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(Communicated by Kenneth S. Berenhaut)

In this paper, we classify and compute the convex foldings of a particular rhombus that are obtained via a zipper folding along the boundary of the shape. In the process, we explore computational aspects of this problem; in particular, we outline several useful techniques for computing both the edge set of the final polyhedron and its three-dimensional coordinates. We partition the set of possible zipper starting points into subintervals representing equivalence classes induced by these edge sets. In addition, we explore nonconvex foldings of this shape which are obtained by using a zipper starting point outside of the interval corresponding to a set of edges where the polygon folds to a convex polyhedron; surprisingly, this results in multiple families of nonconvex and easily computable polyhedra.

## 1. Introduction

A folding of a polygon is a gluing together of the points on the perimeter to form a polyhedron. A theorem of Alexandrov [1958] shows that as long as the sum of the angles at every glued point is no more than  $2\pi$ , every folding of a convex polygon leads to unique convex polyhedron (in which a doubly covered polygon is considered a flat polyhedron). If a folding meets the requirements for Alexandrov's theorem then we are given the existence of a convex polyhedron corresponding to the folding. A more recent constructive proof by Bobenko and Izmistiev [2008] allows for the explicit construction of a polyhedron by solving a certain differential equation. An implementation of the constructive algorithm has been coded by Stefan Sechelmann<sup>1</sup>. Given any input triangulation of the polygon with gluing instructions, the implementation will output the final polyhedron. However, this particular implementation does not return the corresponding triangulation on the polygon. The algorithm runs in pseudopolynomial time since the algorithm must take the initial triangulation and flip it to a geodesic triangulation, which is not, in general, a polynomial time operation [Kane et al. 2009].

MSC2010: 68U05.

Keywords: computational geometry, folding algorithms, combinatorial geometry.

<sup>1</sup>[www3.math.tu-berlin.de/geometrie/ps/software.shtml#AlexandrovPolyhedron](http://www3.math.tu-berlin.de/geometrie/ps/software.shtml#AlexandrovPolyhedron)

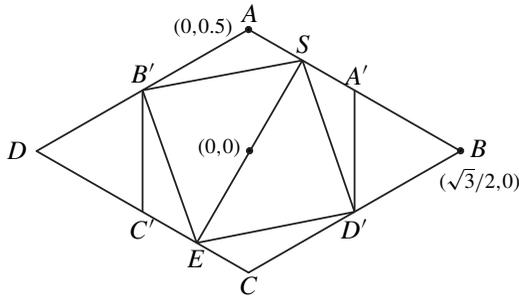
We seek a more combinatorial approach to computing this information. Given a set of gluing rules corresponding to a zipper folding, we outline an approach for computing the crease patterns (i.e., characterize and predict the combinatorial structure of edges and faces) as well as the exact location of the vertices.

**Related work.** Work has been done towards determining all the combinatorially different convex polyhedra obtained via foldings, primarily for regular convex polygons as well as a few other shapes such as the Latin cross [Lubiw and O'Rourke 1996; Alexander et al. 2003; Akiyama and Nakamura 2003; 2004; 2005]. In each work, the authors must determine the set of line segments in the polygon which become edges in the final polyhedron; we refer to these edges as the *crease pattern* for the shape. Note that the crease pattern may not contain all boundary edges of the original polygon; see the left picture in Figure 6 for an example of when the polyhedral edges cross the boundary of the original polygon.

In [Alexander et al. 2003] (and later in [Demaine and O'Rourke 2007]), all (combinatorially distinct) convex polyhedra that are foldable from a square are determined using a combinatorial structure called gluing trees. Crease patterns and reconstructions of the folded polyhedra are also given, making the study of foldings of the square complete. In [Akiyama and Nakamura 2003; 2004; 2005], the focus is on determining all foldings of regular  $n$ -gons, without focusing on reconstructing the actual polyhedron. There is also related work which examines when the Platonic solids can be unzipped to a polygonal net and reziped into a doubly covered flat polygon [O'Rourke 2010]; another paper considers finding different tetrahedra which unzip to a common polygonal net [O'Rourke 2011]. A complete analysis of polyhedra that are zipper foldable from the  $1 \times 2$  rectangle is given in [Schwent 2013], utilizing the techniques outlined here.

As previously mentioned, our primary goal here is to seek a simpler combinatorial approach to verify correct crease patterns in a restricted type of folding. To that end, we consider a restricted class of foldings using the perimeter-halving method, where the perimeter of the polygon is identified starting from a specified point gluing together points equidistant from the starting point (as measured along the perimeter), which zips up the boundary of the polygon into a polyhedron. We will use the term *zipper foldings*, which was first introduced in Demaine et al. [2010]; a related special case is the class of pita polyhedra which arise from zipper folding regular polygons [Demaine and O'Rourke 2007]. As far as the actual resulting polyhedra, surprisingly little is known even about this simple class of foldings aside from the previously mentioned papers which (like ours) consider a particular shape and examine it in detail [Akiyama and Nakamura 2003; 2004; 2005; Alexander et al. 2003].

**Predicting creases.** As was observed by Alexandrov and noted in [Alexander et al. 2003], there are a finite number of possible crease patterns. However, in our



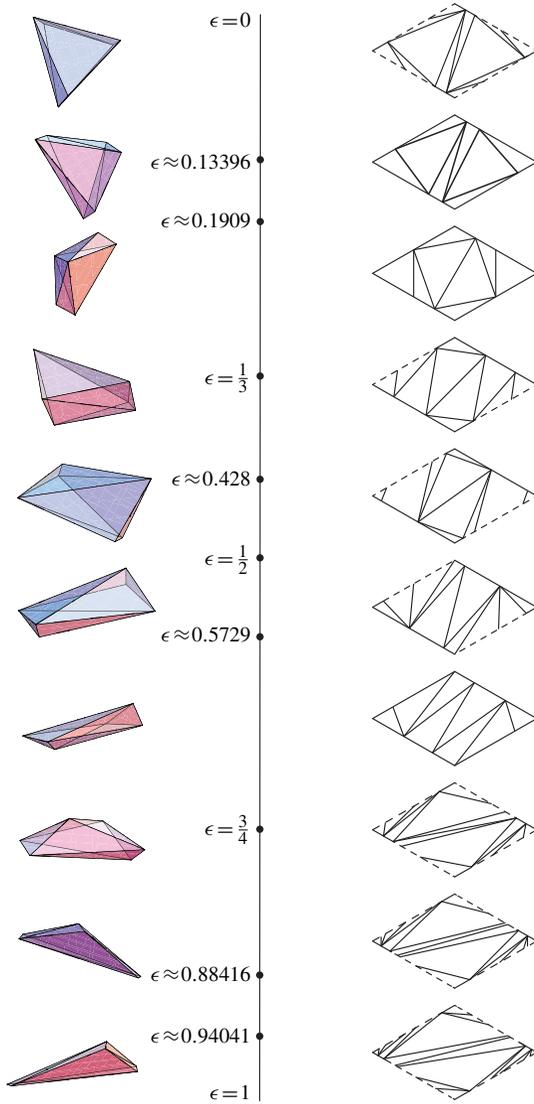
**Figure 1.** The crease pattern at  $\epsilon = \frac{1}{4}$ , which has a 3-regular adjacency graph.

experience, verifying or discounting a crease pattern is surprisingly difficult in more complex polygons since checking a crease pattern either involves seeing if a paper model will fold (highly prone to error) or attempting to compute the folding in a program such as Mathematica (which can lead to numerical issues). As a result, most prior combinatorial work on computing zipper foldings was done using ad hoc methods.

We describe now points on the polygon which will be of interest as we construct the corresponding polyhedron via zipper folding. Our initial polygon is the equilateral rhombus (or diamond) centered on the origin with unit edge length and interior angles  $60^\circ$  and  $120^\circ$ . Label the vertices  $A, B, C, D$ ; see Figure 1 for an example of the labeling. Let the starting point of our zipping  $S$  be a point on edge  $AB$  located at  $(\sqrt{3}\epsilon/2, (1-\epsilon)/2)$ , and refer to the location of  $S$  by this  $\epsilon$ . Note that  $0 \leq \epsilon \leq 1$ , and the location of  $S$  is distance  $\epsilon$  from point  $A$  along edge  $AB$ . The point  $E$  on edge  $CD$  is the reflection of  $S$  through the origin, which is where the zipper ends. For any such folding, we will use  $A', B', C', D'$  to denote the points on the boundary of the polygon which glue to  $A, B, C, D$ , respectively.

As previously mentioned, for polygonal foldings in general, it is known that if the requirements for Alexandrov's theorem are satisfied then there exists a valid folding. Here, we present several computational reconstruction techniques which may be of interest in this area. We also develop several methods to prove that a particular crease pattern is valid as the starting point moves along a continuous interval on the boundary; previous papers seem to have relied on numerical approximation to verify validity, which will reach its limit as the polygon becomes more complex. We also classify all the zipper foldings resulting from the diamond outlined above.

**Theorem 1.** *There are 21 combinatorially distinct convex polyhedra resulting from zipper foldings of a diamond. There are 7 polyhedra which have nontriangular faces and 4 flat polyhedra, all of which occur at isolated points where the crease pattern changes.*



**Figure 2.** All the crease patterns for the zipper foldings of the diamond as the start point  $S$  varies by distance  $\epsilon$  from point  $A$  along edge  $AB$ . Images are taken from sample values between each transition point, marked with solid dots. Dashed lines indicate that a crease extends over an edge. The polyhedra shown between transition points correspond to the respective crease patterns. The symmetry of our input shape allows us to study all zipper foldings by varying the location of  $S$  along one edge; moving  $S$  to another edge will give a combinatorially equivalent folding.

The polyhedra shown in [Figure 2](#) represent the 10 polyhedra with triangular faces, and the solid dots represent the 11 isolated polyhedra noted above. The polyhedra with triangular faces form octahedra. Together, these represent all the zipper foldings of the diamond.

## 2. Computing the foldings

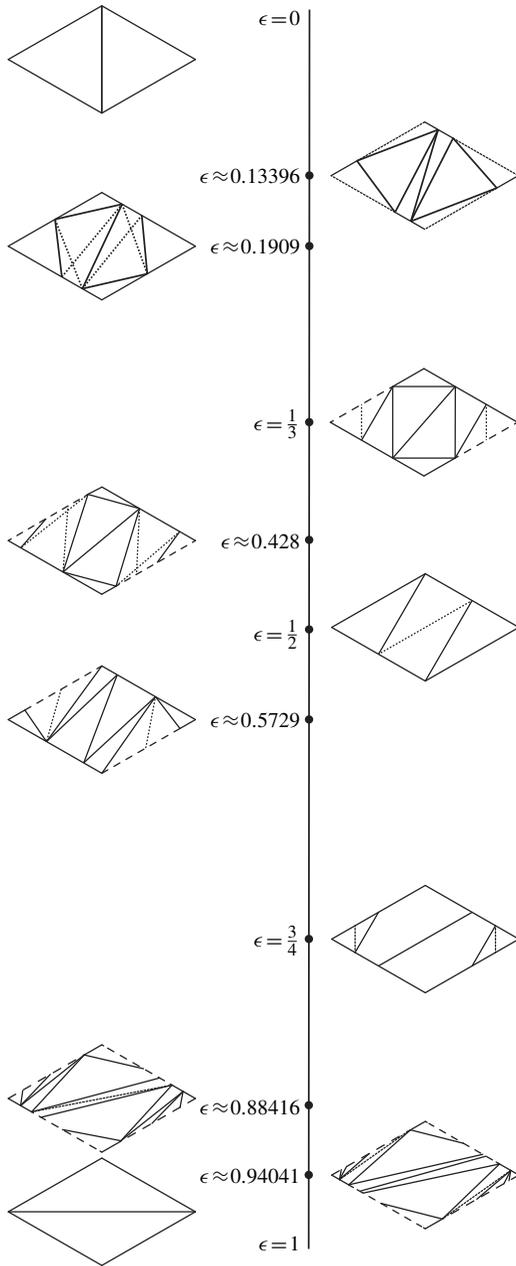
In our foldings, all polyhedra will have at most 6 vertices, resulting from gluing each of  $A$ ,  $B$ ,  $C$ , and  $D$  to some other point on the perimeter, as well as the vertices  $S$  and  $E$ . We are often interested in the actual adjacencies in the final folded polyhedron; this network of edges forms an adjacency graph, often called the graph of the polyhedron, on the (at most) 6 vertices. We refer to this adjacency graph as the *crease pattern*.

Our techniques for computing these foldings break down into several relevant categories. The first (and simplest) are the flat foldings when the entire polygon folds into a doubly covered polygon. For example, when  $\epsilon = 0$ , the vertices  $B$  and  $D$  zip together and the result is a flat doubly covered regular triangle; flat foldings also occur when  $\epsilon = 0.5, 0.75$ , and 1.

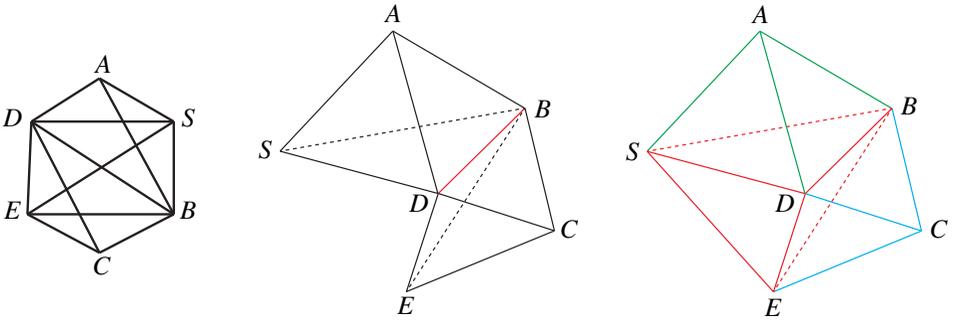
The remaining cases in our computation are handled based on whether the graph of the polyhedron is 4-regular or not; if not, in our shape, as well as in the  $1 \times 2$  rectangle studied in [[Schwent 2013](#)], the graph will always consist of vertices of degrees 3, 4, and 5. When degree-3 vertices exist, as discussed in [Section 2.1](#), computing the crease pattern is much simpler since it is not difficult to verify that the underlying structure of the polyhedron can be decomposed into several tetrahedra. The more complex 4-regular case requires additional techniques to calculate exactly; we detail these techniques in [Section 2.2](#). In addition, further complexity arises when the boundary edges of the initial polygon do not become edges in the final polyhedron; see, for example, the crease patterns in [Figure 2](#) which are nearest to  $\epsilon = 1$ . These patterns, which occur much more often in this shape than previous related work, required an extra set of tools to calculate correct crease patterns and 3-D realizations. In [Section 2.3](#), we examine these tools which require zipper folding a related *nonconvex* polygon to yield the same polyhedron.

In [Figure 3](#), we show the creases with marked points for the places where the crease pattern undergoes a combinatorial change, which we call a *transition*. Note that (as in previous work) at most of these transitions, two triangles become coplanar to form a quadrilateral (indicated as a dotted line) and then the opposite quadrilateral diagonal appears in the polyhedron. All other transitions occur when the polyhedron folds to a flat doubly covered polygon.

**2.1. Degree-3 vertices in the pattern.** Crease patterns with at least one degree-3 vertex are substantially easier to realize in  $\mathbb{R}^3$ , computationally speaking. In this



**Figure 3.** All the crease patterns at each transition point (which are marked by black dots). Dashed lines indicate that a crease extends over an edge. Dotted lines indicate interior diagonals of a quadrilateral face that are realized as polyhedral edges on either side of the transition point.



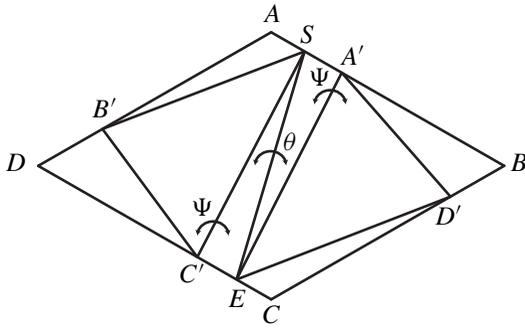
**Figure 4.** Left: The graph of the polyhedron for  $\epsilon = \frac{1}{4}$ . Middle: The two outer tetrahedra of the polyhedron joined along the common edge  $\overline{BD}$ . Right: The final polyhedron decomposed into three tetrahedra.

shape, this results from the fact that when we have such a graph with our setup, we can decompose the final polyhedron into three tetrahedra (two outer tetrahedra and the inner tetrahedron). In Figure 4, we show the adjacency graph of the polyhedron generated when  $\epsilon = \frac{1}{4}$ , the reconstruction of the two outer tetrahedra, and the final polyhedron decomposed into three tetrahedra, where the inner tetrahedron is composed of two triangles from the outer tetrahedra which meet on an edge, plus a single additional edge.

For values of  $\epsilon$  in intervals with a degree-3 vertex in the crease pattern, we wrote code to find exact coordinates for the three-dimensional polyhedron that results. Reconstructing a tetrahedron using adjacencies and edge lengths is not difficult to do, so the general approach we used was to reconstruct the inside tetrahedron shown in Figure 4, and then reconstruct the outer tetrahedra. We next illustrate this process via an example.

Consider the crease pattern at  $\epsilon = \frac{1}{4}$ . This crease pattern contains the edges  $SE$  and  $BD$ . In its initial configuration, we note that points  $B$  and  $D$  are both adjacent to  $SE$  as well as to each other. We can leave edge  $SE$  fixed in the  $z=0$  plane. Rotate points  $B'$  and  $D'$  by  $\theta$  about edge  $SE$  into the positive  $z$ -direction. We solve for the value of  $\theta$  which positions points  $B'$  and  $D'$  at the correct final distance  $|B'D| = 2\epsilon$  from each other; this establishes a central tetrahedron within our final polygon. Vertex  $A$  is adjacent to  $B'$ ,  $D'$ ,  $S$ , and hence can be located by solving a system of three distance equations. Similarly,  $C$  is adjacent to  $B'$ ,  $D'$ ,  $E$ . The resulting figure is convex and has edge lengths that match those from the polygonal net.

To extend from a specific value of  $\epsilon$  to the entire interval containing  $\epsilon$ , we note that the ability to construct the central tetrahedron  $BDSE$  for  $S$  corresponding to a specific  $0.1909 < \epsilon < \frac{1}{3}$  can be verified via an intermediate value theorem argument. Simply measure the distance between  $B$  and  $D$  when the dihedral angle at  $SE$  is 0

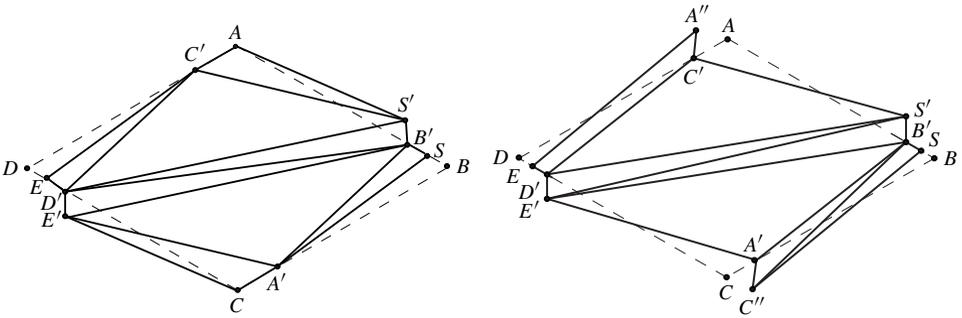


**Figure 5.** A crease pattern whose adjacency graph is 4-regular to illustrate the reconstruction process.

and again when it is  $\pi$ . If the desired length of  $BD$  is between these two values, then folding over  $SE$  by some angle  $0 \leq \theta \leq \pi$  will attain the correct length for  $BD$ . Then, checking the angle criterion given in Lemma 2 (see the Appendix) confirms that locations for points  $A$  and  $C$  can be found that realize all desired distances. It remains only to verify that the resulting polyhedron is convex. Since for  $\epsilon$ -values in this interval, the orthogonal projection of  $A$  onto the plane containing triangle  $BDS$  is interior to triangle  $BDS$ , the final polyhedron will be convex.

**2.2. 4-regular graph of the polyhedron.** In folding patterns where all vertices are degree-4, realization of the vertices in  $\mathbb{R}^3$  is not as simple as the degree-3 case. In [Demaine and O’Rourke 2007], the authors describe a method for constructing an octahedron by splitting it into two smaller hexahedra which share an edge that is an internal diagonal of the octahedron. They vary the length of this edge until the dihedral angles of the faces incident to the edge match. We utilize a different method that also reduces a partial polyhedron to one parameter of change. We illustrate this for  $\epsilon$  in the interval  $0.13396 < \epsilon < 0.1909$ ; the crease pattern for this range is shown in Figure 5.

We consider the following flex over edge  $SE$ . Fold triangles  $SEA'$  and  $SEC'$  upward from the  $z = 0$  plane, each by angle  $\theta$ , leaving edge  $SE$  fixed in the plane. Each choice of  $\theta$  results in a fixed measure of  $\angle A'SC'$  and  $\angle A'EC'$ . The two remaining triangular faces  $SB'C'$  and  $ASB'$  which are incident to  $S$  (or respectively  $E$ ) are uniquely configurable into a shell comprised of faces of the final convex polyhedron. That is, of the two locations in  $\mathbb{R}^3$  for point  $B_\theta$  that give correct distances for segments  $AB$ ,  $BC$ ,  $BS$ , only one is extendable into a convex polyhedron. We similarly find a location for  $D_\theta$ . (The subscripts here serve as a reminder that the locations of  $B_\theta$  and  $D_\theta$  depend on the initial flex by angle  $\theta$ .) Note that this convex shell contains six of the eight faces of the final polyhedron; the two missing faces must share a common edge. We then vary  $\theta$  to realize the correct length for this



**Figure 6.** The crease patterns for  $\epsilon = \frac{7}{8}$  (left) and the range just below 1 (right) rearranged to a nonconvex polygon.

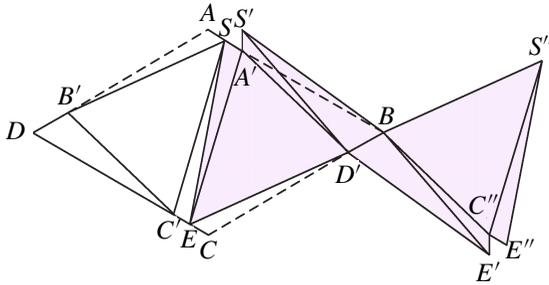
missing common edge; if no such  $\theta$  exists, then we can reject this crease pattern. Moreover, we can also reject the pattern if the final folding results in a nonconvex polyhedron; else, it must realize a convex folding of the initial crease pattern.

**2.3. Creases over the boundary of the polygon.** While the particular approach varies slightly, this process from the previous section can be repeated for any 4-regular graph of the polyhedron. However, some complications arise when the crease pattern is more complex. For example, consider the crease pattern when the source of the zipper  $S$  is near  $B$ . In this case, many of the edges in the final polyhedron actually cross an edge of the initial polygon since not all of the polygon's edges are edges of the final polyhedron. (This also occurs at several other positions; see Figure 2.) Computationally speaking, these patterns are more difficult because a single crease is split into different segments inside the polygon. In order to compute these foldings, we altered the original polygon to be *nonconvex* and verified the crease pattern in this related polygon.

One example occurs when the zipper point reaches near point  $B$  in our shape for the crease patterns above  $\epsilon = 0.75$ ; see Figure 6 for the pattern at  $\epsilon = \frac{7}{8}$ . Here, the creases cross over the edges  $AB'$  (and by symmetry also  $BA'$ ) as well as  $C'D$  (and  $DC'$ ). Using the original gluing information, we reconstructed an equivalent nonconvex polygon which folded to an identical polyhedron and allowed for easier computation, given the symmetry and reduction in the number of creases.

This set of crossings becomes even more drastic as the zipper point nears  $B$ , which is a vertex of high curvature. The crossing edges do not change, but the rearranged figure becomes more complex due to extra crossings, and the nonconvex polygon in turn becomes more complex. See Figure 6 for the final rearranged figure just below  $\epsilon = 1$ .

A very different example occurs for the crease pattern at  $\epsilon = \frac{1}{10}$ ; here, instead of keeping the shape close to the original diamond, we more drastically rearrange



**Figure 7.** A crease pattern for a convex polyhedron when  $\epsilon = \frac{1}{10}$ . The initial net is diamond  $ABCD$ . The shaded faces are shown rearranged to match the algorithmic approach to reconstruction.

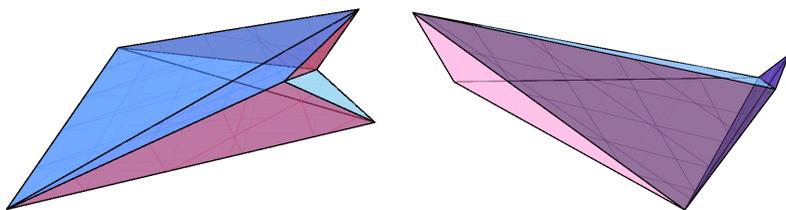
to take advantage of symmetry when computing the folding. This crease pattern contains the edges  $SE$  and  $BD$ . However, in its initial configuration, we note that points  $A'$  and  $C'$  are both adjacent to  $SE$ , but  $A$  and  $C$  are not adjacent to each other in the final polyhedron. We use the gluing instructions to rearrange the triangular faces so that they are as in [Figure 7](#). Now vertices  $S$  and  $E$  are both incident to edge  $BD$ , so we can fold this polygon symmetrically, leaving edge  $BD$  fixed in the  $z = 0$  plane and proceed exactly as outlined in the  $\epsilon = \frac{1}{4}$  case described in [Section 2.1](#).

For nets where the vertices incident to the crease through  $(0, 0)$  are adjacent, this rearranging of the net is not always needed. We do take advantage of this rearrangement technique whenever a three-dimensional edge of our final polyhedron intersects a two-dimensional boundary edge of our initial polygon. Since the gluing instructions are preserved, this is merely a bookkeeping tool that allows for easier computations.

### 3. Nonconvex polyhedra

One interesting result of our investigation of this pattern is a natural classification of some types of *nonconvex* foldings, which to the best of our knowledge have not been a focus of investigation in related work on zipper foldings. It is known, of course, that convex shapes will fold to nonconvex polyhedra, and work has been done on counting the number of foldings of a shape; see, for example, [\[Demaine et al. 2000\]](#). In addition, recent work has focused on unfolding a polyhedron to a convex shape and then refolding it to a different (convex) polyhedron [\[Demaine et al. 2012\]](#); in contrast, our results consider zipper folding a convex planar shape to one or more nonconvex polyhedra, which seems to be an interesting variant of refold rigidity.

The main point of interest is how easy these nonconvex foldings are to find computationally speaking. These foldings result from pushing a particular crease pattern past the point where two faces become coplanar and a flip in the crease



**Figure 8.** Left: A nonconvex folding when the zipper source is at  $\epsilon = .36$ . Right: A nonconvex folding when  $\epsilon = \frac{1}{6}$ ; here, the folding results in a flat flap, indicating that this crease pattern will not fold to a polyhedron at all when pushed lower than  $\frac{1}{6}$ .

pattern occurs. In our experiments, the primary method to establish the validity of a crease pattern is by finding a solution in  $\mathbb{R}^3$  to a system of quadratic equations defining pairwise distances, then checking the convexity of the resulting polyhedron. These nonconvex foldings appeared when the code for computing a solution ran successfully but failed the convexity check.

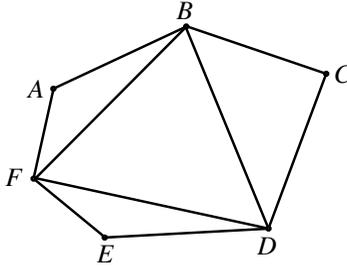
For an example of these, consider [Figure 8](#). In the example shown on the left, we consider when the zipper point is at  $\epsilon = 0.36$ . However, instead of using the correct crease pattern shown in [Figure 2](#), we are instead using the crease pattern for the interval below  $\epsilon = \frac{1}{3}$ . Similarly, on the right side of [Figure 8](#), we have a nonconvex folding when  $\epsilon = \frac{1}{6}$ , but the crease pattern used is the one that is valid for the interval above  $\epsilon \approx 0.1909$ . In this second picture, we have actually pushed the nonconvex folding as far as it will extend, since using this crease pattern for any lower value of  $\epsilon$  will result in an invalid folding (where the polygon self-intersects).

These calculations lead us to conjecture that any valid crease pattern over an interval will fold to a nonconvex polyhedron for some value of  $\epsilon$  close to the interval of convexity. This conjecture is certainly true in our shape (except for near flat foldings), and it seems likely to hold for other shapes since convexity does not impact the existence of a solution.

#### 4. Future directions

We have focused here on zipper foldings of this particular shape. In much of the previous work in this area, all the foldings of a convex shape have been determined using techniques such as gluing trees. Using those techniques to calculate all the convex foldings of this diamond remains an area to address.

Another interesting question is to determine the relationship between the zipper foldings we discuss here with the zipper foldings of the square, as they appear similar [[Alexander et al. 2003](#); [Demaine and O'Rourke 2007](#)]. A link between the diamond and square zipper foldings might give a list of constraints for when two



**Figure 9.** Corresponding figure for [Lemma 2](#).

similar figures have similar foldings. Related lines of questioning could be asked about rhombi in general. For instance, if the edge lengths are similarly defined, are the values of  $\epsilon$  where transitions occur similar?

The nonconvex foldings described in [Section 3](#) are also perhaps worth further investigation in other shapes. It would also be interesting to examine when these nonconvex foldings cease to be valid, and to try to discover how many valid (nonconvex) crease patterns might be present at a particular zipper point.

### Appendix: Realizing tetrahedra

In our discussion of calculating the folding where there is a degree-3 vertex in the graph of the polyhedron, we need a characterization of when a set of vertices and edges can be realized in  $\mathbb{R}^3$  as a tetrahedron. We then use this to help us discover the entire range along which the tetrahedron is present in the final folding. We summarize this tool in the following lemma:

**Lemma 2.** *A net of four triangles as shown in [Figure 9](#) will fold to a tetrahedron if*

- (1) *lengths of corresponding sides are equal ( $|AF| = |EF|$ ,  $|AB| = |BC|$ , and  $|CD| = |DE|$ );*
- (2) *at each vertex, the angle of the base is less than the sum of the other two incident face angles.*

*Proof.* It is clear that the first condition is necessary since, within the tetrahedron, points  $A, C, E$  will all be identified and thus corresponding edge lengths must be the same. To verify the second condition, we show that, without loss of generality, all points incident to  $F$  can be realized in three dimensions. Assume  $\angle BFD \leq \angle AFB + \angle DFE$ . If this condition were not met, then no position of point  $A$  rotated over segment  $BF$  will coincide with any position of point  $E$  rotated over  $DF$ . When the angle criterion is satisfied, let  $X$  be a point in  $\mathbb{R}^3$  where points  $A$  and  $E$  coincide after rotations over  $BF$  and  $DF$  respectively. We know by condition (1) that  $|XB| = |AB| = |CB|$  and  $|XD| = |ED| = |CD|$ , so triangle  $\triangle BCD$  will fold into position over edge  $BD$  with  $C$  identified with point  $X$  to complete the tetrahedron.  $\square$

## Acknowledgements

This research was supported in part by the National Science Foundation under Grant No. CCF 1054779, as well as an REU supplemental to that grant.

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Received: 2014-01-31    Revised: 2014-03-20    Accepted: 2014-07-01

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
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2015

vol. 8

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