2015 Mathematical Sciences Publishers

# involve

2015

vol. 8

no. 3

|   |     |
|---|-----|
| Colorability and determinants of $T(m, n, r, s)$ twisted torus knots for $n \equiv \pm 1 \pmod{m}$                    | 361 |
| MATT DELONG, MATTHEW RUSSELL AND JONATHAN SCHROCK   |     |
| Parameter identification and sensitivity analysis to a thermal diffusivity inverse problem                            | 385 |
| BRIAN LEVENTHAL, XIAOJING FU, KATHLEEN FOWLER AND OWEN ESLINGER   |     |
| A mathematical model for the emergence of HIV drug resistance during periodic bang-bang type antiretroviral treatment | 401 |
| NICOLETA TARFULEA AND PAUL READ   |     |
| An extension of Young's segregation game  | 421 |
| MICHAEL BORCHERT, MARK BUREK, RICK GILLMAN AND SPENCER ROACH  |     |
| Embedding groups into distributive subsets of the monoid of binary operations   | 433 |
| GREGORY MEZERA  |     |
| Persistence: a digit problem  | 439 |
| STEPHANIE PEREZ AND ROBERT STYER  |     |
| A new partial ordering of knots   | 447 |
| ARAZELLE MENDOZA, TARA SARGENT, JOHN TRAVIS SHRONTZ AND PAUL DRUBE  |     |
| Two-parameter taxicab trigonometric functions   | 467 |
| KELLY DELP AND MICHAEL FILIPSKI   |     |
| ${}_3F_2$ -hypergeometric functions and supersingular elliptic curves   | 481 |
| SARAH PITMAN  |     |
| A contribution to the connections between Fibonacci numbers and matrix theory   | 491 |
| MIRIAM FARBER AND ABRAHAM BERMAN  |     |
| Stick numbers in the simple hexagonal lattice   | 503 |
| RYAN BAILEY, HANS CHAUMONT, MELANIE DENNIS, JENNIFER MCLLOUD-MANN, ELISE MCMAHON, SARA MELVIN AND GEOFFREY SCHUETTE   |     |
| On the number of pairwise touching simplices  | 513 |
| BAS LEMMENS AND CHRISTOPHER PARSONS   |     |
| The zipper foldings of the diamond  | 521 |
| ERIN W. CHAMBERS, DI FANG, KYLE A. SYKES, CYNTHIA M. TRAUB AND PHILIP TRETTENERO                                      |     |
| On distance labelings of amalgamations and injective labelings of general graphs                                      | 535 |
| NATHANIEL KARST, JESSICA OEHRLEIN, DENISE SAKAI TROXELL AND JUNJIE ZHU  |     |