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We present a lovely connection between the Fibonacci numbers and the sums of inverses of $(0, 1)$-triangular matrices, namely, a number S is the sum of the entries of the inverse of an $n \times n$ $(0, 1)$-triangular matrix (for $n \geq 3$) if and only if S is an integer between $2 - F_{n-1}$ and $2 + F_{n-1}$. Corollaries include Fibonacci identities and a Fibonacci-type result on determinants of a special family of $(1, 2)$-matrices.

1. Introduction

One of the ways to motivate students' interest in linear algebra is to present interesting connections between matrices and the Fibonacci numbers

$$F_1 = F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \quad n \geq 3.$$

For example, one can prove that $F_n^2 - F_{n-1} F_{n+1} = (-1)^{n+1}$ by using induction and the fact that

$$\det \begin{pmatrix} F_n & F_{n-1} \\ F_{n+1} & F_n \end{pmatrix} = \det \begin{pmatrix} F_n & F_{n-1} \\ F_{n+1} - F_n & F_n - F_{n-1} \end{pmatrix}$$

$$= \det \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix} = -\det \begin{pmatrix} F_{n-1} & F_n \\ F_{n-2} & F_{n-1} \end{pmatrix}.$$

Similarly, one can determine the exact value of the n-th Fibonacci number, by calculating the eigenvalues and the eigenvectors of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and using the equation

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} = \cdots = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

As another example of connections between Fibonacci numbers and matrix theory, consider lower triangular matrices of the form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & -1 & 1 \end{pmatrix}.$$

The inverses of these matrices are of the form

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 2 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 3 & 2 & 1 & 1 & 0 & \ddots & \vdots \\ 5 & 3 & 2 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 5 & 3 & 2 & 1 & 1 \end{pmatrix},$$

which, due to their remarkable structure, are known as Fibonacci matrices. Various properties of these matrices and their generalizations have been studied [Lee et al. 2002; Lee and Kim 2003; Wang and Wang 2008].

Fibonacci numbers are also widely used in algorithms in computer science [Atkins and Geist 1987; Knuth 1997], such as algorithms for finding extrema, merging files, searching in trees, etc. We provide here an example of their use in the searching of ordered arrays, described in [Atkins and Geist 1987]. Suppose that we have a sorted array with $F_n - 1$ elements for some natural number n (we can always pad the array with dummy elements in order to achieve such number of elements); for example, let $A = (0, 1, 2, 3, 5, 6, 9, 11, 15, 18, 20, 23)$ be an array with $F_7 - 1 = 12$ elements. We would like to check whether 15 is in A. First compare 15 with the F_{7-1}-th entry. Since $11 < 15$, we can eliminate all the entries to the left of the F_{7-1}-th entry (including the F_{7-1}-th entry), and we are left with the array $B = (15, 18, 20, 23)$ which contains $F_5 - 1 = 4$ elements. We now compare the F_{5-1}-th entry in B with 15, and since $20 > 15$, we eliminate 20 and 23, and we are left with the array $C = (15, 18)$ that has $F_4 - 1$ entries. Finally, we compare the F_{4-1}-th entry of C to 15, and since $18 > 15$, we are left with 15 and have a match. The full algorithm is described in [Atkins and Geist 1987]. Another interesting connection between Fibonacci numbers and matrices is given in [Li 1993], where it is shown that the maximal determinant of an $n \times n$ $(0, 1)$-Hessenberg matrix is F_n.

Let $S(X)$ denote the sum of the entries of a matrix X. Huang, Tam and Wu [Huang et al. 2013] show, among other results, that a number S is equal to $S(A^{-1})$ for an adjacency matrix (a symmetric $(0, 1)$-matrix with trace zero) A if and only if S is rational. More generally, they ask what can be said about the sum of the entries of the inverse of a $(0, 1)$-matrix. We consider the class of triangular matrices and show that a number S is equal to $S(A^{-1})$ for a triangular $(0, 1)$-matrix A if and only if S is an integer. This follows from our main result which shows that for $n \geq 3$, a number S is equal to $S(A^{-1})$ for an $n \times n$ triangular $(0, 1)$-matrix A if and only if

$$2 - F_{n-1} \leq S \leq 2 + F_{n-1}.$$

We use the following definitions and notation. Let e denote a vector of ones (so $S(A) = e^T A e$) and A_n the set of $n \times n$ invertible $(0, 1)$-upper triangular matrices. We will say that a matrix $A \in A_n$, where $n \geq 3$, is *maximizing* if $S(A^{-1}) = 2 + F_{n-1}$ and *minimizing* if $S(A^{-1}) = 2 - F_{n-1}$, and refer to maximizing and minimizing matrices as *extremal matrices*. For a set of vectors $V \subset R^n$, a vector $v \in V$ is *absolutely dominant* if for every $u \in V$, $|v_i| \geq |u_i|$, where $i = 1, 2, \ldots, n$.

We will use the following well-known properties of Fibonacci numbers (see, for example, [Vorobiev 2002]):

Lemma 1.1.   (i) $1 + \sum_{k=1}^{n} F_k = F_{n+2}$;

(ii) $1 + \sum_{k=1}^{n} F_{2k} = F_{2n+1}$;

(iii) $\sum_{k=1}^{n} F_{2k-1} = F_{2n}$.

The main result of the paper is proved in Section 2. In Section 3, we describe a construction of extremal matrices with a beautiful Fibonacci pattern in their inverses, and use it to obtain several Fibonacci identities. We conclude with a Fibonacci-type result on determinants of $(1, 2)$-matrices in spirit of the result in [Li 1993].

2. The main result

Theorem 2.1. *Let $n \geq 3$. Then $S = S(A^{-1})$ for some $A \in A_n$ if and only if S is an integer between $2 - F_{n-1}$ and $2 + F_{n-1}$; that is, $2 - F_{n-1} \leq S \leq 2 + F_{n-1}$.*

Proof. Obviously, $S(A^{-1})$ must be an integer since $A^{-1} = \text{adj}(A)/\det(A)$ and $\det(A) = 1$. The main part of the proof consists of showing

(a) $\max_{A \in A_n} S(A^{-1}) = 2 + F_{n-1}$,

(b) $\min_{A \in A_n} S(A^{-1}) = 2 - F_{n-1}$, and

(c) for every integer S between $2 - F_{n-1}$ and $2 + F_{n-1}$, there exists $A \in A_n$ such that $S(A^{-1}) = S$.

To show (a) and (b) we prove the following lemma.

Lemma 2.2. *Let $V = \{e^T A^{-1} \mid A \in A_n\}$. For the purposes of this lemma only, we will let $F_0 = -1$ (note that this is not a Fibonacci number). Then $v = (v_i)$, where $v_i = (-1)^i F_{i-1}$, is an absolutely dominant vector of V.*

Proof. For $n = 1$, we have $V = \{(1)\}$; for $n = 2$, we have $V = \{(1\ \ 1), (1\ \ 0)\}$; and for $n = 3$, we have $V = \{(1\ \ 1\ \ 1), (1\ \ 0\ \ 1), (1\ \ 1\ \ 0), (1\ \ 0\ \ 0), (1\ \ 1\ \ -1)\}$. Therefore the statement holds for $n = 1, 2, 3$. To prove the lemma for $n \geq 4$, we will use induction. Suppose the lemma is true for $k < n$.

We will now show that the vector v, defined in the lemma, is an absolutely dominant vector of the set $V = \{e^T A^{-1} \mid A \in A_n\}$. Let $A \in A_n$. Then A is of the form

$$\begin{pmatrix} C & \alpha & \beta \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix},$$

where $C \in A_{n-2}$, $\alpha, \beta \in \{0, 1\}^{n-2}$, and $x \in \{0, 1\}$. Therefore,

$$A^{-1} = \begin{pmatrix} C^{-1} & -C^{-1} (\alpha\ \ \beta) \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \\ 0 & \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} C^{-1} & -C^{-1} (\alpha\ \ \beta - x\alpha) \\ 0 & \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

We will use the following notation:

$$e^T C^{-1} = (c_1\ \ c_2\ \ \cdots\ \ c_{n-2}), \qquad \alpha = (\alpha_1\ \ \alpha_2\ \ \cdots\ \ \alpha_{n-2})^T,$$
$$\beta = (\beta_1\ \ \beta_2\ \ \cdots\ \ \beta_{n-2})^T.$$

So

$$e^T A^{-1} = \left(c_1\ \ c_2\ \ \ldots\ \ c_{n-2}\ \ 1 - \sum_{i=1}^{n-2} \alpha_i c_i\ \ 1 - x - \sum_{i=1}^{n-2} c_i(\beta_i - x\alpha_i) \right).$$

Consider the n-th entry of $e^T A^{-1}$. Since $c_1 = 1$, $n \geq 4$, and $-1 \leq \beta_i - x\alpha_i \leq 1$ for all $1 \leq i \leq n-2$, it is easy to see that

$$-\sum_{i=1}^{n-2} |c_i| \leq 1 - x - \sum_{i=1}^{n-2} c_i(\beta_i - x\alpha_i) \leq \sum_{i=1}^{n-2} |c_i|$$

for all possible $x, \alpha_i, \beta_i \in \{0, 1\}$, where $1 \leq i \leq n-2$. Since $\beta_i - \alpha_i \in \{-1, 0, 1\}$, it is possible to achieve equality in each inequality by taking

$$x = 1 \quad \text{and} \quad \text{sign}(\beta_i - \alpha_i) = \text{sign}(c_i), \quad 1 \leq i \leq n-2 \tag{1}$$

in the first, and

$$x = 1 \quad \text{and} \quad \text{sign}(\beta_i - \alpha_i) = -\text{sign}(c_i), \quad 1 \leq i \leq n-2. \tag{2}$$

in the second. Now, since $\left|-\sum_{i=1}^{n-2}|c_i|\right| = \left|\sum_{i=1}^{n-2}|c_i|\right|$, we get that if $A \in A_n$ is a matrix for which $e^T A^{-1}$ is an absolutely dominant vector, its n-th entry must be equal to either

$$-\sum_{i=1}^{n-2}|c_i| \tag{3}$$

or

$$\sum_{i=1}^{n-2}|c_i|. \tag{4}$$

Note that the maximal value of (3) is obtained by taking C such that $e^T C^{-1}$ is an absolutely dominant vector of the set $V = \{e^T A^{-1} \mid A \in A_{n-2}\}$ (and all the absolutely dominant vectors will give the same value). The same is true of the minimal value of (4). By the inductive hypothesis and using Lemma 1.1, the maximal value of (4) is

$$\sum_{i=1}^{n-2}|c_i| = 1 + \sum_{i=1}^{n-3}F_i = F_{n-1}$$

(and this value may be achieved by choosing an appropriate C). Similarly, the minimal value of (3) is $-F_{n-1}$. Let us now consider the $(n-1)$-th entry of $e^T A^{-1}$. By the inductive hypothesis, its absolute value is bounded from above by F_{n-2}. By taking $C \in A_{n-2}$ such that $e^T C^{-1}$ is an absolutely dominant vector, choosing α, β such that either (1) or (2) is satisfied and using Lemma 1.1 and the inductive hypothesis, we get that the $(n-1)$-th entry of $e^T A^{-1}$ is equal to either

$$1 - \sum_{i=1}^{n-2}\alpha_i c_i = 1 - \sum_{k=1}^{\lfloor\frac{n-3}{2}\rfloor}c_{2k+1} = 1 + \sum_{k=1}^{\lfloor\frac{n-3}{2}\rfloor}F_{2k} = F_{2\lfloor\frac{n-3}{2}\rfloor+1}, \tag{5}$$

or

$$1 - \sum_{i=1}^{n-2}\alpha_i c_i = 1 - c_1 - \sum_{k=1}^{\lfloor\frac{n-2}{2}\rfloor}c_{2k} = -\sum_{k=1}^{\lfloor\frac{n-2}{2}\rfloor}F_{2k-1} = -F_{2\lfloor\frac{n-2}{2}\rfloor}. \tag{6}$$

Note that if n is odd then expression (5) is equal to F_{n-2}, and if n is even then expression (6) is equal to $-F_{n-2}$. In sum, using the inductive hypothesis, we showed that the largest possible absolute value of the n-th entry of $e^T A^{-1}$ (such that $A \in A_n$) is F_{n-1}. In this case, we showed that it is possible to choose α such that the absolute value of the $(n-1)$-th entry of $e^T A^{-1}$ is F_{n-2}, the largest possible absolute value due to the inductive hypothesis. Therefore, we showed that the vector v, defined in the lemma, is an absolutely dominant vector for $V = \{e^T A^{-1} \mid A \in A_n\}$. $\square$

We are now ready to prove (a) and (b). We represent $A \in A_n$ in the same form as in Lemma 2.2, and

$$
e^T A^{-1} e = e^T \begin{pmatrix} C^{-1} & -C^{-1} \left(\alpha \;\; \beta - x\alpha \right) \\ 0 & \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \end{pmatrix} e
$$

$$
= 2 - x + e^T C^{-1} e - e^T C^{-1} \left(\beta + (1-x)\alpha \right)
$$

$$
= 2 - x + e^T C^{-1} (e - \alpha - \beta + x\alpha).
$$

Let $u = e - \alpha - \beta + x\alpha$. Note that if $x = 1$ then $u \in \{0, 1\}^{n-2}$, and if $x = 0$ then $u \in \{-1, 0, 1\}^{n-2}$. In addition, note that

$$
\max\{2 - x + e^T C^{-1} u \mid x = 0, \alpha, \beta \in \{0, 1\}^{n-2}\}
$$
$$
\geq \max\{2 - x + e^T C^{-1} u \mid x = 1, \alpha, \beta \in \{0, 1\}^{n-2}\}. \quad (7)
$$

Now, since $C \in A_{n-2}$, the first entry of $e^T C^{-1}$ is 1. If $x = 0$, then in order to minimize the value of $e^T C^{-1} u$, we have to take the first entries of α and β to be 1. On the other hand, if $x = 1$, then in order to minimize the value of $e^T C^{-1} u$, we have to take the first entries of β to be 1. The difference between these two cases is 1, and therefore

$$
\min\{2 - x + e^T C^{-1} u \mid x = 0, \alpha, \beta \in \{0, 1\}^{n-2}\}
$$
$$
\leq \min\{2 - x + e^T C^{-1} u \mid x = 1, \alpha, \beta \in \{0, 1\}^{n-2}\}. \quad (8)
$$

Since we are only interested in the minimal and the maximal values of $e^T A^{-1} e$, we may assume, by (7) and (8), that $x = 0$. Therefore, $e^T A^{-1} e = 2 + e^T C^{-1} (e - \alpha - \beta)$. Using the notation of Lemma 2.2, we get

$$
\min\{2 + e^T C^{-1} (e - \alpha - \beta) \mid \alpha, \beta \in \{0, 1\}^{n-2}\} = 2 - \sum_{i=1}^{n-2} |c_i| \quad (9)
$$

and

$$
\max\{2 + e^T C^{-1} (e - \alpha - \beta) \mid \alpha, \beta \in \{0, 1\}^{n-2}\} = 2 + \sum_{i=1}^{n-2} |c_i|. \quad (10)
$$

Therefore, the minimal and the maximal values of $e^T A^{-1} e$ are achieved by taking C such that $e^T C^{-1}$ is an absolutely dominant vector of $\{e^T A^{-1} \mid A \in A_{n-2}\}$. Hence,

by Lemmas 2.2 and 1.1,

$$\max_{A \in A_n} S(A^{-1}) = \max\{2 + e^T C^{-1}(e - \alpha - \beta) \mid \alpha, \beta \in \{0, 1\}^{n-2}, C \in A_{n-2}\}$$

$$= 3 + \sum_{i=1}^{n-3} F_i = 2 + F_{n-1},$$

and similarly,

$$\min_{A \in A_n} S(A^{-1}) = 1 - \sum_{i=1}^{n-3} F_i = 2 - F_{n-1}.$$

It is well known that every natural number is the sum of distinct Fibonacci numbers. For the proof of (c), we need a slightly stronger observation.

Lemma 2.3. *Let M be a natural number, and let n be an integer for which $F_{n-1} \leq M < F_n$. Then M can be represented as a sum of distinct Fibonacci elements from the set $\{F_1, F_2, \ldots, F_{n-2}\}$.*

Proof. For $M = 1$, the statement is true. Proceeding by induction, assume that it is true for all integers less than M. Let n be an integer for which $F_{n-1} \leq M < F_n$. Since $M < F_n$, we get that $M < F_{n-2} + F_{n-1}$, and hence $M - F_{n-2} < F_{n-1}$. Therefore, there exists k with $n - 1 \geq k > 0$ such that $F_{k-1} \leq M - F_{n-2} < F_k$, and hence by the inductive hypothesis, $M - F_{n-2}$ can be represented as a sum of distinct Fibonacci elements from the set $\{F_1, F_2, \ldots, F_{k-2}\}$. Since $n - 1 \geq k$, we have $n - 3 \geq k - 2$, and so M can be represented as a sum of distinct Fibonacci elements from the set $\{F_1, F_2, \ldots, F_{n-2}\}$. □

We conclude the proof of Theorem 2.1 by proving (c). Let $S = 2 + T$, where $-F_{n-1} \leq T \leq F_{n-1}$. The cases $T = F_{n-1}$ and $T = -F_{n-1}$ were proved in (a) and (b). For $T = 0$, let A be a triangular Toeplitz matrix with first row $(1\ 0\ 1\ 0\ 0\ \ldots\ 0)$. Then $S(A^{-1}) = 2$. Similarly, it is easy to prove the claim for any S between 1 and n. For the other integers in $[2 - F_{n-1}, 2 + F_{n-1}]$ (and also for $1, 2, \ldots, n$), let us consider the expression in (10). It is easy to see that in fact by choosing appropriate α and β (and C such that $e^T C^{-1}$ is an absolutely dominant vector), $e^T C^{-1}(e - \alpha - \beta)$ can achieve any value of the form

$$\alpha_1 + \sum_{i=2}^{n-2} \alpha_i F_{i-1},$$

where $\alpha_i \in \{0, 1\}$ for all $1 \leq i \leq n - 2$. Note that by Lemma 2.3, there exists appropriate set $\{\alpha_i\}_{i=1}^{n-2}$ such that

$$T = \sum_{i=2}^{n-2} \alpha_i F_{i-1} \quad \text{(we may choose } \alpha_1 = 0\text{)}.$$

Hence, for this choice of C, α and β, we get A such that $S = T + 2 = e^T A^{-1} e$. We obtain a similar result for the case $S = 2 - T$, where $0 \leq T \leq F_{n-1}$, by looking at expression (9), and this completes the proof. $\square$

As an analogy to the result on rational numbers of [Huang et al. 2013] mentioned in the introduction, we now have the following corollary.

Corollary 2.4. *A number S is equal to $S(A^{-1})$ for a $(0, 1)$-triangular matrix A if and only if S is an integer.*

Define G_n to be the set of $n \times n$ matrices of the form $I + B$, where B is an $n \times n$ upper triangular nilpotent matrix with entries from the interval $[0, 1]$. Then, using the fact that for an invertible matrix A, $A^{-1} = \operatorname{adj}(A)/\det(A)$, and that for $A \in G_n$, $\det(A) = 1$, we have $A^{-1} = \operatorname{adj}(A)$ for $A \in G_n$. Thus, since $S(A^{-1})$ is linear in each one of the entries in such a matrix A, we conclude the following:

Corollary 2.5. $\displaystyle\max_{A \in G_n} S(A^{-1}) = 2 + F_{n-1}$ *and* $\displaystyle\min_{A \in G_n} S(A^{-1}) = 2 - F_{n-1}$.

Remark 2.6. For a general $n \times n$ invertible $(0, 1)$-matrix A (which is not necessarily triangular), the question regarding the minimal or the maximal value that $S(A^{-1})$ may obtain is still open. For $n = 3, 4, 5, 6$, the extremal values are exactly the same as in the triangular case. However, for $n = 7$, there exist $n \times n$ invertible $(0, 1)$-matrices M and N (which are presented below) such that $S(M^{-1}) = -7$ and $S(N^{-1}) = 11$, whereas in the triangular case, the minimal and the maximal values are -6 and 10, respectively.

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For larger values of n, the difference between the general and the triangular case gets bigger.

3. Extremal matrices

Recall that an invertible triangular $n \times n$ $(0, 1)$-matrix A is extremal if

$$e^T A^{-1} e = 2 \pm F_{n-1}.$$

The matrices I_3 and I_4 are maximizing matrices. The matrices

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are minimizing matrices.

Following the proof of Theorem 2.1, we can construct extremal matrices for $n \geq 5$ that have a beautiful Fibonacci pattern in their inverses. For $l = 2, 3$, partition the off-diagonal entries of an upper triangular $n \times n$ matrix into $n - l$ sets, $S_0, S_1, \ldots, S_{n-l-1}$. The set S_{n-l-1} consists of the entries in the first two rows of the last l columns. For $i = 1, 2, \ldots, n - l - 2$, the set S_i consists of the entries immediately to the left or immediately below the entries in S_{i+1}, and S_0 consists of all the remaining entries which are above the main diagonal (two if $l = 2$ and four if $l = 3$). For example, in the case that $n = 9$, Figure 1 (left) presents the partition in the case $l = 2$, and Figure 1 (right) presents the partition in the case $l = 3$.

Let A be an invertible $(0, 1)$-upper triangular matrix, where the entries in S_i are taken modulo 2. It follows from the proof of Theorem 2.1 that A^{-1} is an $n \times n$ upper triangular matrix where the diagonal entries are 1, the entries in S_0 are 0, and the entries in S_i for $i \geq 1$ are $(-1)^i F_i$. For example, when $n = 9$, $l = 2$,

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 & -1 & 1 & -2 & 3 & -5 & 8 & 8 \\ 0 & 1 & -1 & 1 & -2 & 3 & -5 & 8 & 8 \\ 0 & 0 & 1 & -1 & 1 & -2 & 3 & -5 & -5 \\ 0 & 0 & 0 & 1 & -1 & 1 & -2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

and when $n = 9$, $l = 3$,

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 & -1 & 1 & -2 & 3 & -5 & -5 & -5 \\ 0 & 1 & -1 & 1 & -2 & 3 & -5 & -5 & -5 \\ 0 & 0 & 1 & -1 & 1 & -2 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & -1 & 1 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
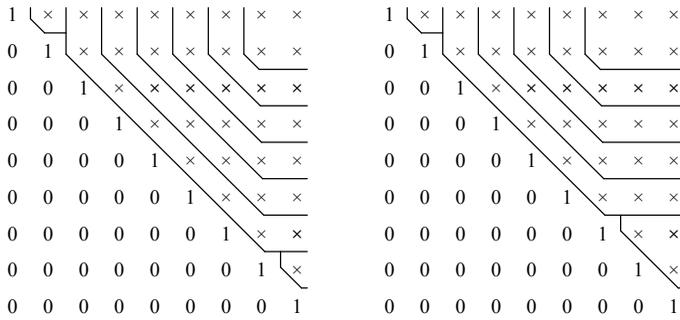
Figure 1. The partition in the cases $l = 2$ (left) and $l = 3$ (right).

In general, if $n + l$ is even, $e^T A^{-1} e = 2 - F_{n-1}$, and hence A is a minimizing extremal matrix (this also includes the case $n = 4$). If $n + l$ is odd, $e^T A^{-1} e = 2 + F_{n-1}$, and hence A is a maximizing extremal matrix. Using these equalities, we obtain the following Fibonacci identities:

Corollary 3.1. $\sum_{i=1}^{n-4}(n-i)(-1)^i F_i + 4(-1)^{n-3} F_{n-3} = (-1)^{n-1} F_{n-1} - (n-2)$.

Corollary 3.2. $\sum_{i=1}^{n-5}(n-i)(-1)^i F_i + 6(-1)^{n-4} F_{n-4} = (-1)^n F_{n-1} - (n-2)$.

4. Determinants of $(1, 2)$-matrices

In [Huang et al. 2013], the following remark, which follows from Cramer's rule and the multilinearity of the determinant, was presented:

Remark 4.1. For any nonsingular matrix A,

$$S(A^{-1}) = \frac{\det(A + J) - \det(A)}{\det(A)},$$

where J is the matrix whose entries are all 1.

Recall that it was proved in [Li 1993] that the maximal determinant of an $n \times n$ Hessenberg (0,1)-matrix is F_n. Using our main result and Remark 4.1, we obtain another family of matrices whose determinants are strongly related to the Fibonacci sequence.

Let W_n be the family of $n \times n$ matrices such that for any $A \in W_n$,

$$A_{ij} = \begin{cases} 1 & \text{if } j > i, \\ 2 & \text{if } j = i, \\ 1 \text{ or } 2 & \text{if } j < i. \end{cases}$$

From Remark 4.1 and Theorem 2.1, we obtain the following corollary:

Corollary 4.2. *Let $n \geq 3$. Then $S = \det(A)$ for some $A \in W_n$ if and only if S is an integer that satisfies $3 - F_{n-1} \leq S \leq 3 + F_{n-1}$.*

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