involve
a journal of mathematics

Stick numbers in the simple hexagonal lattice

Ryan Bailey / Hans Chaumont / Melanie Dennis / Jennifer McLoud-Mann / Elise McMahon / Sara Melvin / Geoffrey Schuette

msp

2015 vol. 8, no. 3

INVOLVE 8:3 (2015)
dx.doi.org/10.2140/involve.2015.8.503

Stick numbers in the simple hexagonal lattice

Ryan Bailey, Hans Chaumont, Melanie Dennis, Jennifer McLoud-Mann,
Elise McMahon, Sara Melvin and Geoffrey Schuette

(Communicated by Colin Adams)

In the simple hexagonal lattice, bridge number is used to establish a lower bound on stick numbers of knots. This result aids in giving a new proof that the minimal stick number is 11. In addition, the authors establish upper bounds for the stick number of a composite knot. Constructions for $(p, p+1)$-torus knots and some 3-bridge knots are given requiring one more stick than the lower bound guarantees.

1. Introduction

Most results concerning lattice knots have focused on knots in the simple cubic lattice, sc or $\mathbb{Z}^3$. Various lower and upper bounds for stick number in the cubic lattice have been given in [Adams et al. 2012; Janse van Rensburg and Promislow 1999; Hong et al. 2013]. Minimal stick numbers for the 3_1 and 4_1 knots are 12 and 14 [Huh and Oh 2005]; see Figure 1. The stick number for a $(p, p+1)$-torus knot is $6p$ for $p \geq 2$ [Adams et al. 2012]. Work has also been done for the minimum stick number of the composition of two knots [Adams et al. 1997; 2012]. Relatively little is known about analogous results in the simple hexagonal lattice. Mann, McLoud-Mann and Milan [Mann et al. 2012] show that the minimum number of sticks to create a nontrivial knot is 11.

In this paper, we will answer some questions regarding the simple hexagonal lattice. In Section 3, we establish a lower bound on the stick number in terms on the bridge number. In Section 4, we give the idea of a new proof of the result in [Mann et al. 2012]. In Section 5, we give an upper bound for the stick number of a composite knot. In Section 6, we catalog results about the stick number of $(p, p+1)$-torus knots, some 3-bridge knots, and particular composite knots.

MSC2010: 57M50.
Keywords: lattice knots, stick number, composition, bridge number.

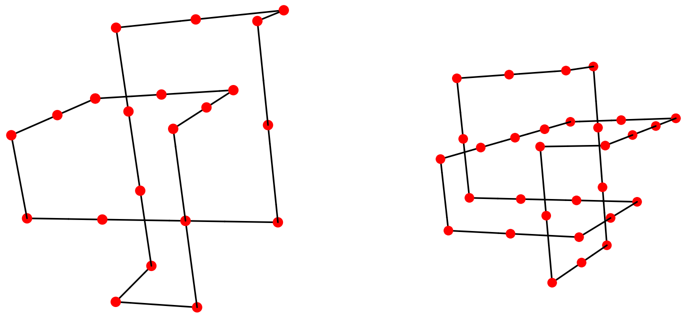

Figure 1. Minimal stick 3_1 (left) and 4_1 (right) knots in the simple cubic lattice.

2. Some preliminaries

We will adopt notation for the simple hexagonal lattice from [Mann et al. 2012], which we include here for completeness. The simple hexagonal lattice is defined to be the set of all integral combinations of vectors

$$x = \langle 1, 0, 0\rangle, \quad y = \left\langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right\rangle, \quad w = \langle 0, 0, 1\rangle;$$

that is,

$$\mathrm{sh} = \left\{ a\langle 1, 0, 0\rangle + b\left\langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right\rangle + c\langle 0, 0, 1\rangle \mid a, b, c \in \mathbb{Z} \right\}.$$

Further, let $X = -x$, $Y = -y$, $W = -w$, $z = \left\langle -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right\rangle$, and $Z = -z$ so that we can describe a polygon by a string of vectors. In Figure 2, the polygon may be written as $x^5 z w^2 X^3 W^3 Z^2 w^2 y^3 X^3 W Y^2$.

A maximal segment in a polygon $\mathcal{P}$ which is parallel to $x = \langle 1, 0, 0\rangle$ will be called an x-stick. Similarly, define y-, z-, and w-sticks to be maximal segments in $\mathcal{P}$ which are parallel to $\left\langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right\rangle$, $\left\langle -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right\rangle$, and $\langle 0, 0, 1\rangle$, respectively. A closed nonintersecting polygon formed from x-, y-, z-, and w-sticks is called an sh lattice knot. The number of x-, y-, z-, and w-sticks in a polygon $\mathcal{P}$ will be denoted $|\mathcal{P}|_x$, $|\mathcal{P}|_y$, $|\mathcal{P}|_z$, and $|\mathcal{P}|_w$, respectively, and the total number of sticks used will be $|\mathcal{P}|$.

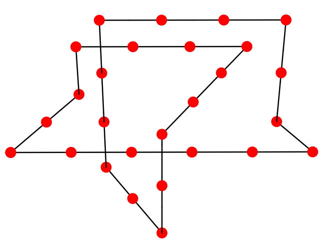

Figure 2. A trefoil knot in the simple hexagonal lattice.

The stick number of a knot type K in the lattice, denoted $s[K]$, is the minimum number of sticks required to form a polygon of type K. In Figure 2, $|\mathcal{P}|_x = 3$, $|\mathcal{P}|_y = 2$, $|\mathcal{P}|_z = 2$, $|\mathcal{P}|_w = 4$, and $|\mathcal{P}| = 11$. Further, observe that $s[3_1] \leq 11$.

3. Lower bound for stick numbers

Janse van Rensburg and Promislow [1999] established the lower bound for the stick number of a knot in the simple cubic lattice with three directions $x = \langle 1, 0, 0\rangle$, $y = \langle 0, 1, 0\rangle$, and $z = \langle 0, 0, 1\rangle$; it was $6b[K]$ where $b[K]$ is the bridge number of the knot K (the minimum number of local maxima of any projection of a knot onto any single vector). The proof guaranteed $2b[K]$ sticks in each of the three directions. Indeed, maximums in the up-down direction, or z-direction, will occur in xy-planes and each maximum will have two z-sticks at the ends of the arc containing the maximum in the xy-plane. We give a similar result here for the simple hexagonal lattice.

Theorem 1 (lower bound for stick numbers). *For any knot K in the simple hexagonal lattice*, $s[K] \geq 5b[K]$.

Proof. A maximum in the w-direction, occurring in an xy-plane, will have two w-sticks at the ends of the arc containing the maximum in the xy-plane. Note that using a z-stick at the end of the arc would keep you in the same xy-plane. Since there are at least $b[K]$ maxima, we have $|\mathcal{P}|_w \geq 2b[K]$.

A maxima occurring in an xw-plane will have two sticks at the ends of the arc containing the maximum in the xw-plane — these sticks can be y- or z-sticks. Since there are at least $b[K]$ maxima, we have $|\mathcal{P}|_y + |\mathcal{P}|_z \geq 2b[K]$. One also considers maxima occurring in yw-planes and zw-planes to get two more inequalities summarized below:

$$|\mathcal{P}|_w \geq 2b[K], \tag{1}$$

$$|\mathcal{P}|_y + |\mathcal{P}|_z \geq 2b[K], \tag{2}$$

$$|\mathcal{P}|_x + |\mathcal{P}|_z \geq 2b[K], \tag{3}$$

$$|\mathcal{P}|_x + |\mathcal{P}|_y \geq 2b[K]. \tag{4}$$

Summing inequalities (2)–(4) and dividing by 2 yields $|\mathcal{P}|_x + |\mathcal{P}|_y + |\mathcal{P}|_z \geq 3b[K]$. Then adding inequality (1) gives $|\mathcal{P}| = |\mathcal{P}|_x + |\mathcal{P}|_y + |\mathcal{P}|_z + |\mathcal{P}|_w \geq 5b[K]$. □

At this point, we can say that the stick number of any nontrivial knot in the simple hexagonal lattice is at least 10. However, in [Mann et al. 2012], it was shown to be 11. In the next section we show that any polygon constructed with ten sticks in the simple hexagonal lattice is the trivial polygon. Before we proceed, we point out what must happen if $|\mathcal{P}| = 5b[K]$.

Corollary 2. *If $|\mathcal{P}| = 5b[K]$, then $|\mathcal{P}|_x = |\mathcal{P}|_y = |\mathcal{P}|_z = \frac{1}{2}|\mathcal{P}|_w = b[K]$.*

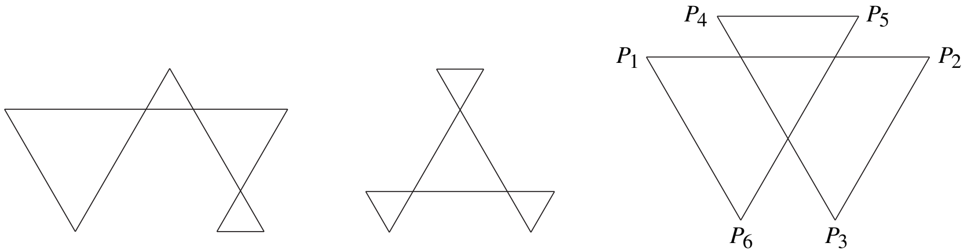

Figure 3. Three crossing projections of ten stick sh knots.

Proof. Suppose $|\mathcal{P}|_x \neq b[K]$, $|\mathcal{P}|_y \neq b[K]$, $|\mathcal{P}|_z \neq b[K]$, or $|\mathcal{P}|_w \neq 2b[K]$. If $|\mathcal{P}|_w > 2b[K]$ is combined with $|\mathcal{P}|_x + |\mathcal{P}|_y + |\mathcal{P}|_z \geq 3b[K]$, the argument above yields $|\mathcal{P}| > 5b[K]$. For the remainder of the argument we may assume $|\mathcal{P}|_w = 2b[K]$.

If $|\mathcal{P}|_x < b[K]$, then $|\mathcal{P}|_x = b[K] - n$ for some $n > 0$. Inequalities (3) and (4) imply that $|\mathcal{P}|_y \geq b[K] + n$ and $|\mathcal{P}|_z \geq b[K] + n$. Thus $|\mathcal{P}| \geq 5b[K] + n > 5b[K]$. Following a similar argument, if $|\mathcal{P}|_y < b[K]$ or $|\mathcal{P}|_z < b[K]$, then $|\mathcal{P}| > 5b[K]$. Hence for the remainder of the argument we may assume $|\mathcal{P}|_x \geq b[K]$, $|\mathcal{P}|_y \geq b[K]$ and $|\mathcal{P}|_z \geq b[K]$. Observe that since one of these inequalities is strict from our original assumption, it must happen that $|\mathcal{P}| > 5b[K]$. □

4. Stick number of the lattice

As mentioned in the previous section, the stick number of any nontrivial knot in the simple hexagonal lattice is at least 10. The work in this section will show that a simple hexagonal knot constructed with ten sticks (necessarily using two x-sticks, two y-sticks, two z-sticks, and four w-sticks from Corollary 2) is the trivial knot. This, along with the eleven-stick trefoil in Figure 2, will establish the following result.

Theorem 3 (minimum stick number in the simple hexagonal lattice). *In the simple hexagonal lattice, the stick number of any nontrivial knot is at least* 11.

Given a ten-stick knot K using two x-sticks, two y-sticks, two z-sticks, and four w-sticks, consider the projection of K onto the xy-plane. If the projection contains two line segments laying on top of one another or multiple crossings at one point, then do a slight perturbation of the knot before projecting. If the projection contains less than three crossings, then the knot is trivial. There are only a few possibilities for projections containing three crossings; see Figure 3 for representative projections.

The first two projections are the trivial knot. For the last projection, it must have alternating crossings to be a nontrivial knot. However, it cannot have alternating crossings in the hexagonal lattice. Indeed, label the endpoints of the projection P_1, P_2, P_3, P_4, P_5, and P_6 as in Figure 3. Without loss of generality, suppose that P_1P_2 on level i crosses over P_3P_4 on level j; that is, $i > j$. Alternating crossings gives

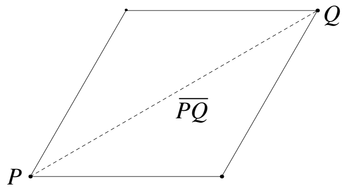

Figure 4. Connecting sh lattice points P and Q with two sticks.

that P_3P_4 on level j crosses over P_5P_6 on level k and P_5P_6 crosses over P_1P_2. This gives $i > j > k > i$.

5. Upper bound for stick composition

In order to compose sh knots we must identify places on the knots to compose them; these will be called *configurations*. To achieve the highest reduction of sticks and edges in the composition of sh lattice knots, we will compose knots with configurations in planes parallel to the xy-plane. In particular, we will compose with configurations in the top xy-plane or the bottom xy-plane of a knot.

Suppose K is a minimal stick conformation in the sh lattice — that is, it can't be constructed with fewer sticks. If K contains more than one connected component in the top xy-plane, then the vertical sticks for one connected component can be lengthened in order to push that connected component to a higher xy-plane without increasing the number of sticks used to create K. Thus one may assume that the top xy-plane (and similarly the bottom xy-plane) contains only one connected component. The two endpoints P and Q of the connected component can either be connected via one stick or two sticks since there are no other components to avoid when creating a path. To see this, consider the angles between the vector $\overrightarrow{PQ}$ and the vectors $\pm\langle 1, 0, 0\rangle$, $\pm\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right\rangle$, $\pm\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right\rangle$. If one of the angles is zero, then P and Q are connected with one stick. If not, then we construct a parallelogram with P and Q on opposite corners using the two vectors which yield the smallest two angles from above. Note that $\overline{PQ}$ forms the major axis of the parallelogram. In this situation P and Q can be connected via two sticks. An example is given in Figure 4.

Thus after possibly rotating the knot around the z-axis, we have two possible configurations occurring in the top or bottom xy-plane as shown in Figure 5.

Theorem 4. *Given knots K and L in the simple hexagonal lattice*,

$$s[K\#L] \le s[K] + s[L] - 3.$$

Proof. Let K and L be two knots in minimal stick conformations in the simple hexagonal lattice. We will compose K along a configuration in the bottom xy-plane and L along a configuration in the top xy-plane. Finally, when expressing K and L

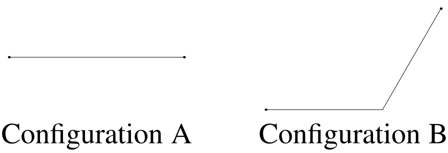

Figure 5. Configurations in sh.

as strings we will choose convenient starting places and orientations to allow for easier composition.

Case 1. Suppose K and L both have type A configurations. Then the bottom and top configurations of K and L, respectively, can be viewed as in Figure 6. Let $K = sx^n$ and $L = X^m t$, where the strings s and t represent what remains of K and L after the type A configurations are removed. Note that s will begin with a w and end with a W, whereas t will begin with a W and end with a w. Assuming that $n \neq m$, we scale K by m and scale L by n. We have $K = \tilde{s}x^{nm}$ and $L = X^{nm}\tilde{t}$, where $\tilde{s}$ represents s scaled by m and $\tilde{t}$ represents t scaled by n. (In the case that $n = m$, $\tilde{s} = s$ and $\tilde{t} = t$.) We may now compose K and L, and write $K\#L = \tilde{s}\tilde{t}$. At first glance it may seem that we have removed only two sticks (from the xs and Xs). However, we have removed two more sticks. The end of $\tilde{s}$ and the beginning of $\tilde{t}$ have combined into one stick instead of two. Similarly the end of $\tilde{t}$ and beginning of $\tilde{s}$ have combined into one stick. Thus we have a reduction of four sticks for this case. That is, $s[K\#L] \leq s[K] + s[L] - 4$.

Case 2. Suppose K has a type A configuration and L has a type B configuration. Then the bottom and top configurations of K and L, respectively, can be viewed as in Figure 7. Let $K = sx^n$ and $L = X^m tY^p$, where strings s and t represent what

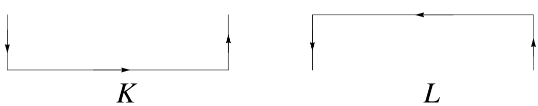

Figure 6. K and L with type A configurations: bottom and top, respectively.

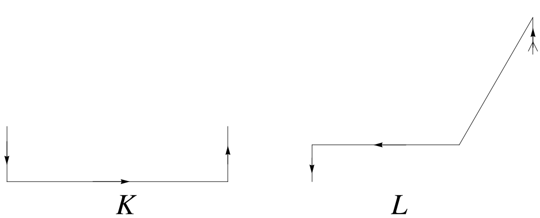

Figure 7. K with type A configuration and L with type B configuration: bottom and top, respectively.

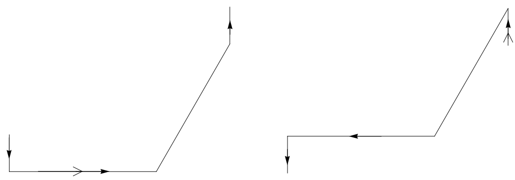

Figure 8. K and L with type B configurations: bottom and top, respectively.

remains of K and L after the type A and B configurations are removed. Note that s will begin with a w and end with a W, whereas t will begin with a W and end with a w. Assuming that $n \neq m$, we scale K by m and scale L by n. We have $K = \tilde{s}x^{nm}$ and $L = X^{nm}\tilde{t}Y^{np}$, where $\tilde{s}$ represents s scaled by m and $\tilde{t}$ represents t scaled by n. (In the case that $n = m$, $\tilde{s} = s$ and $\tilde{t} = t$.) We may now compose K and L, and write $K\#L = \tilde{s}\tilde{t}Y^{np}$. Thus we have a reduction of three sticks for this case — the first for the xs, the second for the Xs and the third for putting end of $\tilde{s}$ together with beginning of $\tilde{t}$. Therefore $s[K\#L] \leq s[K] + s[L] - 3$.

Case 3. Suppose K and L both have type B configurations. Then the bottom and top configurations of K and L, respectively, can be viewed as in Figure 8. Let $K = y^m s x^n$ and $L = X^p t Y^q$, where the strings s and t represent what remains of K and L after the type B configurations are removed. Note that s will begin with a w and end with a W, whereas t will begin with a W and end with a w. Assuming that $n \neq p$, we scale K by p and scale L by n to obtain $K = y^{mp}\tilde{s}x^{np}$ and $L = X^{np}\tilde{t}Y^{nq}$, with $\tilde{s}$ being s scaled by p, and $\tilde{t}$ being t scaled by n. We may now compose K and L, and write

$$K\#L = \begin{cases} y^{mp-nq}\tilde{s}\tilde{t} & \text{if } mp > nq, \\ \tilde{s}\tilde{t}Y^{nq-mp} & \text{if } mp < nq, \\ \tilde{s}\tilde{t} & \text{if } mp = nq. \end{cases}$$

Thus we have a reduction of at least three sticks for $mp \neq nq$ and a reduction of at least six sticks for $mp = nq$. In other words,

$$s[K\#L] \leq \begin{cases} s[K] + s[L] - 3 & \text{if } mp \neq nq, \\ s[K] + s[L] - 6 & \text{if } mp = nq. \end{cases}$$

Thus we have a minimum reduction of three sticks over all cases. Hence,

$$s[K\#L] \leq s[K] + s[L] - 3. \qquad \square$$

6. Knot constructions

Adams, Chu, Crawford, Jensen, Siegel and Zhang [Adams et al. 2012] use constructions combined with the lower bound on stick number to establish that the stick number of the 3-bridge knots 8_{20}, 8_{21}, and 9_{46} are 18 in the simple cubic

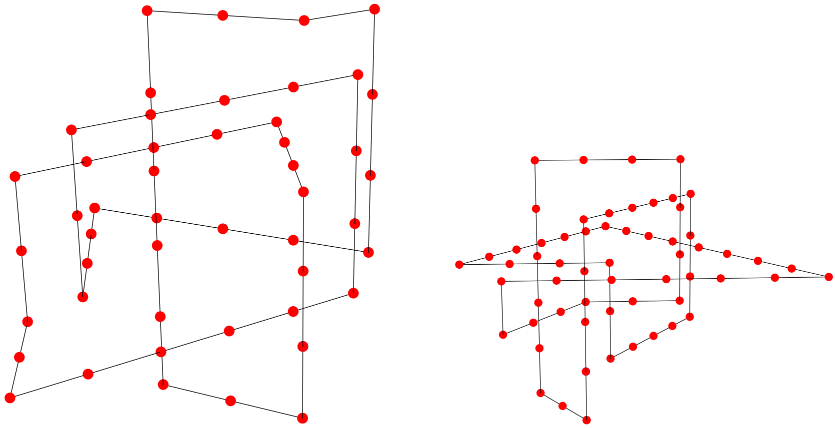

Figure 9. 16-stick hexagonal 8_{20} knot (left) and 8_{21} knot (right).

lattice. In a similar manner, one considers these knots in the simple hexagonal lattice. Figures 9 and 10 show these knots built with 16 sticks. Inspection of these knot constructions does not yield any obvious one stick reductions. Using the constructions and Theorem 1, one gets the following theorem.

Theorem 5. *In the simple hexagonal lattice, knots* 8_{20}, 8_{21}, *and* 9_{46} *have stick number either* 15 *or* 16.

Another use of knot construction combined with using the lower bound for stick number can been seen with $(p, p{+}1)$-torus knots.

Theorem 6 (stick number for $(p, p{+}1)$-torus knots). *For a* $(p, p{+}1)$*-torus knot* K, $5p \leq s[K] \leq 5p+1$.

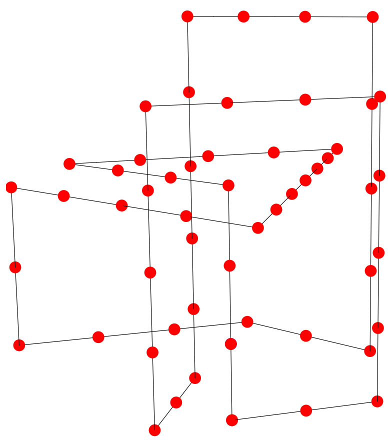

Figure 10. 16-stick hexagonal 9_{46} knot.

Proof. Consider a $(p,\, p{+}1)$-torus knot K which can be constructed in the simple hexagonal lattice in the following way:

$$Yw^pX^{3+p(p-1)/2}y^pWx^{3+\alpha}\prod_{i=0}^{p-2}(Y^{3-i+\alpha}w^{2i+2}z^{2-i+\alpha}W^{2i+3}x^{3-i+\alpha}),$$

where $\alpha=(p-2)(p-1)/2$ and an exponent on a letter refers to the edge length of the stick. Notice there are $5p+1$ sticks used in this construction. In [Schubert 1954], it is shown that $b[K]=p$. Using Theorem 1, we have $s[K]\geq 5p$. Therefore, $s[K]=5p$ or $s[K]=5p+1$. □

Corollary 7. *For a* $(p,\, p{+}1)$*-torus knot* K, $10p-5\leq s[K\#K]\leq 10p-4$.

Proof. Using two configurations of type B, one sees from Theorem 4 that

$$s[K\#K]\leq 2(5p+1)-6=10p-4.$$

On the other hand, [Schubert 1954] says

$$b[K\#K]=2b[K]-1=2p-1,$$

and Theorem 1 yields

$$s[K\#K]\geq 5b[K]\geq 10p-5.$$ □

7. Further work

With all the constructions in the previous section where it is not obvious how to reduce the stick number, it leads one to conjecture that the stick number of a knot is one more than five times its bridge number. It would be nice to prove this improved lower bound or find an example to demonstrate why the standing lower bound is sharp.

Conjecture. *For any knot K in the simple hexagonal lattice*, $s[K]\geq 5b[K]+1$.

One could try to extend the results to other lattices such as the face-centered cubic lattice and the body-centered cubic lattice. Preliminary investigations of lower bounds for minimal stick number are not great; following similar inequality arguments for these two lattices yields lower bounds of 7 and 8 respectively for 2-bridge knots but has been conjectured to be 9 and 12 via knot constructions [Mann et al. 2012]. A cursory inspection of upper bounds for stick numbers of composite knots suggests that one cannot do better than being subadditive. That is, the stick number of a composite knot is less than or equal to the sum of the stick numbers.

Acknowledgements

We would like to thank the reviewer for very helpful comments. We would also like to thank the NSF for its support; all authors were supported by DMS NSF grant 1062740 during the summers of 2011 and 2013.

References

[Adams et al. 1997] C. C. Adams, B. M. Brennan, D. L. Greilsheimer, and A. K. Woo, "Stick numbers and composition of knots and links", *J. Knot Theory Ramifications* **6**:2 (1997), 149–161. MR 98h:57010 Zbl 0884.57005

[Adams et al. 2012] C. Adams, M. Chu, T. Crawford, S. Jensen, K. Siegel, and L. Zhang, "Stick index of knots and links in the cubic lattice", *J. Knot Theory Ramifications* **21**:5 (2012), 1250041. MR 2902272 Zbl 1239.57008

[Hong et al. 2013] K. Hong, S. No, and S. Oh, "Upper bound on lattice stick number of knots", *Math. Proc. Cambridge Philos. Soc.* **155**:1 (2013), 173–179. MR 3065265 Zbl 1270.57022

[Huh and Oh 2005] Y. Huh and S. Oh, "Lattice stick numbers of small knots", *J. Knot Theory Ramifications* **14**:7 (2005), 859–867. MR 2006g:57011 Zbl 1085.57005

[Mann et al. 2012] C. E. Mann, J. C. McLoud-Mann, and D. P. Milan, "The stick number for the simple hexagonal lattice", *J. Knot Theory Ramifications* **21**:14 (2012), 1250120. MR 3021758 Zbl 1270.57029

[Janse van Rensburg and Promislow 1999] E. J. Janse van Rensburg and S. D. Promislow, "The curvature of lattice knots", *J. Knot Theory Ramifications* **8**:4 (1999), 463–490. MR 2000i:57009 Zbl 0940.57013

[Schubert 1954] H. Schubert, "Über eine numerische Knoteninvariante", *Math. Z.* **61** (1954), 245–288. MR 17,292a Zbl 0058.17403

Received: 2013-10-21 Revised: 2014-05-21 Accepted: 2014-05-23

rlb3624@utexas.edu — *The University of Texas at Austin, Department of Mathematics, 1 University Station C1200, Austin, TX 78712, United States*

chaumont@math.wisc.edu — *Department of Mathematics, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706, United States*

melanie.n.dennis.gr@dartmouth.edu — *Department of Mathematics, Dartmouth College, 27 North Main Street, Hanover, NH 03755, United States*

jmcloud@uw.edu — *Division of Engineering and Mathematics, University of Washington Bothell, Box 358538, 18115 Campus Way NE, Bothell, WA 98011, United States*

elisemc93@gmail.com — *Manteca, CA 95337, United States*

smelvin@uttyler.edu — *Department of Mathematics, The University of Texas at Tyler, 3900 University Boulevard, Tyler, TX 75799, United States*

geoffrey.schuette@mavs.uta.edu — *Department of Mathematics, The University of Texas at Arlington, 411 South Nedderman Drive, 478 Pickard Hall, Arlington, TX 76019, United States*

involve

msp.org/involve

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2015 is US $140/year for the electronic version, and $190/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2015 vol. 8 no. 3