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Two-parameter taxicab trigonometric functions

Kelly Delp and Michael Filipski

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In this paper, we review some of the fundamental properties of the ℓ^1, or taxicab, metric on $\mathbb{R}^2$. We define and give explicit formulas for two-parameter sine and cosine functions for this metric space. We also determine the maximum of these functions, which is greater than 1.

1. Introduction

The ℓ^1 metric on $\mathbb{R}^2$, the so-called taxicab metric, is often one of the first non-Euclidean metrics a mathematics student encounters. For any points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $\mathbb{R}^2$, the metric is given by the formula

$$d_T(p, q) = |p_1 - q_1| + |p_2 - q_2|.$$

The ℓ^1 metric is just one metric in a class of metrics defined on $\mathbb{R}^2$ known as *Minkowski metrics*; see [Álvarez Paiva and Thompson 2005] for an introduction to these metric spaces. Let Ω be a closed, bounded convex set in $\mathbb{R}^2$ which contains and is symmetric about the origin. The set Ω defines a norm on $\mathbb{R}^2$, where Ω is the unit disk. Given a norm $\|\cdot\|$, one can define a metric on $\mathbb{R}^2$ by $d(p, q) = \|p - q\|$. Examples of Minkowski metrics include the ℓ^p metrics, the ℓ^∞ or max metric, and metrics with a unit disk that is a regular $2n$-gon.

Length minimizing paths in the taxicab plane are not necessarily unique, so we use the vector space properties of $\mathbb{R}^2$ and define *lines* to be the sets of points of the form $L = \{t\mathbf{v} + \mathbf{b} \mid t \in \mathbb{R}\}$ for some fixed $\mathbf{v}$ and $\mathbf{b}$. We can similarly define *line segments, triangles, rays*, and *angles* (pairs of rays sharing an initial point). We define the length of a line segment $\overline{AB}$ to be the distance between the endpoints, $d_T(A, B)$.

Given a metric d on a set X, a circle C of radius r is the set of all points $p \in X$ equidistant from a given point called the center. A circle in the taxicab metric is a square with diagonals parallel to the x- and y-axes. In Euclidean space there is an intrinsic notion of angle measure, radian measure, which is determined by the

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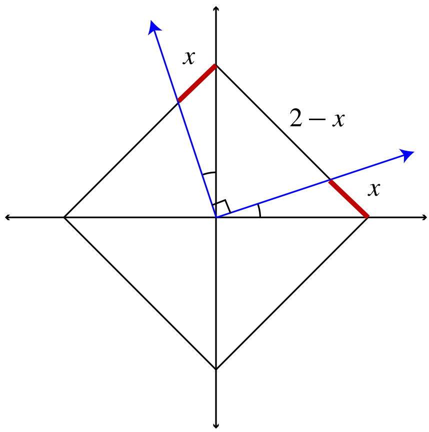

Figure 1. Euclidean right angles have taxicab angle measure of 2.

length of a particular circle arc. We can similarly define an intrinsic angle measure in the taxicab plane, called *t-radians*.

Definition 1. Let C be a circle with radius r and center P. Given an angle with vertex P, let s be the length of the subtended arc. The t-radian measure, θ, of a taxicab angle is given by

$$\theta = \frac{s}{r}.$$

It is this notion of angle measure which was used in these previous works [Akça and Kaya 1997; Brisbin and Artola 1985; Thompson and Dray 2000] on taxicab trigonometry. Another well-studied angle measure in a Minkowski metric uses the *area* of the sector of the circle, rather than arc length, to define the angle measure. (Due to a theorem of Haar, any area measure μ is proportional to Lebesgue measure; see [Álvarez Paiva and Thompson 2004] for a discussion of areas in normed spaces.) By Theorem 1 in [Düvelmeyer 2005], these two notions are equivalent (up to scale) because the taxicab circle is an example of an equiframed curve. See [Düvelmeyer 2005] for the definition of equiframed curve.

Note that an ℓ^1 circle has 8 t-radians, which means in this metric, 4 is the analogue of π. Some of the properties from Euclidean geometry have analogous statements which are true in the taxicab plane. We will use the following propositions.

Proposition 2 [Thompson and Dray 2000, Theorem 4.2]. *The angle sum of a taxicab triangle is* 4 *t-radians.*

We define a taxicab right angle to be an angle with measure 2 t-radians, which, as in Euclidean geometry, is an angle which has measure equal to its supplement.

Proposition 3 [Thompson and Dray 2000, Lemma 2.5]. *A Euclidean right angle has taxicab angle measure of* 2 *t-radians, and the converse is also true.*

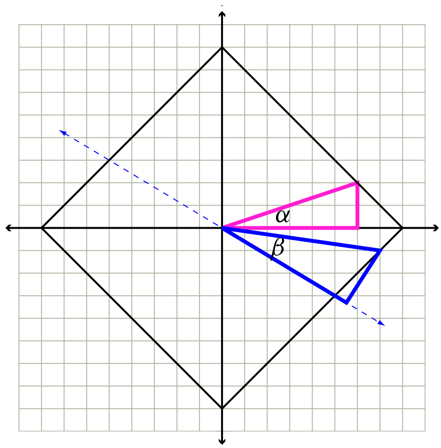

Figure 2. An ℓ^1 circle with two right-angled triangles.

Figure 1 gives a sketch of a proof of Proposition 3.

Proposition 3 implies that the vectors $\mathbf{x}$ and $\mathbf{y}$ form a right angle in the taxicab plane if and only if they are orthogonal in the Euclidean sense. The study of different notions of orthogonality in Minkowski spaces is an active area of research. Two important orthogonality types in Minkowski spaces are Birkhoff orthogonality, ($\mathbf{x} \perp \mathbf{y}$ if and only if $\|\mathbf{x}-\alpha\mathbf{y}\| \geq \|\mathbf{x}\|$ for all α) and James (or isosceles) orthogonality ($\mathbf{x} \perp \mathbf{y}$ if and only if $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}-\mathbf{y}\|$). In the taxicab plane, Birkhoff orthogonality is not symmetric and James orthogonality is not invariant under scalar multiplication, which implies neither notion is equivalent to the definition of right angle that we use above; see the recent survey [Alonso et al. 2012] for an explanation of these facts and extensive discussion of orthogonality in normed linear spaces.

Not all angles in the taxicab geometry behave as nicely as right angles. In Figure 2, the Euclidean angles α and β of the two triangles depicted are not equal, but the taxicab angle measure of both is $\frac{1}{2}$.

A taxicab right triangle is in *standard position* if the base of the triangle is parallel to the x-axis (see α-triangle in Figure 2). For triangles in standard position, we can define the taxicab sine and cosine functions as we do in Euclidean geometry with the $\cos\theta$ and $\sin\theta$ equal to the x- and y-coordinates on the unit circle. Indeed, the piecewise linear formulas for these functions are given in [Thompson and Dray 2000; Akça and Kaya 1997] and with slightly different formulas in [Brisbin and Artola 1985]. However, if we define sine and cosine as ratios of sides of right triangles, considering only triangles in standard position will not give all possible values. To illustrate this, we refer again to Figure 2.

Both triangles are right triangles with hypotenuse (the side opposite the 2 t-radian angle) of length 1. Also, since α and β both have angle measure $\frac{1}{2}$, the other nonright angle is $4-2-\frac{1}{2}=\frac{3}{2}$. In the α-triangle, we compute the cosine of α by taking the ratio of the lengths of the adjacent side and the hypotenuse, which is $\frac{3}{4}$. However, looking at the β-triangle, we see the vertex of the right angle falls outside of the unit circle, which implies that the length of the side adjacent to β, and therefore the cosine of β, is *greater* than 1.

A natural question arises: what is the maximum value of the cosine of an angle in the taxicab plane? In this paper, we define and give explicit formulas for two-parameter sine and cosine functions, describing the possible side ratios of right triangles in the taxicab plane. Using these formulas, we show the maximum value to be $1/2+1/\sqrt{2}$, which is greater than 1. Thus we obtain a quantitative measure of a difference between the Euclidean and taxicab plane.

We would like to thank the referee for pointing out many references on the geometry of Minkowski metric spaces, including [Thompson 1996]. In Chapter 8 of this text, Thompson defines two-parameter sine and cosine functions for general Minkowski spaces. For Thompson's function, the Minkowski cosine of two vectors is zero if and only if the vectors x_1 and x_2 are Birkhoff orthogonal. This property does not hold for our definition of cosine, so our functions are not a special case of those defined by Thompson, even up to scale. Using the sine function, Thompson defines an α which measures how far the Minkowski space is from Euclidean space, leaving us with a question: is this α related to the value we obtain for the maximum of our taxicab sine function?

2. A two-parameter sine and cosine function

Definition 4. Given two metric spaces (X, d_1) and (Y, d_2), a bijection $f : X \to Y$ is an *isometry* if for any two points $p, q \in X$,

$$d_1(p,q) = d_2(f(p), f(q)).$$

Given a metric space X, the set of all isometries $\phi : X \to X$ forms a group, and the set of isometries that fix a point forms a subgroup of this group. An important subgroup is the set of isometries which fix the origin, which, by the Mazur–Ulam theorem (see [Thompson 1996, Chapter 3]), are linear. Using this fact and the fact that isometries map circles to circles with the same radius, one can see that the group of isometries that fix the origin of $(\mathbb{R}^2, d_T)$ is the group of symmetries of a square, also called the dihedral group D_4. This includes the set of rotations (by 0°, 90°, 180°, and 270°) and reflections across the x-axis, y-axis and the lines passing through the origin with slope ± 1. The full group of isometries is the semidirect product $D_4 \rtimes \mathbb{R}^2$, which is proved in [Schattschneider 1984]. This group is generated by translations and isometries that fix the origin.

Two triangles T_1, T_2 in the taxicab plane are *congruent* if there is a taxicab isometry ϕ such that $\phi(T_1) = T_2$. Note that due to the rigidity of the isometry group,

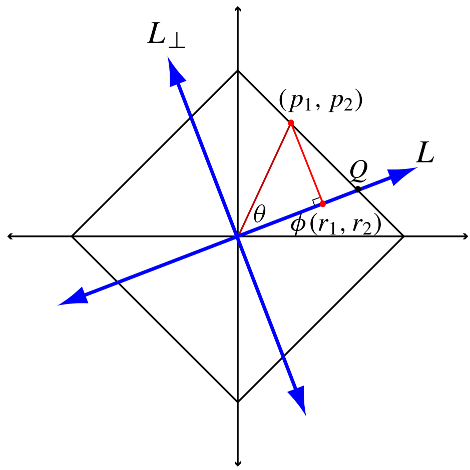

Figure 3. Defining sine and cosine.

there is no taxicab isometry taking the α-triangle in Figure 2 to the β-triangle, so there is no angle-side-angle theorem in taxicab geometry. We will define the taxicab sine and cosine functions to have two angle parameters; one parameter is the usual θ-angle parameter measured from a fixed axis, and the other ϕ-parameter will denote the "direction" of the triangle in the plane (see Figure 3).

Before giving the definition, we describe a notion of orthogonal projection in the taxicab plane. Let L be a line and P be a point. If P is on L, the *orthogonal projection of P onto L* is P. If P is not on L, the orthogonal projection is a unique point R on L for which the line segment $\overline{OPR}$ makes a Euclidean right angle with L; Proposition 3 implies that this point R is also the unique point on L which makes a taxicab right angle. The following definition, which is convenient for later proofs, may seem somewhat unnatural; we refer the reader to Propositions 6 and 7 which justify that this definition gives the desired "signed ratio" of side lengths.

Definition 5. Let L be the line through the origin O which makes reference angle ϕ with the x-axis, where $0 \le \phi < 2$, and let $P = (p_1, p_2)$ be a point on the unit circle so that $\overline{OP}$ makes angle θ with L. Let $R = (r_1, r_2)$ be the orthogonal projection of P onto L. We define the taxicab cosine and sine of angle θ at reference angle ϕ as

$$\mathrm{tcos}_\phi\, \theta = r_1 + r_2, \quad \mathrm{tsin}_\phi\, \theta = (r_1 - p_1) + (p_2 - r_2).$$

Given a right triangle T with hypotenuse of length 1, there is a taxicab isometry which maps T to a triangle of the form $\triangle\, PRO$ given in Definition 5, so T is congruent to $\triangle\, PRO$.

Let $L_\perp$ be the perpendicular to L which also passes through the origin. The lines L and $L_\perp$ divide the plane into four quadrants, which we number I, II, III, IV in the usual way.

Proposition 6. *The value of* $\operatorname{tcos}_\phi \theta$ *is positive for* θ *in* L-$L_\perp$ *quadrants I and IV, and negative for* θ *in quadrants II and III. Similarly* $\operatorname{tsin}_\phi \theta$ *is positive for* θ *in quadrants I and II, and negative for* θ *in quadrants III and IV.*

Proof. Let $P = (p_1, p_2)$ and $R = (r_1, r_2)$ be as given in Definition 5. When θ is in quadrants I and IV, as defined by L and $L_\perp$, the coordinate r_1 is positive and r_2 is nonnegative (when $\phi = 0$, the line L is the x-axis and $r_2 = 0$). Therefore $\operatorname{tcos}_\phi \theta$, which is the sum of these coordinates, is positive. Similarly, when θ is in quadrants II and III, r_1 is negative and r_2 is nonpositive; hence $\operatorname{tcos}_\phi \theta$ is negative.

Recall that $\operatorname{tsin}_\phi \theta = (r_1 - p_1) + (p_2 - r_2)$. For a fixed ϕ, the coordinates of P and R are continuous real-valued functions of θ, and therefore the functions $r_1 - p_1$ and $p_2 - r_2$ are also continuous functions. When $0 < \phi < 2$, each of these functions is zero if and only if $\theta = 4n$ for some integer n. This follows from the fact that the slope of L is positive, which implies that the line through P and R has negative slope; so $p_1 = r_1$ or $p_2 = r_2$ if and only if $P = R$. Therefore the sign of each of these functions, $r_1 - p_1$ and $p_2 - r_2$, is constant for θ in quadrants I and II. Picking a specific angle such as $\theta = 2$ allows us to verify that both are positive, and therefore $\operatorname{tsin}_\phi \theta$ is positive. Choosing an angle in the range $4 < \theta < 8$ shows that both of these functions are negative, and therefore $\operatorname{tsin}_\phi \theta$ is also negative when θ is in quadrants III and IV.

When $\phi = 0$, we have that $r_2 = 0$ and $r_1 = p_1$; then $\operatorname{tsin}_\phi \theta = p_2$, and the result follows. □

Proposition 7. *In the right triangle made by* P, R *and the origin* O, $|\operatorname{tcos}_\phi \theta|$ *gives the length of the side adjacent to* θ, *and* $|\operatorname{tsin}_\phi \theta|$ *gives the length of the opposite side.*

Proof. Fix an angle $0 \le \phi < 2$. The length of the adjacent side is the distance from R to the origin, which is $|r_1| + |r_2|$. When θ is in quadrants I and IV (defined by L and $L_\perp$), both r_1 and r_2 are nonnegative, so

$$|r_1| + |r_2| = r_1 + r_2 = |\operatorname{tcos}_\phi \theta|.$$

When θ lies in quadrants II and III, both r_1 and r_2 are nonpositive, so

$$|r_1| + |r_2| = -r_1 - r_2 = -(r_1 + r_2) = |\operatorname{tcos}_\phi \theta|.$$

The length of the side opposite of θ in triangle OPR is given by the distance between P and R, which is $|p_1 - r_1| + |p_2 - r_2|$. Arguing as in Proposition 6, when θ is in quadrants I and II, we have

$$|p_1 - r_1| + |p_2 - r_2| = (r_1 - p_1) + (p_2 - r_2) = |\operatorname{tsin}_\phi \theta|,$$

and when θ is in quadrants III and IV,

$$\begin{aligned}|p_1 - r_1| + |p_2 - r_2| &= -(r_1 - p_1) - (p_2 - r_2) \\ &= -\big((r_1 - p_1) + (p_2 - r_2)\big) = |\mathrm{tsin}_\phi\,\theta|.\end{aligned}$$ □

Proposition 8. *The following identities hold.*

$$\mathrm{tsin}_\phi(\theta - 4) = -\,\mathrm{tsin}_\phi\,\theta \quad \textit{and} \quad \mathrm{tcos}_\phi(\theta - 4) = -\,\mathrm{tcos}_\phi\,\theta.$$

Proof. Let P and R be the points given in Definition 5 corresponding to θ, and P' and R' the points corresponding to $\theta - 4$. By Proposition 3, taxicab angles of measure 2 are Euclidean right angles, which means P and P' are antipodal points on the unit circle and $P' = -P$. The map $(x, y) \to (-x, -y)$ is an isometry of the taxicab plane which maps P to P'. Angles are defined by the metric, and therefore isometries preserve angle measure. It follows from the definition of R that $R' = -R$. Therefore,

$$\mathrm{tcos}_\phi(\theta - 4) = -r_1 - r_2 = -(r_1 + r_2) = -\,\mathrm{tcos}_\phi\,\theta$$

and

$$\begin{aligned}\mathrm{tsin}_\phi(\theta - 4) &= (-r_1 + p_1) + (-p_2 + r_2) \\ &= -\big[(r_1 - p_1) + (p_2 - r_2)\big] = -\,\mathrm{tsin}_\phi\,\theta.\end{aligned}$$ □

3. Explicit formulas for sine and cosine functions

Theorem 9. *Let ϕ be a taxicab reference angle such that $0 \le \phi < 2$ and let θ be a taxicab angle measured relative to ϕ. Let*

$$\alpha = \frac{1}{\phi^2 - 2\phi + 2},$$

which is well-defined for all ϕ since $\phi^2 - 2\phi + 2 > 0$. The sine and cosine of θ with reference angle ϕ are given by

$$\mathrm{tsin}_\phi\,\theta = \begin{cases} \alpha\,\theta & \textit{if } -\phi \le \theta \le 2 - \phi, \\ 1 + \alpha(\theta - 2)(\phi - 1) & \textit{if } 2 - \phi \le \theta \le 4 - \phi, \\ \alpha(4 - \theta) & \textit{if } 4 - \phi \le \theta \le 6 - \phi, \\ -1 + \alpha(6 - \theta)(\phi - 1) & \textit{if } 6 - \phi \le \theta \le 8 - \phi, \end{cases}$$

and

$$\mathrm{tcos}_\phi\,\theta = \begin{cases} 1 + \alpha\,\theta(\phi - 1) & \textit{if } -\phi \le \theta \le 2 - \phi, \\ \alpha(2 - \theta) & \textit{if } 2 - \phi \le \theta \le 4 - \phi, \\ -1 + \alpha(4 - \theta)(\phi - 1) & \textit{if } 4 - \phi \le \theta \le 6 - \phi, \\ \alpha(\theta - 6) & \textit{if } 6 - \phi \le \theta \le 8 - \phi. \end{cases}$$

Lemma 10. *Let L be a line through the origin that makes angle ϕ with the x-axis, where $0 \le \phi < 2$. The point of intersection between L and the unit taxicab circle is*

$$Q = \left(\frac{2-\phi}{2}, \frac{\phi}{2}\right).$$

Proof. Let $Q = (q_1, q_2)$. Since Q lies on the unit circle and $0 \le \phi < 2$, both coordinates are positive and

$$q_1 + q_2 = 1. \tag{1}$$

Since the radius of the unit circle is 1, the definition of angle implies that ϕ is the distance between Q and $(1, 0)$. This distance is given by

$$|q_1 - 1| + |q_2 - 0| = 1 - q_1 + q_2 = \phi. \tag{2}$$

We solve the system of linear equations consisting of (1) and (2) for q_2 by adding the two equations to get

$$q_2 = \frac{\phi}{2};$$

substituting q_2 into (1) gives us $q_1 = 1 - \phi/2$, which is the desired result. □

3.1. ***Proof of Theorem 9 for $-\phi \le \theta \le 2-\phi$.*** Let $0 \le \phi < 2$ and $-\phi \le \theta \le 2-\phi$. We will determine the coordinates of P and R, given in Definition 5, as functions of ϕ and θ. Lemma 10 implies that the ϕ-axis (line L in Figure 3) intersects the circle at

$$Q = \left(\frac{2-\phi}{2}, \frac{\phi}{2}\right).$$

Since the ϕ-axis passes through the origin, we find that the equation is

$$L(x) = \frac{\phi}{2-\phi}\, x. \tag{3}$$

Next, we determine the coordinates of P, the point of intersection between the circle and the $(\theta+\phi)$-ray. Applying Lemma 10 again with angle $\theta+\phi$ gives coordinates

$$P = \left(\frac{2-\phi-\theta}{2}, \frac{\phi+\theta}{2}\right).$$

Proposition 3 implies that Euclidean right angles are taxicab right angles. Therefore, to find the point R we determine the equation of the line perpendicular (in the usual Euclidean sense) to the ϕ-axis, L_P, through point P. Since the ϕ-axis has slope $\phi/(2-\phi)$, L_P has slope $(\phi-2)/\phi$. Since we know the coordinates of

$P = (p_1, p_2)$ and the slope, we can determine the equation for L_P, which is

$$\begin{aligned} L_P(x) &= \left(\frac{\phi-2}{\phi}\right)(x-p_1)+p_2 \\ &= \frac{(\phi-2)x}{\phi} + \frac{(\phi-2)(\theta+\phi-2)+\phi(\theta+\phi)}{2\phi}. \end{aligned} \tag{4}$$

The point R is the intersection between the ϕ-axis and L_P. Setting equations (3) and (4) equal to each other and solving for the x-coordinate of R yields

$$r_1 = \frac{2-\phi}{2} + \frac{(2-\phi)(\phi\theta-\theta)}{2(\phi^2-2\phi+2)}.$$

Plugging r_1 into L (or L_p) gives the y-coordinate of R,

$$r_2 = \frac{\phi}{2} + \frac{\phi^2\theta-\phi\theta}{2(\phi^2-2\phi+2)}.$$

Thus, the coordinates of R are

$$R = \left(\frac{2-\phi}{2} + \frac{(2-\phi)(\phi\theta-\theta)}{2(\phi^2-2\phi+2)},\ \frac{\phi}{2} + \frac{\phi^2\theta-\phi\theta}{2(\phi^2-2\phi+2)}\right).$$

The result now follows by using the coordinates of R and P to compute $\operatorname{tsin}_\phi \theta$ and $\operatorname{tcos}_\phi \theta$ by the formulas given in Definition 5. □

3.2. ***Proof for $2-\phi \le \theta \le 4-\phi$.*** We again find the coordinates of P and R to compute $\operatorname{tsin}_\phi \theta$ and $\operatorname{tcos}_\phi \theta$. When $2 < \theta+\phi < 4$, the point P is in the second quadrant (as defined by the x- and y-axes). Let θ_1 be the portion of θ measured from the y-axis, so $\theta_1 = \phi+\theta-2$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $(x, y) \mapsto (y, -x)$. This map is an order 4 isometry of the ℓ^1 metric. Note that $f(0, 1) = (1, 0)$ and $f(P)$ is in the first quadrant. Since angle measure is defined by the metric, angle measure is preserved by isometries. We can therefore apply Lemma 10 to $f(P)$ to obtain the coordinates

$$f(P) = \left(\frac{2-\theta_1}{2}, \frac{\theta_1}{2}\right).$$

To obtain the coordinates for P we apply the inverse map:

$$P = f^{-1}\left(\frac{2-\theta_1}{2}, \frac{\theta_1}{2}\right) = \left(-\frac{\theta_1}{2}, \frac{2-\theta_1}{2}\right) = \left(\frac{2-\phi-\theta}{2}, \frac{4-\phi-\theta}{2}\right).$$

To finish the proof for this interval, we use the same procedure as in the proof for the first interval; that is, we find the equation of the line perpendicular to the ϕ-axis through P to determine the coordinates of the point R. The line through P

perpendicular to $L(x)$ is

$$L_P(x) = \left(\frac{\phi-2}{\phi}\right)(x-p_1)+p_2$$
$$= \frac{(\phi-2)x}{\phi} + \frac{(\phi-2)(\theta+\phi-2)+\phi(4-\theta-\phi)}{2\phi}. \tag{5}$$

To find r_1, we set equations (3) and (5) equal to one another and solve for x, which gives

$$r_1 = \frac{(\phi-2)(\theta-2)}{2(\phi^2-2\phi+2)}.$$

Plugging r_1 into $L(x)$ (Equation (3)) gives

$$r_2 = \frac{-\phi(\theta-2)}{2(\phi^2-2\phi+2)}.$$

The sine and cosine functions can now be computed from the formulas given in Definition 5. □

3.3. ***Proof for*** $\boldsymbol{4-\phi \le \theta \le 8-\phi}$**.** We will use the symmetry of the functions to establish the formulas for the third and fourth intervals. Let θ be in the given interval, and $\theta^* = \theta - 4$. Then $-\phi \le \theta^* \le 4-\phi$. We have determined formulas for $\mathrm{tsin}_\phi(\theta^*)$ and $\mathrm{tcos}_\phi(\theta^*)$ in this interval, so applying Proposition 8 gives formulas for angle θ in the remaining two intervals. □

It should be noted that our formulas are a generalization of those formulas in [Thompson and Dray 2000; Akça and Kaya 1997]; if $\phi = 0$, then θ is in standard position and we obtain identical formulas.

4. Properties of the functions

4.1. ***Periodic extensions and graphs.*** In Definition 5, the generalized sine and cosine functions were defined for all real numbers θ and for values of ϕ such that $0 \le \phi < 2$. It is evident from the definition that the θ-period of these functions is 8, so for any integer k,

$$\mathrm{tcos}_\phi(\theta+8k) = \mathrm{tcos}_\phi\,\theta \quad \text{and} \quad \mathrm{tsin}_\phi(\theta+8k) = \mathrm{tsin}_\phi\,\theta.$$

There is a natural ϕ-extension of these functions; since rotation by right angles gives isometries of the ℓ^1 metric, we extend the ϕ-domain of the generalized sine and cosine functions to be ϕ-periodic with period 2. Therefore, for any integer s,

$$\mathrm{tcos}_{\phi+2s}\,\theta = \mathrm{tcos}_\phi\,\theta \quad \text{and} \quad \mathrm{tsin}_{\phi+2s}\,\theta = \mathrm{tsin}_\phi\,\theta.$$

It should be noted that the formulas for $\mathrm{tsin}_\phi\,\theta$ and $\mathrm{tcos}_\phi\,\theta$ given by P and R from Definition 5 are only valid for values of ϕ in the first quadrant. Since Theorem 9

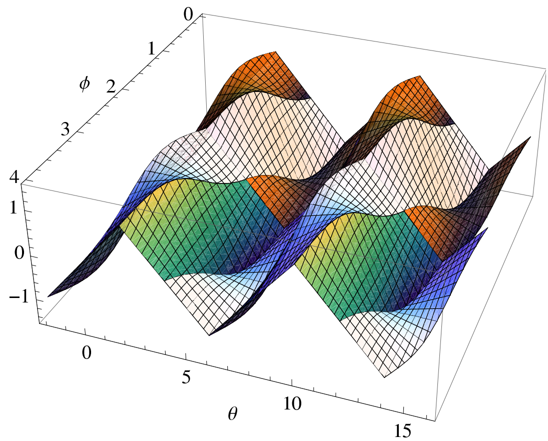

Figure 4. Graph of the generalized sine function.

gives explicit formulas for entire ϕ and θ periods, we may use this theorem and the two periodic properties stated above to give values for $\mathrm{tsin}_\phi\,\theta$ and $\mathrm{tcos}_\phi\,\theta$ for any $(\phi, \theta) \in \mathbb{R} \times \mathbb{R}$. Figure 4 contains a graph of $\mathrm{tsin}_\phi\,\theta$ for two periods of ϕ and two periods of θ.

Table 1 shows a family of cross-sections. Referring to the formulas in Theorem 9, we see that for a fixed ϕ these functions are piecewise linear. We invite the interested reader to verify that these functions are constant when $\theta = 2n$ for some integer n.

Recall that in the Euclidean metric, $\sin(\theta + \pi/2) = \cos\theta$. The cross-sections for the sine and cosine functions when ϕ is fixed suggest a similar identity.

Proposition 11. $$\mathrm{tsin}_\phi(\theta + 2) = \mathrm{tcos}_\phi\,\theta.$$

Proof. While this identity follows from the symmetry of the space, Theorem 9 gives explicit formulas for $\mathrm{tsin}_\phi\,\theta$ and $\mathrm{tcos}_\phi\,\theta$, so we need only check the formulas to verify this identity. Assume that $0 \le \phi < 2$ and $-\phi \le \theta \le 2 - \phi$, which implies $2 - \phi \le \theta + 2 \le 4 - \phi$. For angles in the interval $[2 - \phi, 4 - \phi]$,

$$\mathrm{tsin}_\phi\,\theta = 1 + \alpha(\theta - 2)(\phi - 1).$$

Therefore,

$$\mathrm{tsin}_\phi(\theta + 2) = 1 + \alpha((\theta + 2) - 2)(\phi - 1) = 1 + \alpha\,\theta(\phi - 1),$$

which is equal to $\mathrm{tcos}_\phi\,\theta$ when $-\phi \le \theta \le 2 - \phi$. The other intervals can be verified similarly. □

4.2. *Maximum and minimum values.*

Theorem 12. *The maximum value of* $\mathrm{tsin}_\phi\,\theta$ *and* $\mathrm{tcos}_\phi\,\theta$ *is* $1/2 + 1/\sqrt{2}$; *the minimum value is* $-(1/2 + 1/\sqrt{2})$.

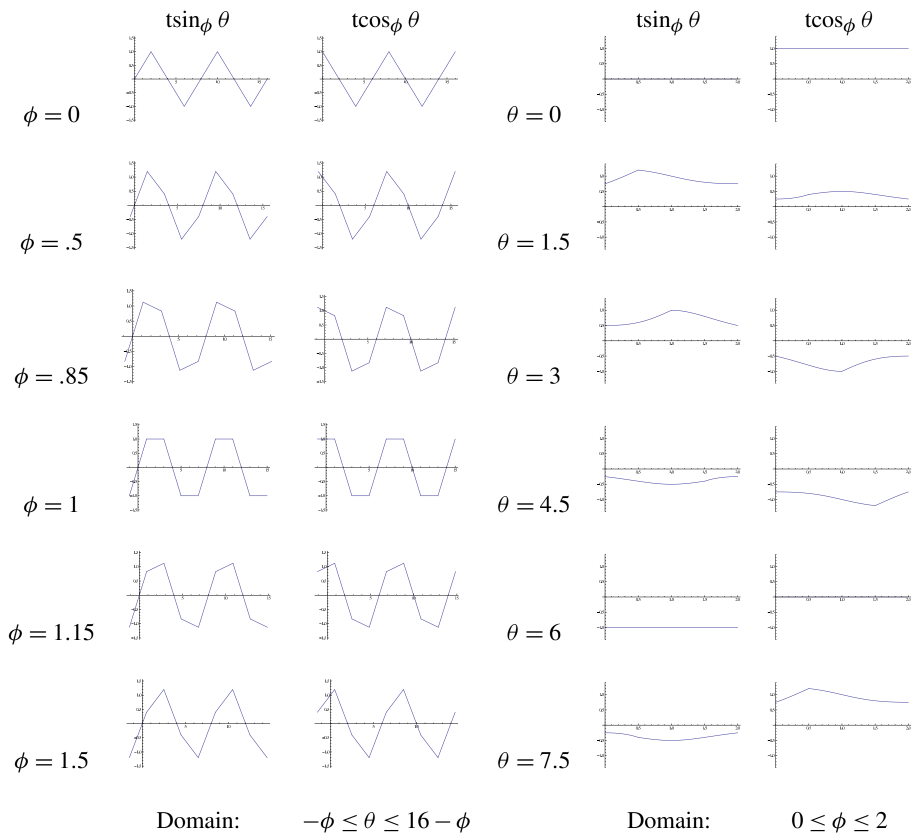

Table 1. Cross sections.

Proof. By Proposition 11, the maximum of the sine function is equal to the maximum of the cosine function. Also, by Proposition 8, the minimum of the sine function is equal to the negative of the maximum. Therefore it is sufficient to verify the maximum of the sine function.

The sine function has a θ-period of 8 and a ϕ-period of 2. However, the maximum of the sine function must occur when sine is positive, and hence θ must be in the interval [0, 4] by Proposition 6. It is therefore sufficient to find the maximum of $\operatorname{tsin}_\phi \theta$ on the region defined by $0 \le \phi \le 2$ and $0 \le \theta \le 4$. We will use standard techniques from multivariable calculus to maximize this function.

As $\operatorname{tsin}_\phi \theta$ is piecewise defined, we will consider the intervals

$$[0, 2-\phi], \quad [2-\phi, 4-\phi], \quad \text{and} \quad [4-\phi, 4].$$

Recall that

$$\alpha = \frac{1}{\phi^2 - 2\phi + 2} = \frac{1}{(\phi-1)^2 + 1},$$

which is positive for all ϕ. When θ is in the interval $[0, 2-\phi]$, we have $\operatorname{tsin}_\phi \theta = \alpha\theta$, and θ in $[4-\phi, 4]$ implies $\operatorname{tsin}_\phi \theta = \alpha(4-\theta)$. The partial derivatives with respect to θ of these functions are α and $-\alpha$; therefore, $\operatorname{tsin}_\phi \theta$ is increasing with respect to θ on $[0, 2-\phi]$ and decreasing in θ on $[4-\phi, 4]$. This implies the absolute maximum of $\operatorname{tsin}_\phi \theta$ occurs when θ is in the middle interval.

When $2-\phi \le \theta \le 4-\phi$,

$$\operatorname{tsin}_\phi \theta = 1 + \frac{(\theta-2)(\phi-1)}{\phi^2 - 2\phi + 2}.$$

The partial derivatives are

$$\frac{\partial}{\partial\phi}\left[1 + \frac{(\theta-2)(\phi-1)}{\phi^2-2\phi+2}\right] = \frac{(2\phi-\phi^2)(\theta-2)}{(\phi^2-2\phi+2)^2},$$
$$\frac{\partial}{\partial\theta}\left[1 + \frac{(\theta-2)(\phi-1)}{\phi^2-2\phi+2}\right] = \frac{\phi-1}{\phi^2-2\phi+2}.$$

These are both zero only when $(\phi, \theta) = (1, 2)$. In this case, $\operatorname{tsin}_1(2) = 1$. We now check the boundary conditions.

When $\phi = 0$, we have $2 \le \theta \le 4$ and $\operatorname{tsin}_\phi \theta = 2 + (-\theta/2)$, which has a maximum of 1. Note that $\operatorname{tsin}_\phi \theta$ has the same maximum when $\phi = 2$ because of the ϕ-periodic property previously stated.

When $\theta = 2-\phi$, we have

$$g(\phi) = \operatorname{tsin}_\phi(2-\phi) = 1 - \frac{(\phi-1)\phi}{\phi^2-2\phi+2}.$$

The derivative of this function is

$$g'(\phi) = \frac{\phi^2-4\phi+2}{(\phi^2-2\phi+2)^2}.$$

This function is zero when $\phi = 2 \pm \sqrt{2}$. Only one of these values, $\phi = 2-\sqrt{2}$, is in the region under consideration. For this value of ϕ, we have $\theta = \sqrt{2}$ and we see the value of the sine function is

$$\operatorname{tsin}_{2-\sqrt{2}} \sqrt{2} = 1/2 + 1/\sqrt{2}.$$

When $\theta = 4-\phi$, we have

$$h(\phi) = \operatorname{tsin}_\phi(4-\phi) = 1 - \frac{(\phi-2)(\phi-1)}{\phi^2-2\phi+2}.$$

The derivative of this function is

$$h'(\phi) = \frac{2-\phi^2}{(\phi^2-2\phi+2)^2}.$$

For values of ϕ in the interval [0, 2], this derivative is zero when $\phi = \sqrt{2}$. Then $\theta = 4 - \sqrt{2}$, and

$$\operatorname{tsin}_{\sqrt{2}}(4-\sqrt{2}) = \frac{1}{2} + \frac{1}{\sqrt{2}}.$$

We can therefore conclude for values in the region $0 \le \phi \le 2$ and $0 \le \theta \le 4$, the function $\operatorname{tsin}_\phi \theta$ achieves its absolute maximum, $1/2 + 1/\sqrt{2}$, in two locations: $(2-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, 4-\sqrt{2})$. □

Corollary 13. *The hypotenuse of a right triangle in taxicab space is not always the longest side of the triangle.*

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