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and injective labelings of general graphs
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An $L(2, 1)$-labeling of a graph $G$ is a function assigning a nonnegative integer to each vertex such that adjacent vertices are labeled with integers differing by at least 2 and vertices at distance two are labeled with integers differing by at least 1. The minimum span across all $L(2, 1)$-labelings of $G$ is denoted $\lambda(G)$. An $L'(2, 1)$-labeling of $G$ and the number $\lambda'(G)$ are defined analogously, with the additional restriction that the labelings must be injective. We determine $\lambda(H)$ when $H$ is a join-page amalgamation of graphs, which is defined as follows: given $p \geq 2$, $H$ is obtained from the pairwise disjoint union of graphs $H_0, H_1, \ldots, H_p$ by adding all the edges between a vertex in $H_0$ and a vertex in $H_i$ for $i = 1, 2, \ldots, p$. Motivated by these join-page amalgamations and the partial relationships between $\lambda(G)$ and $\lambda'(G)$ for general graphs $G$ provided by Chang and Kuo, we go on to show that $\lambda'(G) = \max\{n_G - 1, \lambda(G)\}$, where $n_G$ is the number of vertices in $G$.

1. Introduction

In a well-studied model of the classic channel assignment problem introduced in [Hale 1980], each vertex of a graph $G$ represents a transmitter in a communications network, and edges connect vertices corresponding to transmitters operating in close proximity which must receive sufficiently different frequencies to avoid interference. In a simplified instance of the problem, a frequency assignment is represented by an $L(2, 1)$-labeling of $G$, which is a function $f$ from the vertex set to the nonnegative integers such that $|f(x) - f(y)| \geq 2$ if vertices $x$ and $y$ are adjacent and $|f(x) - f(y)| \geq 1$ if $x$ and $y$ are at distance two. $L(2, 1)$-labelings and their variations have been studied extensively since their introduction in [Griggs and Yeh 1992] (see the surveys [Calamoneri 2011; Griggs and Král 2009; Yeh 2006]) and continue to generate a rich literature to this date (see a sample of the

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most recent works in [Calamoneri 2013; Franks 2015; Karst et al. 2015; Li and Zhou 2013; Lin and Dai 2015; Lu and Zhou 2013; Shao and Solis-Oba 2013]).

An $L(2, 1)$-labeling of a graph $G$ that uses labels in the set $\{0, 1, \ldots, k\}$ will be called a $k$-$L(2, 1)$-labeling. The minimum $k$ so that $G$ has a $k$-$L(2, 1)$-labeling is called the $\lambda$-number of $G$, denoted by $\lambda(G)$. Griggs and Yeh [1992] conjectured that $\lambda(G) \leq \Delta^2(G)$, where $\Delta(G)$ denotes the maximum degree of $G$. This conjecture holds for $\Delta(G) \geq 10^6$ [Havet et al. 2012], but it remains open even when $\Delta(G) = 3$. The best general upper bound yet established is $\lambda(G) \leq \Delta^2(G) + \Delta(G) - 2$ [Gonçalves 2008]. Recently, it has been proven that this conjecture also holds for small enough graphs, namely, graphs with at most $(\lceil \Delta(G)/2 \rceil + 1)(\Delta^2(G) - \Delta(G) + 1) - 1$ vertices [Franks 2015]. As the general problem of determining $\lambda(G)$ is NP-hard [Georges et al. 1994], a significant body of literature has focused on finding bounds or exact $\lambda$-numbers for particular classes of graphs. In particular, [Adams et al. 2013] focused on the amalgamations of graphs.

**Definition 1.1.** Let $H_1, H_2, \ldots, H_p$ be $p \geq 2$ graphs each containing a fixed induced subgraph isomorphic to a graph $H_0$. The amalgamation of $H_1, H_2, \ldots, H_p$ along $H_0$ is the simple graph $H = \text{Amalg}(H_0; H_1, H_2, \ldots, H_p)$ obtained by identifying $H_1, H_2, \ldots, H_p$ at the vertices in the fixed subgraphs isomorphic to $H_0$ in each $H_1, H_2, \ldots, H_p$ respectively. $H_0$ is referred to as the spine and $H_k$ as the $k$-th page of the amalgamation for $k = 1, 2, \ldots, p$. (We refer the reader to [Adams et al. 2013] for some concrete examples.)

In [Adams et al. 2013], upper bounds for the $\lambda$-number of the amalgamation of graphs along a given graph were established by determining the exact $\lambda$-number of amalgamations of complete graphs along a complete graph. They also provided the exact $\lambda$-numbers of amalgamations of rectangular grids along a path, or more specifically, of the Cartesian products of a path and a star with spokes of arbitrary lengths. This focus on the Cartesian products motivated us to investigate amalgamations of the join of graphs.

**Definition 1.2.** Let $G_1$ and $G_2$ be two disjoint graphs. The union $G_1 \cup G_2$ is the graph with vertex (resp., edge) set equal to the union of the vertex (resp., edge) sets of $G_1$ and $G_2$. The join $G_1 + G_2$ is obtained from $G_1 \cup G_2$ by adding an edge between each vertex in $G_1$ and each vertex in $G_2$.

**Definition 1.3.** Let $G_0, G_1,$ and $G_2$ be pairwise disjoint graphs. The graph $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2)$ is called a join-page amalgamation of $G_1, G_2$ along $G_0$. Note that $G$ is isomorphic to $G_0 + (G_1 \cup G_2)$.

Definitions 1.2 and 1.3 can be extended for more than two graphs $G_1, G_2$. The $\lambda$-numbers of the union and join of graphs are well known as stated in the next two results.
Result 1.4 [Chang and Kuo 1996, Lemma 3.1]. For any two graphs $G$ and $H$, 
\[ \lambda(G \cup H) = \max(\lambda(G), \lambda(H)). \]

Result 1.5 [Georges et al. 1994, Corollary 4.6]. For any two graphs $G$ and $H$ with $n_G$ and $n_H$ vertices respectively,
\[ \lambda(G + H) = \max\{n_G - 1, \lambda(G)\} + \max\{n_H - 1, \lambda(H)\} + 2. \]

In Section 2, we provide the exact $\lambda$-number for all join-page amalgamations. Motivated by a connection between this $\lambda$-number and the minimum span over injective $L(2, 1)$-labelings, Section 3 revisits these labelings for general graphs which were first introduced in [Chang and Kuo 1996]. More specifically, we establish a new exact relationship between the $\lambda$-number of a graph and the minimum span over all injective $L(2, 1)$-labelings of this graph.

2. The $\lambda$-number of join-page amalgamations

Theorem 2.1. Let $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \ldots, G_0 + G_p)$ be a join-page amalgamation, where $G_i$ is a graph with $n_i \geq 1$ vertices for $i = 0, 1, \ldots, p \geq 2$ so that $n_1 \geq n_j$ for $j = 2, 3, \ldots, p$, and let $n = n_1 + n_2 + \cdots + n_p$. Then,
\[ \lambda(G) = \max\{n_0 - 1, \lambda(G_0)\} + \max\{n - 1, \lambda(G_1)\} + 2. \]

Proof. Since $G$ is isomorphic to $G_0 + (G_1 \cup G_2 \cup \cdots \cup G_p)$, using Results 1.4 and 1.5,
\[
\begin{align*}
\lambda(G) &= \lambda(G_0 + (G_1 \cup G_2 \cup \cdots \cup G_p)) \\
&= \max\{n_0 - 1, \lambda(G_0)\} + \max\{n - 1, \lambda(G_1 \cup G_2 \cup \cdots \cup G_p)\} + 2 \\
&= \max\{n_0 - 1, \lambda(G_0)\} + \max\{n - 1, \lambda(G_1), \lambda(G_2), \ldots, \lambda(G_p)\} + 2.
\end{align*}
\]
For $i = 2, 3, \ldots, p$, we have $\lambda(G_i) \leq \lambda(K_{n_i}) = 2n_i - 2 \leq n_1 + n_i - 2 < n - 1$, where $K_{n_i}$ denotes the complete graph with $n_i$ vertices, and therefore
\[ \max\{n - 1, \lambda(G_1), \lambda(G_2), \ldots, \lambda(G_p)\} = \max\{n - 1, \lambda(G_1)\}, \]
and the desired result follows. }

It is worth noting that Theorem 2.1 implies that $\lambda(G)$ depends on the number of vertices in $G_2, G_3, \ldots, G_p$ but not on their particular $\lambda$-numbers.

The following corollary is equivalent to Theorem 2.3 in [Adams et al. 2013] but with an alternative and more compact proof.

Corollary 2.2. Let $G = \text{Amalg}(K_0; K_0 + K_1, K_0 + K_2, \ldots, K_0 + K_p)$ be a join-page amalgamation, where $K_i$ is the complete graph with $n_i \geq 1$ vertices for $i = 0, 1, \ldots, p \geq 2$ so that $n_1 \geq n_j$ for $j = 2, 3, \ldots, p$, and let $n = n_1 + n_2 + \cdots + n_p$. Then $\lambda(G) = 2n_0 + \max\{n - 1, 2n_1 - 2\}.$
Proof. By Theorem 2.1,
\[ \lambda(G) = \max\{n_0 - 1, \lambda(K_0)\} + \max\{n - 1, \lambda(K_1)\} + 2 \]
\[ = \max\{n_0 - 1, 2n_0 - 2\} + \max\{n - 1, 2n_1 - 2\} + 2 \]
\[ = 2n_0 - 2 + \max\{n - 1, 2n_1 - 2\} + 2 \]
\[ = 2n_0 + \max\{n - 1, 2n_1 - 2\}. \] □

3. A connection between join-page amalgamation and injective \(L(2, 1)\)-labelings

When examining the \(L(2, 1)\)-labelings of a join-page amalgamation of the form \(G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \ldots, G_0 + G_p)\), as described in Theorem 2.1 in Section 2, we noticed that we could extend an injective \(L(2, 1)\)-labeling of \(G_0\) of minimum span over all its injective labelings to a \(\lambda(G)\)-\(L(2, 1)\)-labeling of the entire \(G\). We suspected that this was not a coincidence, which led us to revisit the following variation of \(L(2, 1)\)-labelings introduced in [Chang and Kuo 1996].

Definition 3.1. An \(L'(2, 1)\)-labeling of a graph \(G\) is an injective \(L(2, 1)\)-labeling of \(G\). The definitions of \(k-L'(2, 1)\)-labeling, \(\lambda'\)-number and \(\lambda'(G)\) are analogous to those of \(k-L(2, 1)\)-labeling, \(\lambda\)-number, and \(\lambda(G)\) when restricted to injective labelings.

The following basic properties were previously known.

Result 3.2 [Chang and Kuo 1996, Lemmas 2.1, 2.2, 2.3]. For any graph \(G\) with \(n_G\) vertices, (i) \(\lambda'(H) \leq \lambda'(G)\) for any subgraph \(H\) of \(G\); (ii) \(\lambda(G) \leq \lambda'(G)\) with equality if \(G\) has diameter at most two; and (iii) \(c(G) = \lambda'(G^c) - n_G + 2\), where \(c(G)\) is the path covering number of \(G\), i.e., the smallest number of vertex-disjoint paths needed to cover all the vertices of the graph \(G\), and \(G^c\) is the complement of \(G\).

In Theorem 3.4, we will strengthen Result 3.2(ii) by providing a surprisingly simple exact relationship between \(\lambda(G)\) and \(\lambda'(G)\) for any graph \(G\). We will be using the following auxiliary result in the proof of Theorem 3.4.

Result 3.3 [Georges et al. 1994, Theorem 1.1]. For any graph \(G\) on \(n_G\) vertices, (i) \(\lambda(G) \leq n_G - 1\) if and only if \(c(G^c) = 1\); and (ii) \(\lambda(G) = n_G + c(G^c) - 2\) if and only if \(c(G^c) \geq 2\).

Theorem 3.4. For any graph \(G\) with \(n_G\) vertices,
\[ \lambda'(G) = \max\{n_G - 1, \lambda(G)\}. \]
Proof. Suppose $\lambda(G) \leq n_G - 1$. By Result 3.3(i), $c(G^c) = 1$, and Result 3.2(iii) implies $1 = c(G^c) = \lambda'(G) - n_G + 2$. Therefore,

$$\lambda'(G) = n_G - 1 = \max\{n_G - 1, \lambda(G)\}.$$ 

Assume, on the other hand, that $\lambda(G) > n_G - 1$. Item (i) in Result 3.3 implies $c(G^c) \geq 2$, and item (ii) implies $\lambda(G) = n_G + c(G^c) - 2$, or equivalently, $c(G^c) = \lambda(G) - n_G + 2$. Finally, Result 3.2(iii) implies

$$\lambda'(G) = c(G^c) + n_G - 2 = (\lambda(G) - n_G + 2) + n_G - 2 = \lambda(G) = \max\{n_G - 1, \lambda(G)\}. \quad \Box$$

In view of Theorem 3.4, the general problem of determining the $\lambda'$-number of graphs is as complex as determining their $\lambda$-numbers, which, as mentioned previously, is known to be an NP-hard problem. Furthermore, the exact $\lambda'$-numbers of families of graphs, such as the ones derived in [Chang and Kuo 1996] using more involved techniques (e.g., paths, cycles, union and join of two graphs), can be readily obtained using Theorem 3.4 and the vast list of known exact $\lambda$-numbers in the $L(2, 1)$-labeling literature.

If $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \ldots, G_0 + G_p)$ and we apply Theorem 3.4 to $G_0$ in Theorem 2.1, we obtain a relationship between $\lambda(G)$ and $\lambda'(G_0)$, confirming the connection between injective $L(2, 1)$-labelings of $G_0$ and $L(2, 1)$-labelings of $G$ we mentioned in the first paragraph of this section. The following corollary provides this relationship.

**Corollary 3.5.** Let $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \ldots, G_0 + G_p)$ be a join-page amalgamation, where $G_i$ is a graph with $n_i$ vertices for $i = 0, 1, \ldots, p \geq 2$ so that $n_1 \geq n_j$ for $j = 2, 3, \ldots, p$, and let $n = n_1 + n_2 + \cdots + n_p$. Then $\lambda(G) = \lambda'(G_0) + \max\{n - 1, \lambda(G_1)\} + 2$.

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