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with log-gamma function

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In this paper, we give a general formula for the multiple integral

$$I = \int_0^1 \int_0^1 \dots \int_0^1 f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n.$$

As an application, the integral I with $f(x) = \log \Gamma(x)$ is evaluated for all $n \in \mathbb{N}$. The subsidiary computational challenges are interesting in their own right.

1. Introduction

A general idea, when faced with a multiple integral, is to lower its dimension. A well-known example, (see [Chang and Shi 2003], for instance) is the n -dimensional integral

$$\int_{\substack{x_1+x_2+\dots+x_n \leq 1 \\ x_1, x_2, \dots, x_n \geq 0}} \dots \int f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n, \quad (1-1)$$

which can be simplified to a one-dimensional integral

$$\frac{1}{(n-1)!} \int_0^1 t^{n-1} f(t) dt.$$

However, to the best of our knowledge, a similar integral,

$$I = \int_0^1 \int_0^1 \dots \int_0^1 f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n, \quad (1-2)$$

has no such formula.

The aim of this paper is to find a formula for the above integral I and apply it to the special case when $f(x) = \log \Gamma(x)$. The main results are as follows. A general formula of I is obtained in [Theorem 4.1](#).

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Theorem 4.1. *The integral I satisfies*

$$\begin{aligned}
 I &= \int_0^1 \int_0^1 \dots \int_0^1 f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n \\
 &= \frac{1}{(n-1)!} \sum_{m=1}^n \int_0^1 G_m(t) f(t+m-1) dt,
 \end{aligned}
 \tag{1-3}$$

where

$$G_m(t) = \sum_{i=1}^m (-1)^{i-1} (t+m-i)^{n-1} \binom{n}{i-1}.$$

When $f(x) = \log \Gamma(x)$, the value of I is given in [Theorem 5.1](#). The main challenge of the proof is to find appropriate combinatorial identities to simplify I .

Theorem 5.1.

$$\begin{aligned}
 I &= I(n) = \int_0^1 \int_0^1 \dots \int_0^1 \log \Gamma(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n \\
 &= \frac{1}{2} \log(2\pi) - \frac{n-1}{2} H_n + \sum_{k=2}^{n-1} \frac{(-1)^{n+k+1} k^n}{n!} \binom{n-1}{k} \log k,
 \end{aligned}$$

where the last sum is missing when $n = 2$ and $H_n = \sum_{k=1}^n 1/k$.

The paper is organized as follows. In [Sections 2 and 3](#), we explain the main ideas by using the cases $n = 2$ and 3 . One can see from [Figures 1 and 2](#) how we cut the square and the cube so that the integral I over each subset becomes a simple one-dimensional integral. In [Section 4](#), a formula of I is derived in [Theorem 4.1](#), and in [Section 5](#), we evaluate I when $f(x) = \log \Gamma(x)$.

2. The case $n = 2$

When $n = 2$, the integral I becomes $\int_0^1 \int_0^1 f(x+y) dx dy$, where the integral domain is a unit square. Let $t = x+y$. The unit square can then be divided into two domains, D_1 and D_2 as in [Figure 1](#), where

$$\begin{aligned}
 D_1 &= \{(x, y) : 0 \leq x+y \leq 1, 0 \leq x \leq 1, 0 \leq y \leq 1\}, \\
 D_2 &= \{(x, y) : 1 \leq x+y \leq 2, 0 \leq x \leq 1, 0 \leq y \leq 1\}.
 \end{aligned}$$

The following lemma shows that $t \int_0^1 \int_0^1 f(x+y) dx dy$ is the sum of two one-dimensional integrals.

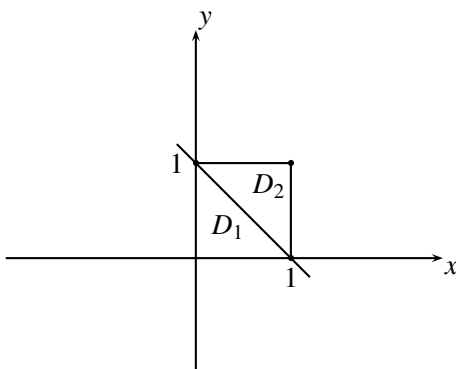


Figure 1. Domains D_1 and D_2 .

Lemma 2.1.

$$\begin{aligned} \int_0^1 \int_0^1 f(x + y) \, dx \, dy &= \iint_{D_1} f(x + y) \, dx \, dy + \iint_{D_2} f(x + y) \, dx \, dy \\ &= \int_0^1 t f(t) \, dt + \int_0^1 (1 - t) f(t + 1) \, dt. \end{aligned} \tag{2-1}$$

Proof. It is clear that

$$\int_0^1 \int_0^1 f(x + y) \, dx \, dy = \iint_{D_1} f(x + y) \, dx \, dy + \iint_{D_2} f(x + y) \, dx \, dy.$$

We first consider $\iint_{D_1} f(x + y) \, dx \, dy$. Note that $t = x + y$, and consider the transformation $(x, y) \mapsto (x, t)$. It is clear that the Jacobian is 1. Then

$$\iint_{D_1} f(x + y) \, dx \, dy = \int_0^1 \int_0^t f(t) \, dx \, dt = \int_0^1 t f(t) \, dt. \tag{2-2}$$

For the integral over domain D_2 , we set $x_1 = 1 - x$ and $y_1 = 1 - y$. Then $(x_1, y_1) \in D_1$ and

$$\begin{aligned} \iint_{D_2} f(x + y) \, dx \, dy &= \iint_{D_1} f(2 - x_1 - y_1) \, dx_1 \, dy_1 \\ &= \int_0^1 t f(2 - t) \, dt. \end{aligned} \tag{2-3}$$

If one sets $u = 1 - t$, it follows that $\int_0^1 t f(2 - t) \, dt = \int_0^1 (1 - u) f(u + 1) \, du$. Then

$$\iint_{D_2} f(x + y) \, dx \, dy = \int_0^1 (1 - u) f(u + 1) \, du. \tag{2-4}$$

Then, identity (2-1) follows by identities (2-2) and (2-4). □

3. The case $n = 3$

When $n = 3$, the integral domain of I is a unit cube. The main idea is to cut the unit cube into several simplexes so that we can apply the integral formula (1-1) over each one.

Let $E = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$ be the unit cube. Set

$$E_1 = \{(x, y, z) : 0 \leq x + y + z \leq 1, 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\},$$

$$E_2 = \{(x, y, z) : 1 \leq x + y + z \leq 2, 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\},$$

$$E_3 = \{(x, y, z) : 2 \leq x + y + z \leq 3, 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}.$$

Then $E = E_1 \cup E_2 \cup E_3$ and the integral I satisfies

$$\begin{aligned} I &= \int_0^1 \int_0^1 \int_0^1 f(x + y + z) dx dy dz \\ &= \int_{E_1} f(x + y + z) dx dy dz + \int_{E_2} f(x + y + z) dx dy dz \\ &\quad + \int_{E_3} f(x + y + z) dx dy dz. \end{aligned}$$

Using formula (1-1), it follows that $\int_{E_1} f(x + y + z) dx dy dz = \frac{1}{2} \int_0^1 t^2 f(t) dt$. The difficult parts are the integrals over E_2 and E_3 . The following lemma explains how to simplify these two integrals to one-dimensional integrals.

Lemma 3.1.

$$\begin{aligned} &\int_0^1 \int_0^1 \int_0^1 f(x + y + z) dx dy dz = \\ &\frac{1}{2} \int_0^1 t^2 f(t) dt + \frac{1}{2} \int_0^1 (-2t^2 + 2t + 1) f(t + 1) dt + \frac{1}{2} \int_0^1 (1 - t)^2 f(t + 2) dt. \end{aligned} \quad (3-1)$$

Proof. We introduce the transformation $(x, y, z) \mapsto (x, y, t)$. By formula (1-1),

$$\int_{E_1} f(x + y + z) dx dy dz = \frac{1}{2} \int_0^1 t^2 f(t) dt. \quad (3-2)$$

Note that integral (3-2) can be applied to calculate the integral over E_3 . Let $x_1 = 1 - x$, $y_1 = 1 - y$ and $z_1 = 1 - z$. The integral over E_3 becomes

$$\begin{aligned} \int_{E_3} f(x + y + z) dx dy dz &= \int_{E_1} f(3 - x_1 - y_1 - z_1) dx_1 dy_1 dz_1 \\ &= \frac{1}{2} \int_0^1 t^2 f(3 - t) dt. \end{aligned} \quad (3-3)$$

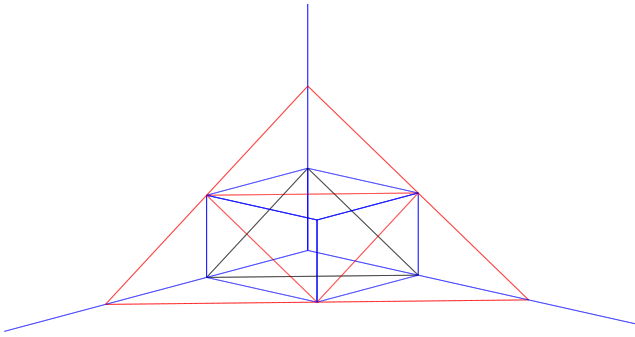


Figure 2. Region E_{20} and its partition: $E_2, E_{21}, E_{22}, E_{23}$.

If one sets $u = 1 - t$, it implies that $\frac{1}{2} \int_0^1 t^2 f(3-t) dt = \frac{1}{2} \int_0^1 (1-u)^2 f(2+u) du$. Hence,

$$\int_{E_3} f(x + y + z) dx dy dz = \frac{1}{2} \int_0^1 (1-t)^2 f(t+2) dt. \quad (3-4)$$

By equalities (3-2) and (3-4), it is sufficient to show that

$$\int_{E_2} f(x + y + z) dx dy dz = \frac{1}{2} \int_0^1 (-2t^2 + 2t + 1) f(t+1) dt. \quad (3-5)$$

Consider the domain

$$E_{20} = \{(x, y, z) : 1 \leq x + y + z \leq 2, 0 \leq x \leq 2, 0 \leq y \leq 2, 0 \leq z \leq 2\}.$$

Similar to Figure 1, we can cut E_{20} into 4 different domains, E_2, E_{21}, E_{22} and E_{23} , so that the integral over each domain can be handled easily. A picture of this partition is shown in Figure 2.

$$E_2 = \{(x, y, z) : 1 \leq x + y + z \leq 2, 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\},$$

$$E_{21} = \{(x, y, z) : 1 \leq x + y + z \leq 2, 1 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 1\},$$

$$E_{22} = \{(x, y, z) : 1 \leq x + y + z \leq 2, 0 \leq x \leq 1, 1 \leq y \leq 2, 0 \leq z \leq 1\},$$

$$E_{23} = \{(x, y, z) : 1 \leq x + y + z \leq 2, 0 \leq x \leq 1, 0 \leq y \leq 1, 1 \leq z \leq 2\},$$

where $E_{20} = E_2 \cup E_{21} \cup E_{22} \cup E_{23}$.

Again by using formula (1-1), the integral over E_{20} is

$$\int_{E_{20}} f(x + y + z) dx dy dz = \int_1^2 \frac{1}{2} t^2 f(t) dt = \frac{1}{2} \int_0^1 (t+1)^2 f(t+1) dt. \quad (3-6)$$

On the other hand, the integral over E_{20} satisfies

$$\begin{aligned} \int_{E_{20}} f(x+y+z) dx dy dz &= \int_{E_{21}} f(x+y+z) dx dy dz + \int_{E_{22}} f(x+y+z) dx dy dz \\ &\quad + \int_{E_{23}} f(x+y+z) dx dy dz + \int_{E_2} f(x+y+z) dx dy dz. \end{aligned} \quad (3-7)$$

By the definitions of E_{21} , E_{22} and E_{23} , it is clear that

$$\int_{E_{21}} f(x+y+z) dx dy dz = \int_{E_{22}} f(x+y+z) dx dy dz = \int_{E_{23}} f(x+y+z) dx dy dz.$$

So we only need to consider $\int_{E_{21}} f(x+y+z) dx dy dz$. Let $\tilde{x} = x - 1$; then by equality (3-2),

$$\begin{aligned} \int_{E_{21}} f(x+y+z) dx dy dz &= \int_{E_1} f(\tilde{x}+y+z+1) d\tilde{x} dy dz \\ &= \frac{1}{2} \int_0^1 t^2 f(t+1) dt. \end{aligned} \quad (3-8)$$

Therefore, (3-6), (3-7) and (3-8) imply that

$$\begin{aligned} \int_{E_2} f(x+y+z) dx dy dz &= \int_{E_{20}} f(x+y+z) dx dy dz - 3 \int_{E_{21}} f(x+y+z) dx dy dz \\ &= \frac{1}{2} \int_0^1 (t+1)^2 f(t+1) dt - \frac{3}{2} \int_0^1 t^2 f(t+1) dt \\ &= \frac{1}{2} \int_0^1 (-2t^2 + 2t + 1) f(t+1) dt, \end{aligned}$$

which shows equality (3-5). □

4. The general case

In this section, we give a general formula for

$$I = \int_0^1 \int_0^1 \dots \int_0^1 f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n$$

in [Theorem 4.1](#). In order to prove it, we first find a recursive formula for I in [Theorem 4.3](#). The proof of [Theorem 4.1](#) then follows by [Theorem 4.4](#) and [Theorem 4.3](#).

Theorem 4.1. *The integral I satisfies*

$$\begin{aligned}
 I &= \int_0^1 \int_0^1 \dots \int_0^1 f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n \\
 &= \frac{1}{(n-1)!} \sum_{m=1}^n \int_0^1 G_m(t) f(t+m-1) dt,
 \end{aligned}
 \tag{4-1}$$

where

$$G_m(t) = \sum_{i=1}^m (-1)^{i-1} (t+m-i)^{n-1} \binom{n}{i-1}.$$

The idea is to divide the n -dimensional unit box into n different polyhedrons and the integral I over each polyhedron can be simplified to a one-dimensional integral by applying the ideas in the 2D or 3D cases. The n different polyhedrons are defined as follows:

$$\begin{aligned}
 K_1 &= \{(x_1, x_2, \dots, x_n) : 0 \leq x_1 + x_2 + \dots + x_n \leq 1, \\
 &\hspace{15em} 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, \dots, 0 \leq x_n \leq 1\},
 \end{aligned}$$

$$\begin{aligned}
 K_2 &= \{(x_1, x_2, \dots, x_n) : 1 \leq x_1 + x_2 + \dots + x_n \leq 2, \\
 &\hspace{15em} 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, \dots, 0 \leq x_n \leq 1\},
 \end{aligned}$$

⋮

$$\begin{aligned}
 K_n &= \{(x_1, x_2, \dots, x_n) : n-1 \leq x_1 + x_2 + \dots + x_n \leq n, \\
 &\hspace{15em} 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, \dots, 0 \leq x_n \leq 1\}.
 \end{aligned}$$

By formula (1-1), the integral over K_1 satisfies the following proposition.

Proposition 4.2.

$$\int_{K_1} f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n = \frac{1}{(n-1)!} \int_0^1 t^{n-1} f(t) dt.$$

Let

$$I_m = \int_{K_m} f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n, \quad m = 1, 2, \dots, n.$$

It is obvious that $I = \sum_{m=1}^n I_m$. Then the integral I reduces to the calculation of each I_m ($1 \leq m \leq n$). Define

$$J_{s,m} = \int_{K_s} f(x_1 + \dots + x_n + m-s) dx_1 dx_2 \dots dx_n, \tag{4-2}$$

where s is an integer and $1 \leq s \leq m$. Note that $J_{m,m} = I_m$. For any $1 \leq s \leq m-1$, $J_{s,m}$ can be calculated by I_s . The following theorem shows that I_m satisfies a recursive formula.

Theorem 4.3.

$$I_m = \frac{1}{(n-1)!} \int_0^1 (t+m-1)^{n-1} f(t+m-1) dt - a_1 J_{1,m} - a_2 J_{2,m} - \dots - a_{m-1} J_{m-1,m}, \quad (4-3)$$

where

$$a_i = \binom{m+n-i-1}{n-1}, \quad i = 1, 2, \dots, m-1.$$

Proof. We consider the region

$$K_{m0} = \{(x_1, x_2, \dots, x_n) : m-1 \leq x_1 + x_2 + \dots + x_n \leq m, 0 \leq x_1 \leq m, 0 \leq x_2 \leq m, \dots, 0 \leq x_n \leq m\}.$$

By Proposition 4.2,

$$\begin{aligned} \int_{K_{m0}} f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n &= \frac{1}{(n-1)!} \int_{m-1}^m t^{n-1} f(t) dt \\ &= \frac{1}{(n-1)!} \int_0^1 (t+m-1)^{n-1} f(t+m-1) dt. \end{aligned} \quad (4-4)$$

We define the subset $K_{i_1 i_2 \dots i_n} \subset K_{m0}$ as follows:

$$K_{i_1 i_2 \dots i_n} = \{(x_1, x_2, \dots, x_n) : m-1 \leq x_1 + x_2 + \dots + x_n \leq m, i_1 - 1 \leq x_1 \leq i_1, i_2 - 1 \leq x_2 \leq i_2, \dots, i_n - 1 \leq x_n \leq i_n\},$$

where $i_1, i_2, \dots, i_n \in [1, m]$ are positive integers. It is easily seen that the intersection of any two subsets $K_{i_1 i_2 \dots i_n}$ only happens on their boundaries. We then classify all possible $K_{i_1 i_2 \dots i_n}$ so that the integral over each one can be evaluated easily. Note that by definition, $K_{1,1,\dots,1} = K_m$. To find the integral over K_m , we need to subtract the integrals over all the other nonempty subsets $K_{i_1 i_2 \dots i_n}$ ($i_1, i_2, \dots, i_n \in [1, m]$) from $\int_{K_{m0}} f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n$.

The first step is to determine when $K_{i_1 i_2 \dots i_n}$ ($i_1, i_2, \dots, i_n \in [1, m]$) is nonempty. For any set $K_{i_1 i_2 \dots i_n}$, let

$$\tilde{x}_1 = x_1 - (i_1 - 1), \quad \tilde{x}_2 = x_2 - (i_2 - 1), \quad \dots, \quad \tilde{x}_n = x_n - (i_n - 1). \quad (4-5)$$

Then $K_{i_1 i_2 \dots i_n}$ becomes

$$\begin{aligned} \tilde{K}_{i_1 i_2 \dots i_n} = \{(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) : m+n-\alpha-1 \leq \tilde{x}_1 + \tilde{x}_2 + \dots + \tilde{x}_n \leq m+n-\alpha, 0 \leq \tilde{x}_1 \leq 1, 0 \leq \tilde{x}_2 \leq 1, \dots, 0 \leq \tilde{x}_n \leq 1\}. \end{aligned}$$

where $\alpha = i_1 + i_2 + \dots + i_n$. Let $s = m + n - \alpha$. It is clear that $K_{i_1 i_2 \dots i_n} \cong \tilde{K}_{i_1 i_2 \dots i_n} = K_s$. Since $m + n - s = \sum_{j=1}^n i_j \geq n$, it follows that $s \leq m$. Note that if $s = m$, by equality (4-2), $J_{m,m} = I_m$. If $s = 0$, $K_{i_1 i_2 \dots i_n} \cong \tilde{K}_{i_1 i_2 \dots i_n} = \{0\}$, and if $s < 0$, $K_{i_1 i_2 \dots i_n} \cong \tilde{K}_{i_1 i_2 \dots i_n} = \emptyset$. So we only need to consider the case $1 \leq s \leq m - 1$. For any given $s \in [1, m - 1]$, it follows that

$$\begin{aligned} \int_{K_{i_1 i_2 \dots i_n}} f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n &= \int_{\tilde{K}_{i_1 i_2 \dots i_n}} f(\tilde{x}_1 + \dots + \tilde{x}_n + i_1 + \dots + i_n - n) d\tilde{x}_1 \dots d\tilde{x}_n \\ &= \int_{K_s} f(x_1 + \dots + x_n + m - s) dx_1 \dots dx_n \\ &= J_{s,m}. \end{aligned} \tag{4-6}$$

It implies that the subsets $K_{i_1 i_2 \dots i_n}$ ($i_1, i_2, \dots, i_n \in [1, m]$, $i_1 + i_2 + \dots + i_n \neq n$) with nonzero measure can be classified into $m - 1$ classes. In each class, every element is identical to some subset K_s after a shifting transformation in (4-5): $(x_1, x_2, \dots, x_n) \mapsto (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$.

Next step is to fix m and s ($1 \leq s \leq m - 1$), and find out how many subsets are identical to K_s . Since $s = m + n - (i_1 + i_2 + \dots + i_n)$, we have

$$m + n - s = i_1 + i_2 + \dots + i_n, \quad \text{where } i_1, i_2, \dots, i_n \text{ are positive integers.} \tag{4-7}$$

The number of positive integer solutions (i_1, i_2, \dots, i_n) for (4-7) is $\binom{m+n-s-1}{n-1}$. It follows that the total number of subsets identical to K_s ($s \in [1, m - 1]$) is

$$a_s = \binom{m+n-s-1}{n-1}. \tag{4-8}$$

Therefore, by equalities (4-4), (4-6) and (4-8), I_m satisfies

$$\begin{aligned} I_m &= \int_{K_m} f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n \\ &= \frac{1}{(n-1)!} \int_0^1 (t+m-1)^{n-1} f(t+m-1) dt \\ &\quad - a_1 J_{1,m} - a_2 J_{2,m} - \dots - a_{m-1} J_{m-1,m}, \end{aligned} \tag{4-9}$$

where a_s ($s = 1, \dots, m - 1$) is defined by (4-8). □

By using the cases $n = 2$ and 3 , we can show by induction that

$$I_m = \frac{1}{(n-1)!} \int_0^1 G_m(t) f(t+m-1) dt, \tag{4-10}$$

where $G_m(t)$ is a polynomial. It follows that

$$\begin{aligned} J_{s,m} &= \int_{K_s} f(x_1 + \dots + x_n + m - s) dx_1 \dots dx_n \\ &= \frac{1}{(n-1)!} \int_0^1 G_s(t) f(t + m - 1) dt, \end{aligned} \tag{4-11}$$

where s is an integer and $1 \leq s \leq m$. The integral I satisfies

$$I = \int_0^1 \int_0^1 \dots \int_0^1 f(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n = \sum_{m=1}^n I_m. \tag{4-12}$$

In order to find a formula for I , we only need to compute the polynomial $G_m(t)$ in equality (4-10) for all $1 \leq m \leq n$. For $m = 1, 2$ and 3 , a direct calculation shows that

$$\begin{aligned} G_1(t) &= t^{n-1}, \\ G_2(t) &= (t + 1)^{n-1} - \binom{n}{1} t^{n-1}. \end{aligned} \tag{4-13}$$

By Theorem 4.3 and equality (4-11),

$$\begin{aligned} G_3(t) &= (t + 2)^{n-1} - \binom{n+1}{n-1} G_1(t) - \binom{n}{n-1} G_2(t) \\ &= (t + 2)^{n-1} - \binom{n}{1} (t + 1)^{n-1} + \binom{n}{2} t^{n-1}. \end{aligned}$$

Similarly,

$$G_4(t) = (t + 3)^{n-1} - \binom{n}{1} (t + 2)^{n-1} + \binom{n}{2} (t + 1)^{n-1} - \binom{n}{3} t^{n-1}.$$

It is reasonable to believe that $G_m(t)$ follows a pattern. The following theorem actually proves this fact.

Theorem 4.4.
$$G_m(t) = \sum_{i=1}^m (-1)^{i-1} (t + m - i)^{n-1} \binom{n}{i-1}. \tag{4-14}$$

Proof. The proof is based on the recursive formula (4-3) in Theorem 4.3 and the identity (4-11). By formula (4-3),

$$\begin{aligned} I_m &= \frac{1}{(n-1)!} \int_0^1 (t + m - 1)^{n-1} f(t + m - 1) dt - \sum_{i=1}^{m-1} a_i J_{i,m} \\ &= \frac{1}{(n-1)!} \int_0^1 G_m(t) f(t + m - 1) dt, \end{aligned}$$

where

$$G_m(t) = (t + m - 1)^{n-1} - \sum_{i=1}^{m-1} a_i G_i(t), \quad \text{and} \quad a_i = \binom{m+n-i-1}{n-1}. \quad (4-15)$$

We show this theorem by induction. It is clear that formula (4-14) of $G_m(t)$ holds for $m = 1$. Assume that it holds for any $1 \leq m \leq k$. We need to show that formula (4-14) also holds for $m = k + 1$.

By (4-15) and the induction assumption, the polynomial $G_{k+1}(t)$ satisfies

$$G_{k+1}(t) = (t + k)^{n-1} + \sum_{i=1}^k \binom{k+1+n-i-1}{n-1} \sum_{j=1}^i (-1)^j (t + i - j)^{n-1} \binom{n}{j-1}. \quad (4-16)$$

By formula (4-14), we can consider each $G_m(t)$ ($1 \leq m \leq k$) as a polynomial of $(t + m - j)^{n-1}$ ($j = 1, 2, \dots, m$) with coefficient $(-1)^{j-1} \binom{n}{j-1}$. Then identity (4-16) implies that the coefficient of $(t + p)^{n-1}$ in $G_{k+1}(t)$ is

$$L_p(G_{k+1}(t)) = \sum_{i=p+1}^k \binom{k+1+n-i-1}{n-1} (-1)^{i-p} \binom{n}{i-p-1}, \quad (4-17)$$

where $p \in [0, k - 1]$ is an integer. Similarly, $G_k(t)$ satisfies

$$G_k(t) = (t + k - 1)^{n-1} + \sum_{i=1}^{k-1} \binom{k+n-i-1}{n-1} \sum_{j=1}^i (-1)^j (t + i - j)^{n-1} \binom{n}{j-1},$$

and the coefficient of $(t + p)^{n-1}$ ($p \in [0, k - 2]$) in $G_k(t)$ is

$$\sum_{i=p+1}^{k-1} \binom{k+n-i-1}{n-1} (-1)^{i-p} \binom{n}{i-p-1}. \quad (4-18)$$

Note that $G_k(t) = \sum_{i=1}^k (-1)^{i-1} (t + k - i)^{n-1} \binom{n}{i-1}$. It follows that

$$\sum_{i=p+1}^{k-1} \binom{k+n-i-1}{n-1} (-1)^{i-p} \binom{n}{i-p-1} = (-1)^{k-p-1} \binom{n}{k-p-1}. \quad (4-19)$$

If $p \neq 0$, let $q = p - 1$. By identity (4-19), the coefficient of $(t + p)^{n-1}$ in (4-17) satisfies

$$\begin{aligned}
 L_p(G_{k+1}(t)) &= \sum_{i=p+1}^k \binom{k+1+n-i-1}{n-1} (-1)^{i-p} \binom{n}{i-p-1} \\
 &= \sum_{i=q+2}^k \binom{k+1+n-i-1}{n-1} (-1)^{i-q-1} \binom{n}{i-q-2} \\
 &= \sum_{i=q+1}^{k-1} \binom{k+n-i-1}{n-1} (-1)^{i-q} \binom{n}{i-q-1} \\
 &= (-1)^{k-q-1} \binom{n}{k-q-1} = (-1)^{k-p} \binom{n}{k-p}. \tag{4-20}
 \end{aligned}$$

Identity (4-20) holds for all integers $p \in [1, k - 1]$. It remains to consider the case when $p = 0$.

If $p = 0$, by (4-17), the coefficient of t^{n-1} in $G_{k+1}(t)$ is

$$L_0(G_{k+1}(t)) = \sum_{i=1}^k \binom{k+1+n-i-1}{n-1} (-1)^i \binom{n}{i-1}. \tag{4-21}$$

Next, we show that $L_0(G_{k+1}(t)) = (-1)^k \binom{n}{k}$. Note that by the binomial theorem, the coefficient of the term x^{k+1} in $(1+x)^{-n}(1+x)^n$ is

$$\begin{aligned}
 &\sum_{i=0}^k (-1)^i \binom{n+i-1}{i} \binom{n}{k-i} \\
 &= \sum_{i=0}^k (-1)^i \binom{n+i-1}{n-1} \binom{n}{k-i} \\
 &= \sum_{j=1}^{k+1} \binom{k+1+n-j-1}{n-1} (-1)^{k+1-j} \binom{n}{j-1} \quad (j = k+1-i) \\
 &= (-1)^{k+1} \left(L_0(G_{k+1}(t)) + (-1)^{k+1} \binom{n}{k} \right). \tag{4-22}
 \end{aligned}$$

On the other hand, for a nonnegative integer k , the coefficient of the term x^{k+1} in $(1+x)^{-n}(1+x)^n = 1$ is always 0. Hence, (4-22) implies that

$$L_0(G_{k+1}(t)) = (-1)^k \binom{n}{k}. \tag{4-23}$$

Therefore, by identities (4-20) and (4-23), it follows that

$$\begin{aligned}
 G_{k+1}(t) &= (t+k)^{n-1} + \sum_{p=0}^{k-1} (-1)^{k-p} \binom{n}{k-p} (t+p)^{n-1} \\
 &= \sum_{i=1}^{k+1} (-1)^{i-1} (t+k+1-i)^{n-1} \binom{n}{i-1}.
 \end{aligned}
 \tag{4-24}$$

This concludes the proof. □

5. Application to log-gamma function

In this section, we consider the integral of log-gamma function

$$I = \int_0^1 \int_0^1 \dots \int_0^1 \log \Gamma(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n. \tag{5-1}$$

The integral of log-gamma function has its own importance in many parts of mathematics [Amdeberhan et al. 2011; Choi and Srivastava 2005]. Actually, the case when $n = 2$ is a problem proposed by Ovidiu Furdui [2010] in the Problems and Solutions section of *The College Mathematics Journal*, and one of its solutions is proposed by Geng-zhe Chang [2011]. When it comes to general dimension n , it is quite a challenge to evaluate it.

After the preparation of Theorem 4.1 in Section 4, we can evaluate the integral (5-1). A nice formula is given in Theorem 5.1.

Theorem 5.1.

$$\begin{aligned}
 I = I(n) &= \int_0^1 \int_0^1 \dots \int_0^1 \log \Gamma(x_1 + x_2 + \dots + x_n) dx_1 dx_2 \dots dx_n \\
 &= \frac{1}{2} \log(2\pi) - \frac{n-1}{2} H_n + \sum_{k=2}^{n-1} \frac{(-1)^{n+k+1} k^n}{n!} \binom{n-1}{k} \log k,
 \end{aligned}
 \tag{5-2}$$

where the last sum is missing when $n = 2$ and $H_n = \sum_{k=1}^n 1/k$.

The proof of this theorem is based on Theorem 4.1 and several combinatorial identities in Jihuai Shi’s book [2009].

Note that $\Gamma(t+1) = t\Gamma(t)$ and $G_m(t) = \sum_{i=1}^m (-1)^{i-1} (t+m-i)^{n-1} \binom{n}{i-1}$. By [Theorem 4.1](#), the integral I becomes

$$\begin{aligned} I &= \frac{1}{(n-1)!} \sum_{m=1}^n \int_0^1 G_m(t) \log \Gamma(t+m-1) dt \\ &= \frac{1}{(n-1)!} \int_0^1 \sum_{m=1}^n G_m(t) \log \Gamma(t) dt \\ &\quad + \frac{1}{(n-1)!} \int_0^1 \sum_{k=2}^n \sum_{m=k}^n G_m(t) \log(t+k-2) dt. \quad (5-3) \end{aligned}$$

Several combinatorial identities are introduced to simplify [\(5-3\)](#).

Lemma 5.2.

$$\sum_{m=k}^n G_m(t) = (n-1)! - \sum_{m=1}^{k-1} \binom{n-1}{k-m-1} (-1)^{k-m-1} (t+m-1)^{n-1},$$

and when $k=1$, $\sum_{m=1}^n G_m(t) = (n-1)!$.

Proof. Note that $G_m(t) = \sum_{i=1}^m (-1)^{i-1} (t+m-i)^{n-1} \binom{n}{i-1}$. It follows that

$$\begin{aligned} \sum_{m=1}^k G_m(t) &= \sum_{m=1}^k \sum_{i=1}^m (-1)^{i-1} (t+m-i)^{n-1} \binom{n}{i-1} \\ &= \sum_{m=1}^k \sum_{i=0}^{k-m} (-1)^i \binom{n}{i} (t+m-1)^{n-1}. \end{aligned}$$

By the combinatorial identity $\sum_{i=0}^m (-1)^i \binom{n}{i} = (-1)^m \binom{n-1}{m}$ ($m < n$), we have

$$\sum_{m=1}^k \sum_{i=0}^{k-m} (-1)^i \binom{n}{i} (t+m-1)^{n-1} = \sum_{m=1}^k \binom{n-1}{k-m} (-1)^{k-m} (t+m-1)^{n-1}.$$

Hence,

$$\sum_{m=1}^k G_m(t) = \sum_{m=1}^k \binom{n-1}{k-m} (-1)^{k-m} (t+m-1)^{n-1}.$$

In the case when $k = n$, the combinatorial identity $\sum_{k=0}^n (-1)^k \binom{n}{k} (x+n-k)^n = n!$ implies

$$\begin{aligned} \sum_{m=1}^n G_m(t) &= \sum_{m=1}^n \binom{n-1}{n-m} (-1)^{n-m} (t+m-1)^{n-1} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (t+n-1-k)^{n-1} \\ &= (n-1)!. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{m=k}^n G_m(t) &= \sum_{m=1}^n G_m(t) - \sum_{m=1}^{k-1} G_m(t) \\ &= (n-1)! - \sum_{m=1}^{k-1} \binom{n-1}{k-m-1} (-1)^{k-m-1} (t+m-1)^{n-1}. \quad \square \end{aligned}$$

Let

$$T_k = \sum_{m=1}^k \binom{n-1}{k-m} (-1)^{k-m} (t+m-1)^{n-1} = \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m (t+k-m-1)^{n-1}.$$

Then

$$\sum_{m=k}^n G_m(t) = (n-1)! - T_{k-1}.$$

By applying Lemma 5.2, (5-3) becomes

$$\begin{aligned} I &= \int_0^1 \log \Gamma(t) dt + \int_0^1 \sum_{k=0}^{n-2} \log(t+k) dt - \frac{1}{(n-1)!} \int_0^1 \sum_{k=1}^{n-1} T_k \log(t+k-1) dt \\ &= \frac{1}{2} \log(2\pi) + (n-1) \log(n-1) - n + 1 - \frac{1}{(n-1)!} \int_0^1 \sum_{k=1}^{n-1} T_k \log(t+k-1) dt. \end{aligned} \tag{5-4}$$

Then, the calculation of I reduces to the calculation of

$$\int_0^1 \sum_{k=1}^{n-1} T_k \log(t+k-1) dt.$$

Note that $T_1 = t^{n-1}$ and

$$\begin{aligned} \int_0^1 T_k \log(t+k-1) dt \\ = \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \int_0^1 (t+k-m-1)^{n-1} \log(t+k-1) dt. \end{aligned}$$

When $k > 1$,

$$\begin{aligned} \int_0^1 (t+k-m-1)^{n-1} \log(t+k-1) dt \\ = \frac{(k-m)^n \log k - (k-m-1)^n \log(k-1)}{n} - \int_0^1 \frac{(t+k-m-1)^n}{n(t+k-1)} dt \\ = \frac{(k-m)^n - (-m)^n}{n} \log k - \frac{(k-m-1)^n - (-m)^n}{n} \\ - \frac{1}{n} \sum_{r=1}^n \frac{k^r - (k-1)^r}{r} \binom{n}{r} (-m)^{n-r}. \end{aligned}$$

Let $S_1(1) = 0$,

$$S_1(k) = \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \left(\frac{(k-m)^n - (-m)^n}{n} \log k - \frac{(k-m-1)^n - (-m)^n}{n} \log(k-1) \right),$$

and

$$S_2(k) = \frac{1}{n} \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \sum_{r=1}^n \frac{k^r - (k-1)^r}{r} \binom{n}{r} (-m)^{n-r}.$$

It follows that

$$\int_0^1 \sum_{k=1}^{n-1} T_k \log(t+k-1) dt = \sum_{k=1}^{n-1} S_1(k) - \sum_{k=1}^{n-1} S_2(k). \quad (5-5)$$

The next lemma calculates $\sum_{k=1}^{n-1} S_1(k)$.

Lemma 5.3.

$$\sum_{k=1}^{n-1} S_1(k) = \frac{1}{n} \sum_{k=2}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \log k + \frac{\log(n-1)}{n} (n!(n-1) - (n-1)^n).$$

Proof. Note that $S_1(1) = 0$.

$$\begin{aligned} & \sum_{k=1}^{n-1} S_1(k) \\ &= \sum_{k=1}^{n-1} \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \left(\frac{(k-m)^n - (-m)^n}{n} \log k - \frac{(k-m-1)^n - (-m)^n}{n} \log(k-1) \right) \\ &= \frac{1}{n} \sum_{k=2}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \log k \\ & \quad + \frac{1}{n} \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m ((n-m-1)^n - (-m)^n) \log(n-1). \end{aligned}$$

Using the combinatorial identity $\sum_{k=0}^n \binom{n}{k} (-1)^k (x-k)^{n+1} = (x-n/2)(n+1)!$, we have

$$\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m (n-1-m)^n = \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m (n-1-m)^n = \frac{n-1}{2} n!,$$

and

$$\begin{aligned} & \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m (-m)^n \\ &= \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m (-m)^n - (-1)^{n-1} (1-n)^n = (n-1)^n - \frac{n-1}{2} n!. \end{aligned}$$

Hence,

$$\sum_{k=1}^{n-1} S_1(k) = \frac{1}{n} \sum_{k=2}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \log k + \frac{\log(n-1)}{n} (n!(n-1) - (n-1)^n).$$

□

The following lemma calculates $\sum_{k=1}^{n-1} S_2(k)$. Here we only give the result. For reader's convenience, the proof of it is given in the [Appendix](#).

Lemma 5.4.

$$\sum_{k=1}^{n-1} S_2(k) = (n-1)!(n-1) - \frac{n-1}{2} H_n(n-1)!,$$

where $H_n = \sum_{k=1}^n 1/k$.

Using [Lemma 5.3](#) and [Lemma 5.4](#), we can prove [Theorem 5.1](#) below.

Proof of Theorem 5.1. Let $H_n = \sum_{k=1}^n 1/k$. By identity (5-5), Lemma 5.3 and Lemma 5.4, we have that

$$\begin{aligned}
 & \int_0^1 \sum_{k=1}^{n-1} T_k \log(t+k-1) dt \\
 &= \sum_{k=1}^{n-1} S_1(k) - \sum_{k=1}^{n-1} S_2(k) \\
 &= \frac{1}{n} \sum_{k=2}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \log k + \frac{\log(n-1)}{n} (n!(n-1) - (n-1)^n) \\
 & \quad - (n-1)!(n-1) + \frac{n-1}{2} H_n (n-1)!. \quad (5-6)
 \end{aligned}$$

By identities (5-4) and (5-6), it follows that

$$\begin{aligned}
 I &= \frac{1}{2} \log(2\pi) + (n-1) \log(n-1) - n + 1 - \frac{1}{(n-1)!} \int_0^1 \sum_{k=1}^{n-1} T_k \log(t+k-1) dt \\
 &= \frac{1}{2} \log(2\pi) - \frac{n-1}{2} H_n + \frac{1}{n!} \sum_{k=2}^{n-1} \binom{n-1}{k} (-1)^{k+n+1} k^n \log k. \quad \square
 \end{aligned}$$

When $n = 2, 3$ and 4 , the values of the integral I are

$$\begin{aligned}
 I(2) &= -\frac{3}{4} + \frac{1}{2} \log(2\pi), \\
 I(3) &= \frac{1}{2} \log(2\pi) + \frac{4}{3} \log 2 - \frac{11}{6}, \\
 I(4) &= \frac{1}{2} \log(2\pi) - 2 \log 2 + \frac{27}{8} \log 3 - \frac{25}{8}.
 \end{aligned}$$

Appendix.

For reader's convenience, the proof of Lemma 5.4 is given here.

Lemma 5.4.

$$\sum_{k=1}^{n-1} S_2(k) = (n-1)!(n-1) - \frac{n-1}{2} H_n (n-1)!,$$

where $H_n = \sum_{k=1}^n 1/k$.

Proof. Note that

$$\begin{aligned} \sum_{k=1}^{n-1} S_2(k) &= \frac{1}{n} \sum_{k=1}^{n-1} \left(\sum_{m=0}^{k-1} \binom{n-1}{m} \right) (-1)^m \sum_{r=1}^n \frac{k^r - (k-1)^r}{r} \binom{n}{r} (-m)^{n-r} \\ &= -\frac{1}{n} \sum_{k=1}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \sum_{r=1}^n \binom{n}{r} \frac{(-1)^r}{r} \\ &\quad + \frac{1}{n} \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m \sum_{r=1}^n \frac{(-m)^{n-r} (n-1)^r}{r} \binom{n}{r}. \end{aligned}$$

Let

$$R_1 = -\frac{1}{n} \sum_{k=1}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \sum_{r=1}^n \binom{n}{r} \frac{(-1)^r}{r} \tag{A-1}$$

and

$$R_2 = \frac{1}{n} \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m \sum_{r=1}^n \frac{(-m)^{n-r} (n-1)^r}{r} \binom{n}{r}. \tag{A-2}$$

Then

$$\sum_{k=1}^{n-1} S_2(k) = R_1 + R_2. \tag{A-3}$$

By applying the combinatorial identities

$$\sum_{k=0}^n \binom{n}{k} (-1)^k (x-k)^{n+1} = \left(x - \frac{n}{2}\right) (n+1)! \quad \text{and} \quad \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} = H_n,$$

the sum R_1 can be simplified to

$$\begin{aligned} R_1 &= \frac{1}{n} \sum_{k=1}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \sum_{r=1}^n \binom{n}{r} \frac{(-1)^{r+1}}{r} \\ &= \frac{H_n}{n} \left((n-1)^n - \frac{n-1}{2} n! \right). \end{aligned} \tag{A-4}$$

To simplify R_2 , we apply the combinatorial identity

$$\sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} (1 - (1-x)^k) = \sum_{k=1}^n \frac{x^k}{k},$$

and it follows that

$$\begin{aligned}
 \sum_{r=1}^n \frac{(-m)^{n-r} (n-1)^r}{r} \binom{n}{r} &= -(-m)^n \sum_{r=1}^n \frac{(-1)^{r+1}}{r} \binom{n}{r} \left(1 - \frac{m-n+1}{m}\right)^r \\
 &= (-m)^n \left(\sum_{r=1}^n \frac{1}{r} \left(\frac{m-n+1}{m}\right)^r - \sum_{r=1}^n \frac{(-1)^{r+1}}{r} \binom{n}{r} \right) \\
 &= \sum_{r=1}^n \frac{1}{r} (m-n+1)^r m^{n-r} (-1)^n - (-m)^n H_n.
 \end{aligned}$$

Recalling the formula of R_2 in (A-2), we have

$$nR_2 = \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m+n} \sum_{k=1}^n \frac{1}{k} (m-n+1)^k m^{n-k} - H_n \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m (-m)^n. \quad (\text{A-5})$$

By the combinatorial identity $\sum_{k=0}^n \binom{n}{k} (-1)^k (x-k)^{n+1} = (x-n/2)(n+1)!$, we see that

$$\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m (-m)^n = (n-1)^n - \frac{n-1}{2} n!. \quad (\text{A-6})$$

We then simplify $\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m+n} \sum_{k=1}^n \frac{1}{k} (m-n+1)^k m^{n-k}$. Note that

$$\begin{aligned}
 &\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m+n} \sum_{k=1}^n \frac{1}{k} (m-n+1)^k m^{n-k} \\
 &= \sum_{k=1}^n \frac{(-1)^n}{k} \left(\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m \sum_{i=0}^k \binom{k}{i} m^{n-k+i} (n-1)^{k-i} (-1)^{k-i} \right). \quad (\text{A-7})
 \end{aligned}$$

Let

$$P(m) = \sum_{i=0}^k \binom{k}{i} m^{n-k+i} (n-1)^{k-i} (-1)^{k-i}.$$

We apply the combinatorial identity $\sum_{k=0}^n (-1)^k \binom{n}{k} \mathbf{P}(k) = 0$ for any polynomial $\mathbf{P}(k)$ with $\deg \mathbf{P}(k) < n$, and it follows that

$$\begin{aligned} \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m \sum_{i=0}^k \binom{k}{i} m^{n-k+i} (n-1)^{k-i} (-1)^{k-i} \\ = \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m P(m) \\ = \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m P(m) - (-1)^{n-1} P(n-1) \\ = \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m (-k(n-1)m^{n-1} + m^n). \end{aligned} \tag{A-8}$$

By the combinatorial identity $\sum_{k=0}^n (-1)^k \binom{n}{k} (x+n-k)^n = n!$, we have

$$-k(n-1) \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m m^{n-1} = k(n-1)(-1)^n (n-1)!.$$

By the combinatorial identity $\sum_{k=0}^n \binom{n}{k} (-1)^k (x-k)^{n+1} = (x-n/2)(n+1)!$, we see that

$$\sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m m^n = (-1)^{n-1} \frac{n-1}{2} n!.$$

Then equality (A-7) becomes

$$\begin{aligned} \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m+n} \sum_{k=1}^n \frac{1}{k} (m-n+1)^k m^{n-k} \\ = \sum_{k=1}^n \frac{(-1)^n}{k} \left(k(n-1)(-1)^n (n-1)! + (-1)^{n-1} \frac{n-1}{2} n! \right) \\ = n!(n-1) - \frac{n-1}{2} n! H_n, \end{aligned} \tag{A-9}$$

where $H_n = \sum_{k=1}^n 1/k$.

Hence, by equalities (A-9) and (A-6), nR_2 in (A-5) can be simplified to

$$nR_2 = n!(n-1) - (n-1)^n H_n. \tag{A-10}$$

That is,

$$R_2 = (n-1)!(n-1) - \frac{H_n}{n} (n-1)^n. \tag{A-11}$$

Therefore, by equalities (A-3), (A-4) and (A-11), it follows that

$$\begin{aligned} \sum_{k=1}^{n-1} S_2(k) &= R_1 + R_2 \\ &= \frac{H_n}{n} \left((n-1)^n - \frac{n-1}{2} n! \right) + (n-1)!(n-1) - \frac{H_n}{n} (n-1)^n \\ &= (n-1)!(n-1) - \frac{n-1}{2} H_n (n-1)!, \end{aligned}$$

where $H_n = \sum_{i=1}^n 1/i$. □

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