involve
a journal of mathematics

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Nancy Scherich

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2015 vol. 8, no. 5

INVOLVE 8:5 (2015)
dx.doi.org/10.2140/involve.2015.8.721

A simplification of grid equivalence

Nancy Scherich

(Communicated by Kenneth S. Berenhaut)

In the work of Cromwell and Dynnikov, grid equivalence is given by the grid moves commutation, (de-)stabilization and cyclic permutation. This paper gives a proof that cyclic permutation is a sequence of (de-)stabilization and commutation grid moves.

1. Introduction

A *grid diagram* is a two-dimensional square grid such that each square within the grid is decorated with an $\times$, $\circ$ or is left blank. This is done in a manner such that every column and every row has exactly one $\times$ and one $\circ$ decoration. The *grid number* of a grid diagram is the number of columns (or rows) in the grid. See Figure 1 for an example. This paper follows the grid notation used by Manolescu, Ozsváth, Szabó and Thurston [Manolescu et al. 2007] (see also [Manolescu et al. 2009]) with the convention that the rows and columns are numbered top to bottom and left to right, respectively.

A grid diagram is associated with a knot, or link, by connecting the $\times$ and $\circ$ decorations in each column and row by a straight line with the convention that vertical lines cross over horizontal lines. These lines form strands of the knot, and removing the grid leaves a projection of the knot. As a result, grid diagrams represent particular planar projections of knots, or links. This process is illustrated in Figure 2. The *knot type of a grid* is the knot type of the knot associated with the grid.

It is important to note that the $\times$ and $\circ$ decorations can specify an orientation of the knot, but more importantly they mark the end points of the strands of the knot in that column or row. So, if two grid diagrams are the same up to opposite labeling of the $\times$ and $\circ$ decorations, then the grid diagrams are considered the same even though the labeling might suggest opposite orientations. Also, because grid diagrams are square, any result established for the columns of a grid is also understood for the rows by rotating the grid by 90 degrees, and vice versa.

MSC2010: 57M27.
Keywords: knot theory, grid diagrams.

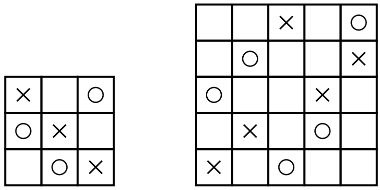


Figure 1. Grid diagrams with grid numbers 3 (left) and 5 (right).

There are three grid moves used to relate grid diagrams: commutation, cyclic permutation and (de-)stabilization. These play a role analogous to the Reidemeister moves [1932] for knot diagrams. Following the notation from [Manolescu et al. 2007], the three grid moves are as follows:

(1) Commutation interchanges two consecutive rows or columns of a grid diagram. This move preserves the grid number, as shown in Figure 3. Even though commutation may be defined for any two consecutive rows or columns, it is only permitted if the commutation preserves the knot type of the grid; refer to Section 2 for details. Throughout the introduction, it is assumed that all commutations preserve the knot type.

(2) Cyclic permutation preserves the grid number and removes an outer row/column and places it on the opposite side of the grid. See Figure 4.

(3) The third grid move has two different names depending on how the move is being used. Stabilization is the addition of a kink while destabilization is the removal. It is important to note that (de-)stabilization does not preserve the grid number. A kink may be added to the right or left of a column, and above or below a row. To

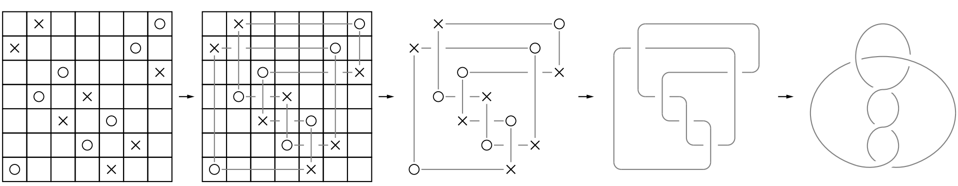


Figure 2. The process of finding the knot associated to a given grid diagram.

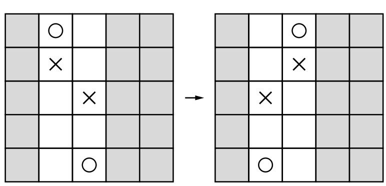


Figure 3. An example of column commutation.

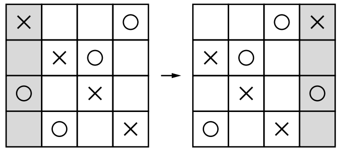


Figure 4. An example of column permutation.

add a kink to column c, insert an empty row between the $\times$ and $\circ$ markers of the column c. Then insert an empty column to the right or left of column c. Move either the $\times$ or $\circ$ decoration in column c into the adjacent grid square in the added column. Complete the added row and column with $\times$ and $\circ$ decorations appropriately. See Figure 5. To add a kink to a row, switch the notions of column and row. To remove a kink, follow these instructions in reverse order. As shown, stabilization increases the grid number by 1 while destabilization reduces the grid number by 1.

The following theorem explicates the relationship between grid diagrams, knots and the three grid moves.

Theorem 1.1 [Cromwell 1995; Dynnikov 2006]. *Let G_1, G_2 be a grid diagrams representing knots K_1, K_2 respectively. Then K_1 and K_2 are equivalent knots if and only if there exists a sequence of commutation, (de-)stabilization and cyclic permutation grid moves to relate G_1 to G_2.*

In other words, the three grid moves form an equivalence relation on the set of grid diagrams, and two grid diagrams are equivalent if and only if they represent the same knot. The three grid moves play a role similar to the Reidemeister moves [1932] for knot diagrams.

Grid diagrams have become increasingly widespread since the use of grids to give a combinatorial definition of knot Floer homology [Manolescu et al. 2007]. From the approach of knot Floer homology, invariance under cyclic permutation is trivial when viewed as diagrams on a torus. However, this paper will show that in any context, cyclic permutation is an unnecessary hypothesis of Theorem 1.1. In other words, the equivalence given by Theorem 1.1 can be strengthened so that two grid diagrams

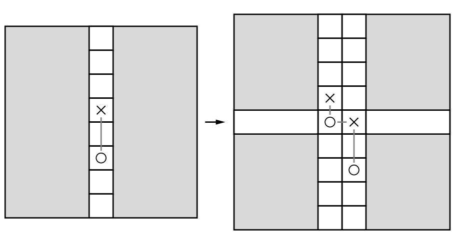


Figure 5. An example of stabilization, or kink addition.

are equivalent if there exists a sequence of commutation and (de-)stabilization grid moves to relate the two grid diagrams. This implies that invariance for any object defined using grids may be confirmed by checking invariance under only two moves: commutation and (de-)stabilization. This strengthened equivalence of grid diagrams is an immediate corollary to the following theorem.

Theorem 1.2. *There exists a sequence of commutation and* (*de-*)*stabilization grid moves that perform the cyclic permutation grid move.*

Corollary 1.3. *Let G_1, G_2 be grid diagrams representing knots K_1, K_2 respectively. Then K_1 and K_2 are equivalent knots if and only if there exists a sequence of commutation and* (*de-*)*stabilization grid moves to relate G_1 to G_2.*

The result of Theorem 1.2 is well known to certain experts. For example, the computer implementation of knot Floer homology available as part of KnotTheory`[1], due to Jean-Marie Droz, makes use of such a simplification. More concretely, after completing this project the author learned that Theorem 1.2 is proved in the work of Ozsváth, Szabó and Thurston [Ozsváth et al. 2008, Lemma 4.3]. However, since Theorem 1.2 is an interesting result in combinatorial knot theory in its own right, an independent proof is of value. Further, an illustrated proof of Theorem 1.2 may serve as a useful introduction to grid diagrams. The main goal of this paper is to provide a constructive proof of Theorem 1.2.

Organization of the paper. To prove Theorem 1.2, Section 2 addresses a subtlety of the commutation grid move required to preserve grid equivalence. Section 3 introduces four intermediate grid moves that when applied sequentially perform a cyclic permutation in terms of commutations and (de-)stabilizations. Lastly, Section 4 formalizes the proof of Theorem 1.2.

Terminology. The word *grid* will be used synonymously with *grid diagram* throughout the paper.

2. Commutation in detail

The commutation grid move is defined to interchange any two consecutive rows or columns in a grid. However, in some instances, commutation does not preserve the knot type of the grid. Since grids are useful as representations of knots with an equivalence relation generated by the grid moves, it is important to identify the exact conditions under which commutation preserves this equivalence relation. These conditions will be established for column commutation.

Figure 6 shows the four possible relative positions of two consecutive columns, up to different $\times$ and $\circ$ labeling and exact spacing. Denote these possibilities as *nonshared, total-shared, partial-shared and point-shared*, see Figure 6.

[1] *KnotTheory*` is a Mathematica package and is available from www.katlas.org.

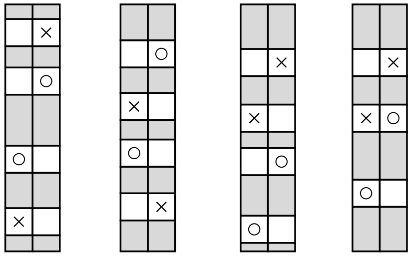


Figure 6. From left to right: nonshared, total-shared, partial-shared, and point-shared columns.

Lemma 2.1. *Commutation of nonshared, total-shared and point-shared columns preserves the knot type of the grid diagram.*

Proof. To prove these conditions preserve the knot type, consider the knot associated with the grid. The following will show that the associated knot is only altered by a Reidemeister I move, a Reidemeister II move or isotopy, thus preserving the knot equivalence class.

For nonshared columns, there are three scenarios, all resulting in isotopy. See Figure 7. For total-shared, there are three scenarios, two resulting in a Reidemeister II move and the other in isotopy. See Figure 8.

For point-shared there are four scenarios, two resulting in isotopy and two resulting in a Reidemeister I move. See Figure 9. □

Corollary 2.2. *Commutation of a column that has* $\times$ *and* $\circ$ *decorations in adjacent grid squares will preserve the knot type of the grid.*

Proof. This column will only be nonshared, point-shared or total-shared with a consecutive column. □

Corollary 2.3. *Commutation of a column that has* $\times$ *and* $\circ$ *decorations in the top and bottom grid squares will preserve the knot type of the grid.*

Proof. This column will always be total-shared with any consecutive column. □

Remark 2.4. Commutation of columns that are partial-shared may change the knot type of the grid.

Figure 10 shows two scenarios of partial-shared columns that, when commuted, change the crossings of the knot associated with the grid in a complicated way. The left scenario shows two strands that are not linked but become linked after the column commutation. The right shows how commutation changes an over-crossing to an under-crossing. In both of these scenarios, more knowledge about the knot would be needed to determine if the knot type was preserved.

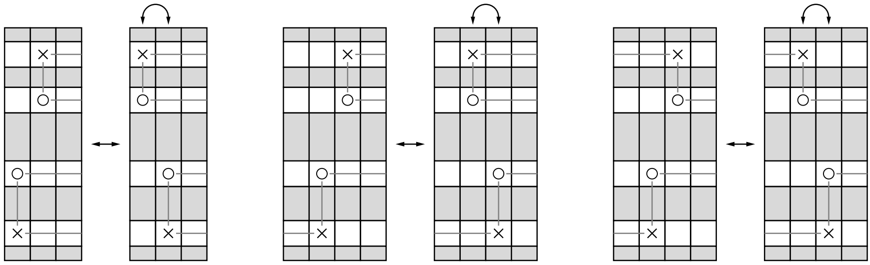


Figure 7. Three scenarios for nonshared columns.

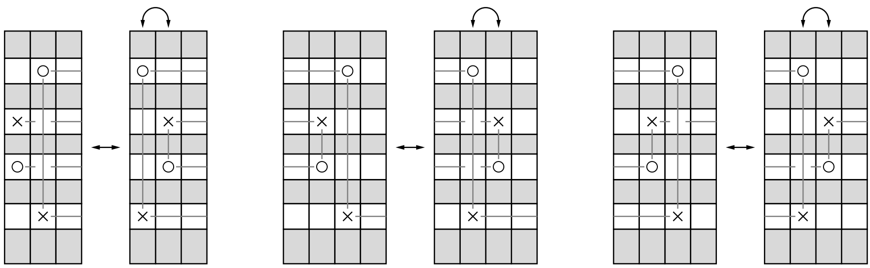


Figure 8. Three scenarios for total-shared columns. The left and middle result in a Reidemeister II move, and the right in isotopy.

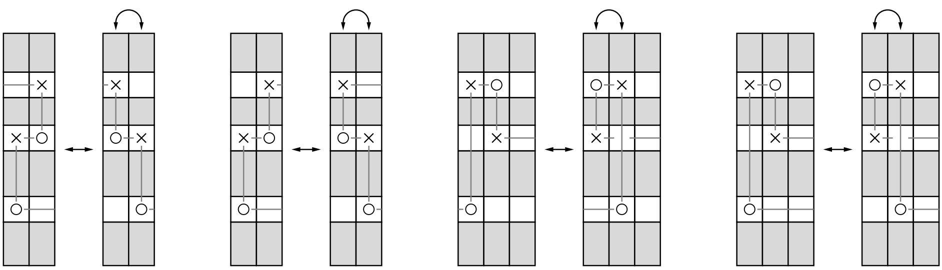


Figure 9. Four scenarios for point-shared columns. From left to right, the first two result in isotopy, and the second two in a Reidemeister I move.

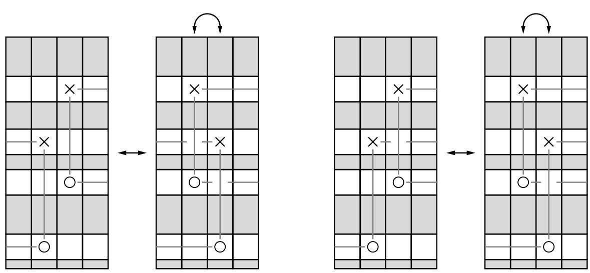


Figure 10. Partial-shared columns.

Remark 2.5. Point-shared commutation is not considered a standard grid move. In fact, it can be accomplished by a single destabilization followed by a single stabilization. This paper considers point-shared commutation with the sole interests of simplifying the proof of Corollary 2.2 and exhibiting a Reidemeister I move via grid moves in Figure 9. Often in the literature, namely [Manolescu et al. 2007] and [Ozsváth et al. 2008], point-shared commutation is not considered an allowable grid move. Throughout the remainder of the paper, point-shared commutation will not be used and the main result does not require this type of commutation.

3. Intermediate grid moves

The goal of the intermediate grid moves is to accomplish a column permutation from left to right using only commutations and (de-)stabilizations. A column permutation preserves the size of the grid and relative positions of the $\times$ and $\circ$ decorations in the permuted column. So throughout the construction of the intermediate moves, any change in grid size or relative positioning of the $\times$ and $\circ$ decorations in the permuted column will be noted.

The intermediate grid moves are independent from each other, but to simplify the proof of Theorem 1.2, each intermediate move will be described starting from the ending position of the previous intermediate move. Thus, when applied sequentially, it will be clear that a cyclic permutation is accomplished.

The first intermediate grid move I_1.

Definition 3.1. The I_1 move increases the grid number by 2 and moves the $\times$ and $\circ$ decorations in the first column to occupy the top and bottom grid squares of the first column, as shown in Figure 11.

Proposition 3.2. *The I_1 move can be accomplished by a sequence of commutation and (de-)stabilization moves that preserve the grid equivalence class.*

Proof. Fix a grid diagram with grid number n. Assume that the row containing the $\times$ in the first column is above the row containing the $\circ$. For alternate labeling, switch the roles of the $\times$ and $\circ$. Let $\times$ be in row m and $\circ$ be in row k with standard

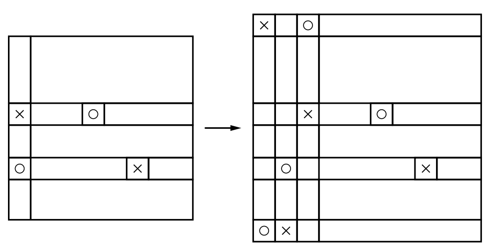
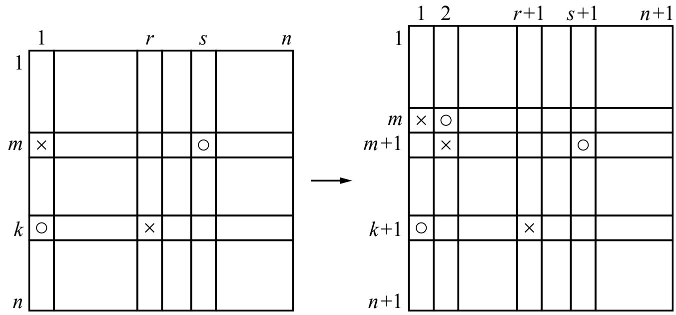


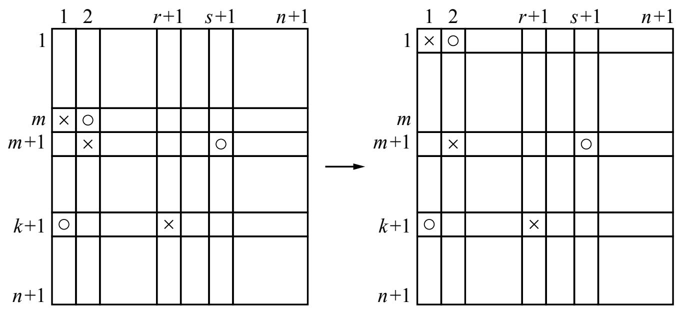
Figure 11. An illustration of the I_1 move.

top to bottom labeling. Let the $\circ$ in the m-th row be in column s and the $\times$ in the k-th row be in column r.

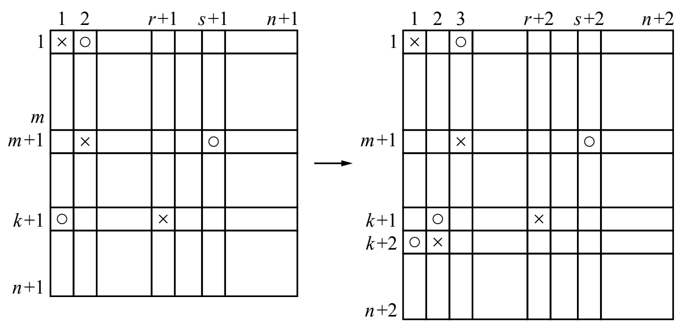
(1) Start by adding a kink above the m-th column. This increases the grid size by 1, resulting in a grid number of $n+1$.



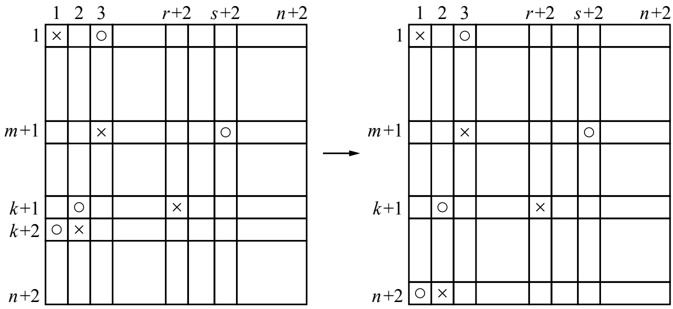
(2) The $\times$ and $\circ$ in the first and second columns of row m are adjacent. By Corollary 2.2, commutation of row m preserves the grid equivalence class. So commute the row m upwards $m-1$ times making the $\times$ in the first column occupy the top row.



(3) After adding the kink, the $\circ$ in the first column has been shifted down one row moving the $\circ$ to the $(k+1)$-th row. Add a kink above the $(k+1)$-th row, moving the $\circ$ to the $(k+2)$-th row. This increases the grid number by 1, resulting in a grid number of $n+2$.



(4) The ○ and × in the $(k+2)$-th row are adjacent, so by Corollary 2.2, commuting row $k+2$ preserves the knot type. Commute the $(k+2)$-th row downwards $(n+2)-(k+2)$ times until the ○ in the first column is in the bottom row.



Now the grid number increased to $n+2$ and the × and ○ decorations in the first column occupy the top and bottom grid squares of the first column. Since all commutations preserved the knot type, the grid equivalence class was preserved. □

The second intermediate move I_2.

Definition 3.3. Starting from the ending position of the I_1 move, where the × and ○ decorations in the first column occupy the top and bottom grid squares, the I_2 move cyclically permutes the first column to become the last column of the grid. (This is a special case of cyclic permutation). The I_2 move preserves the grid number. This is shown in Figure 12.

Proposition 3.4. *The I_2 move can be accomplished in a series of commutation grid moves and preserves the grid equivalence class.*

Proof. Since the × and ○ decorations in the first column are in the top and bottom grid squares, by Corollary 2.3, commutation of this column preserves the knot type of the grid. So commute the first column to the right $n-1$ times until it

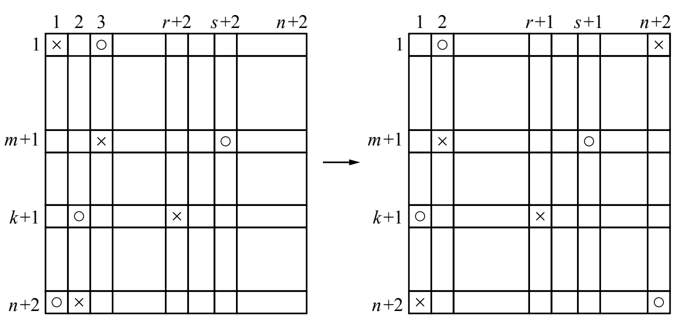


Figure 12. An illustration of the I_2 move.

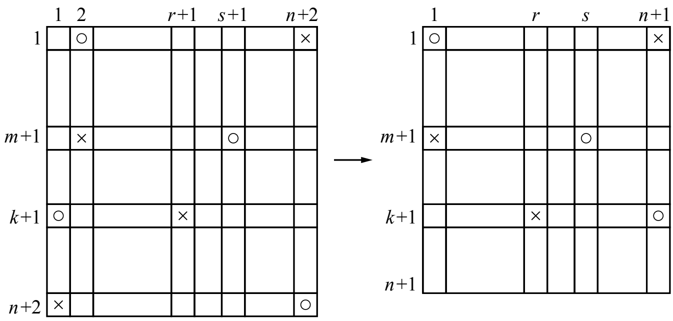


Figure 13. An illustration of the I_3 move.

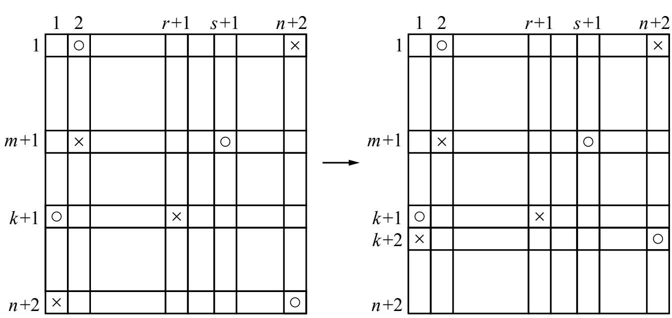
becomes the outermost right column. This clearly preserves the grid number and grid equivalence class. □

Third intermediate move I_3.

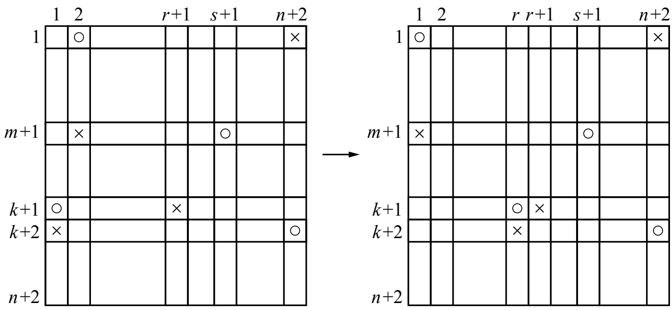
Definition 3.5. Starting from the ending position of the I_2 move, the move I_3 reduces the grid number by 1 and simplifies the bottom portion of the grid as shown in Figure 13.

Proposition 3.6. *The I_3 move can be accomplished by a sequence of commutation and* (*de-*)*stabilization grid moves and preserves the grid equivalence class.*

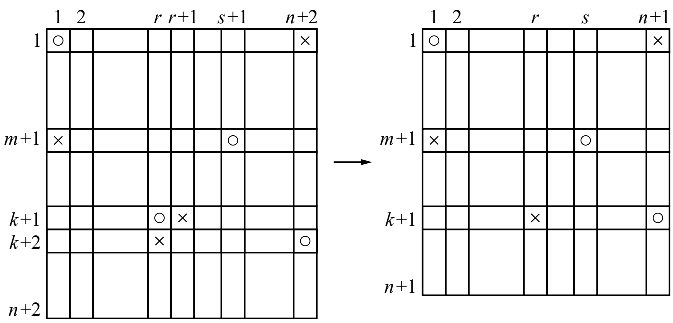
Proof. (1) Since the $\times$ and $\circ$ decorations in the $(n+2)$-th row occupy the first and last grid squares, by Corollary 2.3 commuting this row preserves the knot type. So, commute the $(n+2)$-th row upwards $(n+2)-(k+1)-1$ times, until the $\times$ and $\circ$ decorations in the $(k+1)$-th and $(k+2)$-th rows in the first column are adjacent.



(2) Since the $\times$ and $\circ$ decorations in the first column are in adjacent grid squares, by Corollary 2.2 commuting this column preserves the knot type. So commute the first column to the right $r-1$ times until the $\times$ and $\circ$ decorations in the $(k+1)$-th row are adjacent.



(3) Remove the kink in the r-th column and the $(n+2)$-th row, reducing the grid number to $n+1$.



Since all commutations preserved the knot type, the grid equivalence class was preserved and the grid number was reduced to $n+1$. □

Fourth intermediate move I_4.

Definition 3.7. Starting from the ending position of the I_3 move, the move I_4 mirrors the move I_3 and decreases the grid number to n as shown in Figure 14.

Proposition 3.8. *The* I_4 *move can be accomplished by a sequence of commutation and (de-)stabilization grid moves that preserve the grid equivalence class.*

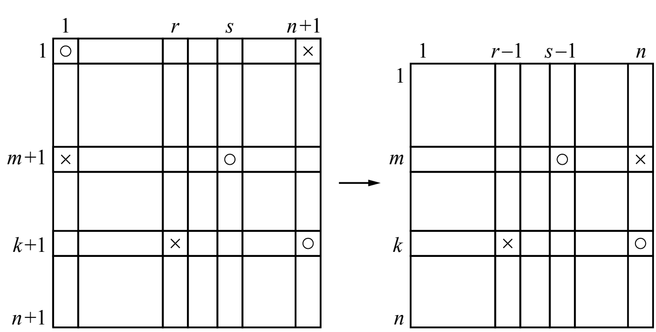
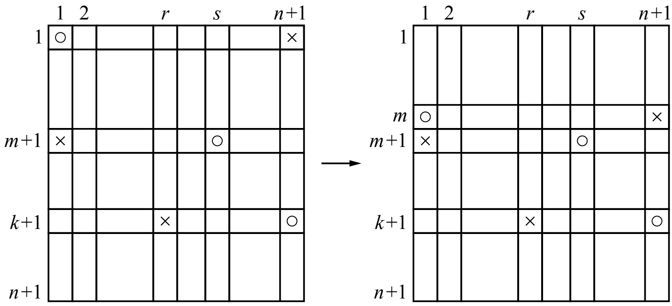
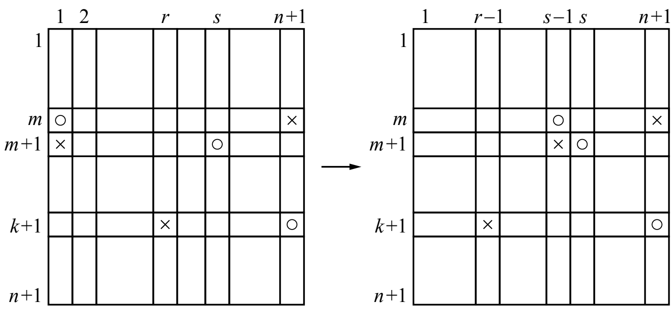


Figure 14. An illustration of the I_4 move.

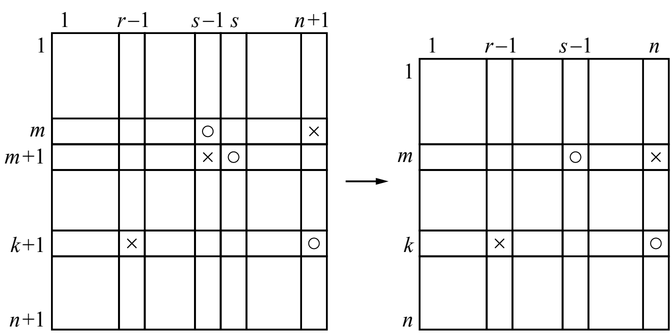
Proof. (1) Since the $\times$ and $\circ$ decorations in the first row occupy the first and last grid squares, by Corollary 2.3 commuting this row preserves the knot type. So, commute the top row down $m-1$ times, so that the $\times$ and the $\circ$ in the first column are adjacent.



(2) Since the $\times$ and $\circ$ decorations in the first column are in adjacent grid squares, by Corollary 2.2 commuting this column preserves the knot type. So, commute the first column to the right s times until the $\times$ and $\circ$ decorations in the $(m+1)$-th row are adjacent.



(3) Lastly, remove the kink in the $(s-1)$-th column and $(m+1)$-th row reducing the grid back to its original grid number n.



After the I_4 move, the grid number returns to the original value n, and the $\times$ and the $\circ$ in the last column are in the same relative row positions as before the

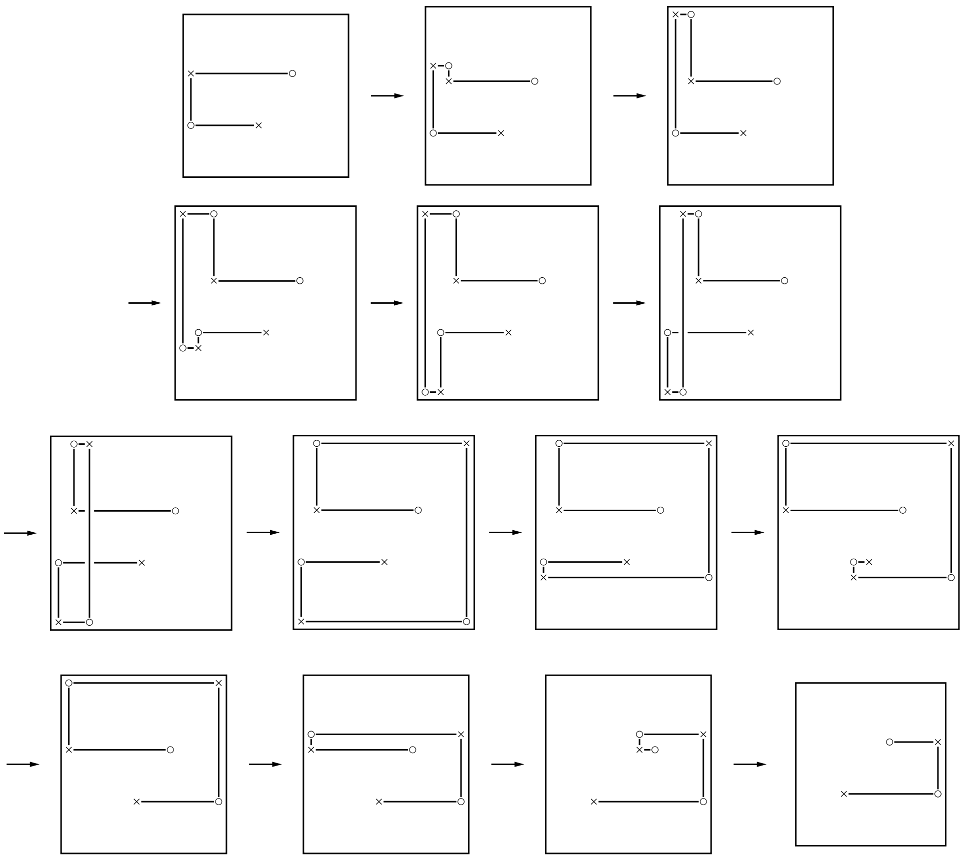


Figure 15. An illustration of the application of the intermediate grid moves used to produce a cyclic permutation grid move.

intermediate grid moves were applied. Since all commutations preserved the knot type, the grid equivalence class was preserved. □

4. Proof of Theorem 1.2

Theorem 1.2. *There exists a sequence of commutation and* (*de-*)*stabilization grid moves that perform the cyclic permutation grid move.*

Proof. Given a grid diagram, apply the intermediate grid moves I_1, I_2, I_3 and I_4 sequentially. As shown by construction, this sequence of intermediate moves preserves the grid number and relative row position of the $\times$ and $\circ$ decorations in the permuted column. Thus this sequence of intermediate moves performs a column permutation with only commutations and (de-)stabilizations. Figure 15 is a stylized diagram following the strand of the knot through the sequential application of the intermediate grid moves to explicate this construction. This process can be applied with an appropriate change of orientation to accomplish a cyclic permutation for a row or column in any direction. □

Acknowledgements

This paper formed part of a VIGRE funded undergraduate research project at UCLA (summer 2010). I would like to thank faculty advisor Liam Watson for significant guidance, inspiration, editing and choice of topic.

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Received: 2012-11-20 Revised: 2014-10-21 Accepted: 2015-01-16

nancy.scherich@gmail.com *Department of Mathematics, University of California, Santa Barbara, Santa Barbara, CA 93106, United States*

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2015 vol. 8 no. 5