

involve

a journal of mathematics

A simplification of grid equivalence

Nancy Scherich



A simplification of grid equivalence

Nancy Scherich

(Communicated by Kenneth S. Berenhaut)

In the work of Cromwell and Dynnikov, grid equivalence is given by the grid moves commutation, (de-)stabilization and cyclic permutation. This paper gives a proof that cyclic permutation is a sequence of (de-)stabilization and commutation grid moves.

1. Introduction

A *grid diagram* is a two-dimensional square grid such that each square within the grid is decorated with an \times , \circ or is left blank. This is done in a manner such that every column and every row has exactly one \times and one \circ decoration. The *grid number* of a grid diagram is the number of columns (or rows) in the grid. See [Figure 1](#) for an example. This paper follows the grid notation used by Manolescu, Ozsváth, Szabó and Thurston [[Manolescu et al. 2007](#)] (see also [[Manolescu et al. 2009](#)]) with the convention that the rows and columns are numbered top to bottom and left to right, respectively.

A grid diagram is associated with a knot, or link, by connecting the \times and \circ decorations in each column and row by a straight line with the convention that vertical lines cross over horizontal lines. These lines form strands of the knot, and removing the grid leaves a projection of the knot. As a result, grid diagrams represent particular planar projections of knots, or links. This process is illustrated in [Figure 2](#). The *knot type of a grid* is the knot type of the knot associated with the grid.

It is important to note that the \times and \circ decorations can specify an orientation of the knot, but more importantly they mark the end points of the strands of the knot in that column or row. So, if two grid diagrams are the same up to opposite labeling of the \times and \circ decorations, then the grid diagrams are considered the same even though the labeling might suggest opposite orientations. Also, because grid diagrams are square, any result established for the columns of a grid is also understood for the rows by rotating the grid by 90 degrees, and vice versa.

MSC2010: 57M27.

Keywords: knot theory, grid diagrams.

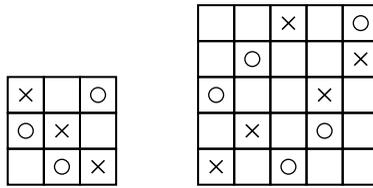


Figure 1. Grid diagrams with grid numbers 3 (left) and 5 (right).

There are three grid moves used to relate grid diagrams: commutation, cyclic permutation and (de-)stabilization. These play a role analogous to the Reidemeister moves [1932] for knot diagrams. Following the notation from [Manolescu et al. 2007], the three grid moves are as follows:

(1) Commutation interchanges two consecutive rows or columns of a grid diagram. This move preserves the grid number, as shown in Figure 3. Even though commutation may be defined for any two consecutive rows or columns, it is only permitted if the commutation preserves the knot type of the grid; refer to Section 2 for details. Throughout the introduction, it is assumed that all commutations preserve the knot type.

(2) Cyclic permutation preserves the grid number and removes an outer row/column and places it on the opposite side of the grid. See Figure 4.

(3) The third grid move has two different names depending on how the move is being used. Stabilization is the addition of a kink while destabilization is the removal. It is important to note that (de-)stabilization does not preserve the grid number. A kink may be added to the right or left of a column, and above or below a row. To

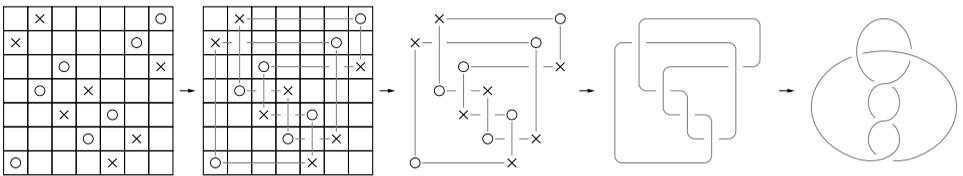


Figure 2. The process of finding the knot associated to a given grid diagram.

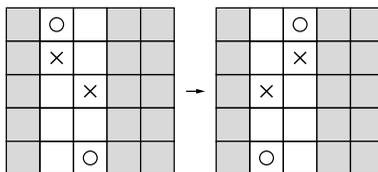


Figure 3. An example of column commutation.

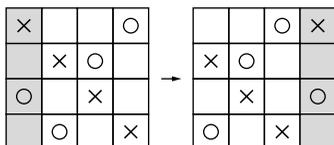


Figure 4. An example of column permutation.

add a kink to column c , insert an empty row between the \times and \circ markers of the column c . Then insert an empty column to the right or left of column c . Move either the \times or \circ decoration in column c into the adjacent grid square in the added column. Complete the added row and column with \times and \circ decorations appropriately. See Figure 5. To add a kink to a row, switch the notions of column and row. To remove a kink, follow these instructions in reverse order. As shown, stabilization increases the grid number by 1 while destabilization reduces the grid number by 1.

The following theorem explicates the relationship between grid diagrams, knots and the three grid moves.

Theorem 1.1 [Cromwell 1995; Dynnikov 2006]. *Let G_1, G_2 be a grid diagrams representing knots K_1, K_2 respectively. Then K_1 and K_2 are equivalent knots if and only if there exists a sequence of commutation, (de-)stabilization and cyclic permutation grid moves to relate G_1 to G_2 .*

In other words, the three grid moves form an equivalence relation on the set of grid diagrams, and two grid diagrams are equivalent if and only if they represent the same knot. The three grid moves play a role similar to the Reidemeister moves [1932] for knot diagrams.

Grid diagrams have become increasingly widespread since the use of grids to give a combinatorial definition of knot Floer homology [Manolescu et al. 2007]. From the approach of knot Floer homology, invariance under cyclic permutation is trivial when viewed as diagrams on a torus. However, this paper will show that in any context, cyclic permutation is an unnecessary hypothesis of Theorem 1.1. In other words, the equivalence given by Theorem 1.1 can be strengthened so that two grid diagrams

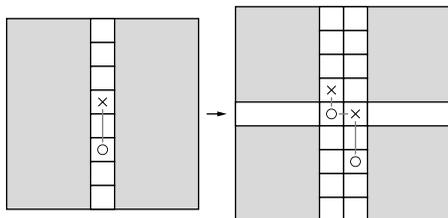


Figure 5. An example of stabilization, or kink addition.

are equivalent if there exists a sequence of commutation and (de-)stabilization grid moves to relate the two grid diagrams. This implies that invariance for any object defined using grids may be confirmed by checking invariance under only two moves: commutation and (de-)stabilization. This strengthened equivalence of grid diagrams is an immediate corollary to the following theorem.

Theorem 1.2. *There exists a sequence of commutation and (de-)stabilization grid moves that perform the cyclic permutation grid move.*

Corollary 1.3. *Let G_1, G_2 be grid diagrams representing knots K_1, K_2 respectively. Then K_1 and K_2 are equivalent knots if and only if there exists a sequence of commutation and (de-)stabilization grid moves to relate G_1 to G_2 .*

The result of [Theorem 1.2](#) is well known to certain experts. For example, the computer implementation of knot Floer homology available as part of `KnotTheory`¹, due to Jean-Marie Droz, makes use of such a simplification. More concretely, after completing this project the author learned that [Theorem 1.2](#) is proved in the work of Ozsváth, Szabó and Thurston [[Ozsváth et al. 2008](#), Lemma 4.3]. However, since [Theorem 1.2](#) is an interesting result in combinatorial knot theory in its own right, an independent proof is of value. Further, an illustrated proof of [Theorem 1.2](#) may serve as a useful introduction to grid diagrams. The main goal of this paper is to provide a constructive proof of [Theorem 1.2](#).

Organization of the paper. To prove [Theorem 1.2](#), [Section 2](#) addresses a subtlety of the commutation grid move required to preserve grid equivalence. [Section 3](#) introduces four intermediate grid moves that when applied sequentially perform a cyclic permutation in terms of commutations and (de-)stabilizations. Lastly, [Section 4](#) formalizes the proof of [Theorem 1.2](#).

Terminology. The word *grid* will be used synonymously with *grid diagram* throughout the paper.

2. Commutation in detail

The commutation grid move is defined to interchange any two consecutive rows or columns in a grid. However, in some instances, commutation does not preserve the knot type of the grid. Since grids are useful as representations of knots with an equivalence relation generated by the grid moves, it is important to identify the exact conditions under which commutation preserves this equivalence relation. These conditions will be established for column commutation.

[Figure 6](#) shows the four possible relative positions of two consecutive columns, up to different \times and \circ labeling and exact spacing. Denote these possibilities as *nonshared*, *total-shared*, *partial-shared* and *point-shared*, see [Figure 6](#).

¹*KnotTheory* is a Mathematica package and is available from www.katlas.org.

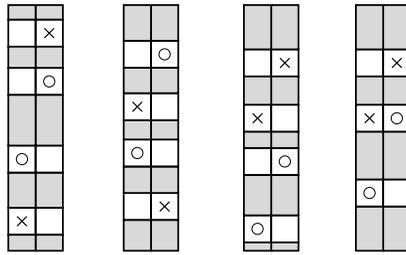


Figure 6. From left to right: nonshared, total-shared, partial-shared, and point-shared columns.

Lemma 2.1. *Commutation of nonshared, total-shared and point-shared columns preserves the knot type of the grid diagram.*

Proof. To prove these conditions preserve the knot type, consider the knot associated with the grid. The following will show that the associated knot is only altered by a Reidemeister I move, a Reidemeister II move or isotopy, thus preserving the knot equivalence class.

For nonshared columns, there are three scenarios, all resulting in isotopy. See Figure 7. For total-shared, there are three scenarios, two resulting in a Reidemeister II move and the other in isotopy. See Figure 8.

For point-shared there are four scenarios, two resulting in isotopy and two resulting in a Reidemeister I move. See Figure 9. □

Corollary 2.2. *Commutation of a column that has \times and \circ decorations in adjacent grid squares will preserve the knot type of the grid.*

Proof. This column will only be nonshared, point-shared or total-shared with a consecutive column. □

Corollary 2.3. *Commutation of a column that has \times and \circ decorations in the top and bottom grid squares will preserve the knot type of the grid.*

Proof. This column will always be total-shared with any consecutive column. □

Remark 2.4. Commutation of columns that are partial-shared may change the knot type of the grid.

Figure 10 shows two scenarios of partial-shared columns that, when commuted, change the crossings of the knot associated with the grid in a complicated way. The left scenario shows two strands that are not linked but become linked after the column commutation. The right shows how commutation changes an over-crossing to an under-crossing. In both of these scenarios, more knowledge about the knot would be needed to determine if the knot type was preserved.

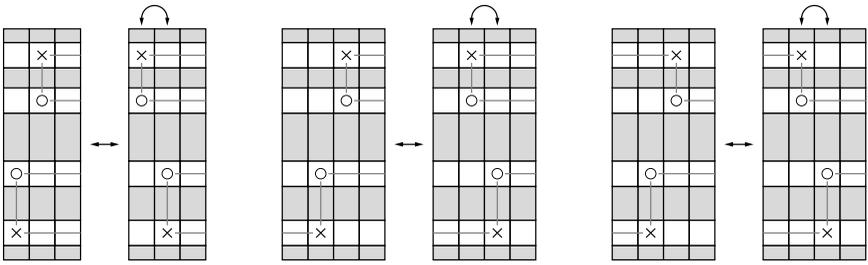


Figure 7. Three scenarios for nonshared columns.

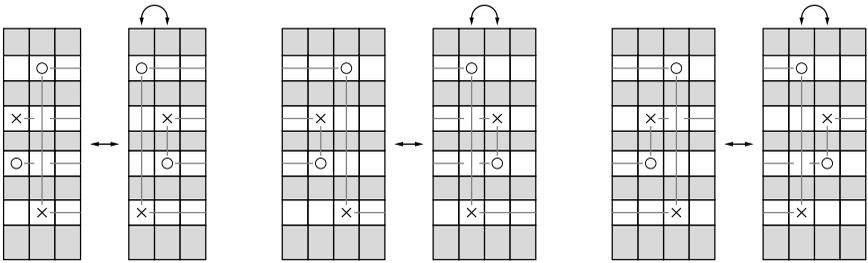


Figure 8. Three scenarios for total-shared columns. The left and middle result in a Reidemeister II move, and the right in isotopy.

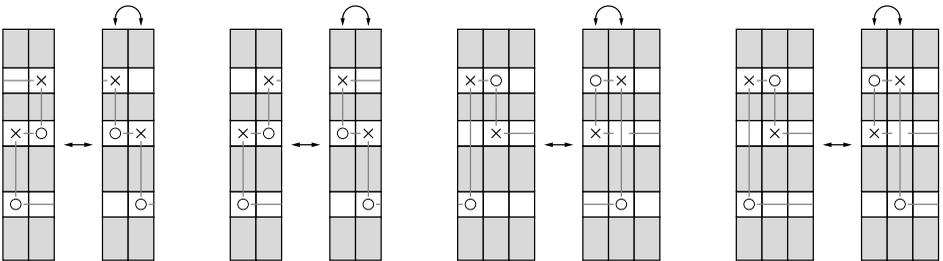


Figure 9. Four scenarios for point-shared columns. From left to right, the first two result in isotopy, and the second two in a Reidemeister I move.

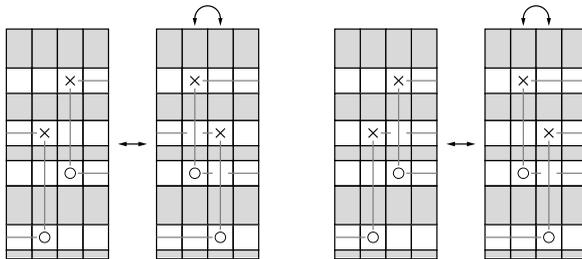


Figure 10. Partial-shared columns.

Remark 2.5. Point-shared commutation is not considered a standard grid move. In fact, it can be accomplished by a single destabilization followed by a single stabilization. This paper considers point-shared commutation with the sole interests of simplifying the proof of [Corollary 2.2](#) and exhibiting a Reidemeister I move via grid moves in [Figure 9](#). Often in the literature, namely [[Manolescu et al. 2007](#)] and [[Ozsváth et al. 2008](#)], point-shared commutation is not considered an allowable grid move. Throughout the remainder of the paper, point-shared commutation will not be used and the main result does not require this type of commutation.

3. Intermediate grid moves

The goal of the intermediate grid moves is to accomplish a column permutation from left to right using only commutations and (de-)stabilizations. A column permutation preserves the size of the grid and relative positions of the \times and \circ decorations in the permuted column. So throughout the construction of the intermediate moves, any change in grid size or relative positioning of the \times and \circ decorations in the permuted column will be noted.

The intermediate grid moves are independent from each other, but to simplify the proof of [Theorem 1.2](#), each intermediate move will be described starting from the ending position of the previous intermediate move. Thus, when applied sequentially, it will be clear that a cyclic permutation is accomplished.

The first intermediate grid move I_1 .

Definition 3.1. The I_1 move increases the grid number by 2 and moves the \times and \circ decorations in the first column to occupy the top and bottom grid squares of the first column, as shown in [Figure 11](#).

Proposition 3.2. *The I_1 move can be accomplished by a sequence of commutation and (de-)stabilization moves that preserve the grid equivalence class.*

Proof. Fix a grid diagram with grid number n . Assume that the row containing the \times in the first column is above the row containing the \circ . For alternate labeling, switch the roles of the \times and \circ . Let \times be in row m and \circ be in row k with standard

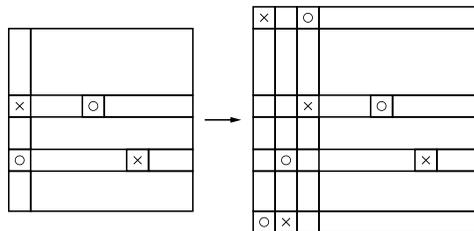
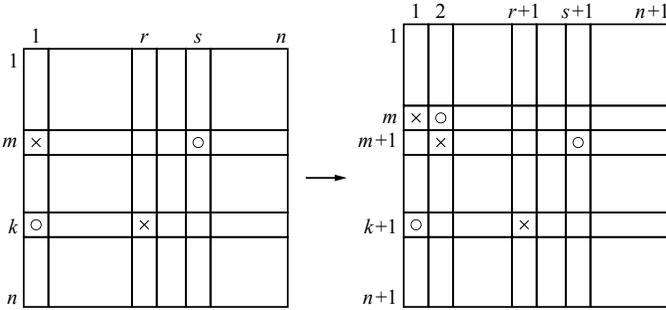


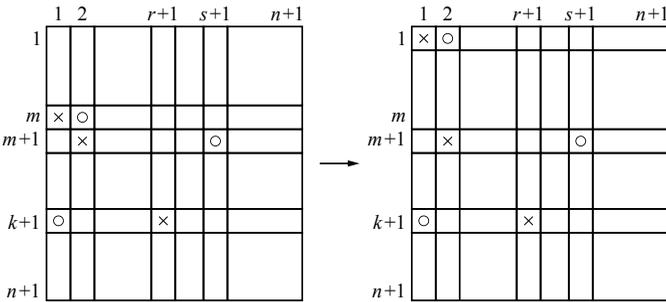
Figure 11. An illustration of the I_1 move.

top to bottom labeling. Let the \circ in the m -th row be in column s and the \times in the k -th row be in column r .

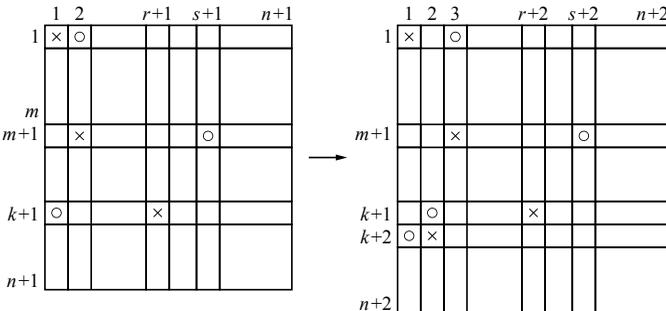
(1) Start by adding a kink above the m -th column. This increases the grid size by 1, resulting in a grid number of $n + 1$.



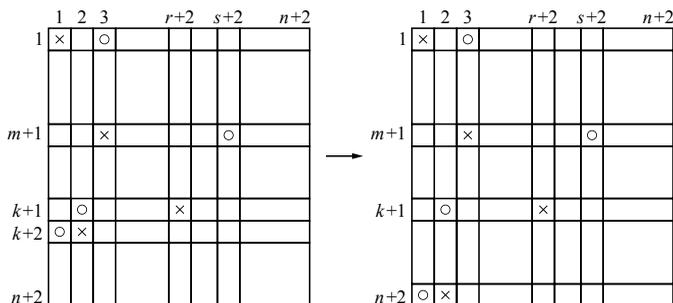
(2) The \times and \circ in the first and second columns of row m are adjacent. By [Corollary 2.2](#), commutation of row m preserves the grid equivalence class. So commute the row m upwards $m - 1$ times making the \times in the first column occupy the top row.



(3) After adding the kink, the \circ in the first column has been shifted down one row moving the \circ to the $(k+1)$ -th row. Add a kink above the $(k+1)$ -th row, moving the \circ to the $(k+2)$ -th row. This increases the grid number by 1, resulting in a grid number of $n + 2$.



(4) The \circ and \times in the $(k+2)$ -th row are adjacent, so by [Corollary 2.2](#), commuting row $k+2$ preserves the knot type. Commute the $(k+2)$ -th row downwards $(n+2) - (k+2)$ times until the \circ in the first column is in the bottom row.



Now the grid number increased to $n+2$ and the \times and \circ decorations in the first column occupy the top and bottom grid squares of the first column. Since all commutations preserved the knot type, the grid equivalence class was preserved. \square

The second intermediate move I_2 .

Definition 3.3. Starting from the ending position of the I_1 move, where the \times and \circ decorations in the first column occupy the top and bottom grid squares, the I_2 move cyclically permutes the first column to become the last column of the grid. (This is a special case of cyclic permutation). The I_2 move preserves the grid number. This is shown in [Figure 12](#).

Proposition 3.4. The I_2 move can be accomplished in a series of commutation grid moves and preserves the grid equivalence class.

Proof. Since the \times and \circ decorations in the first column are in the top and bottom grid squares, by [Corollary 2.3](#), commutation of this column preserves the knot type of the grid. So commute the first column to the right $n-1$ times until it

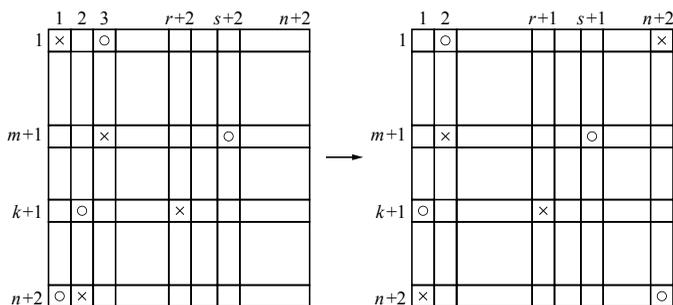


Figure 12. An illustration of the I_2 move.

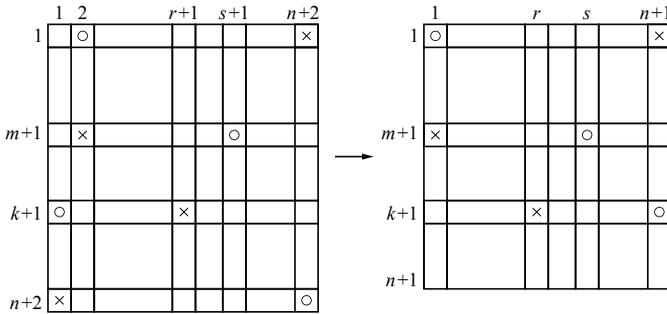


Figure 13. An illustration of the I_3 move.

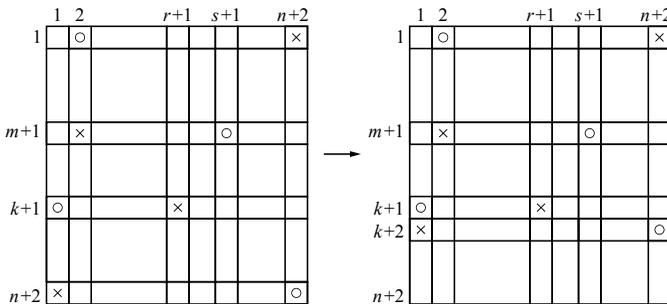
becomes the outermost right column. This clearly preserves the grid number and grid equivalence class. \square

Third intermediate move I_3 .

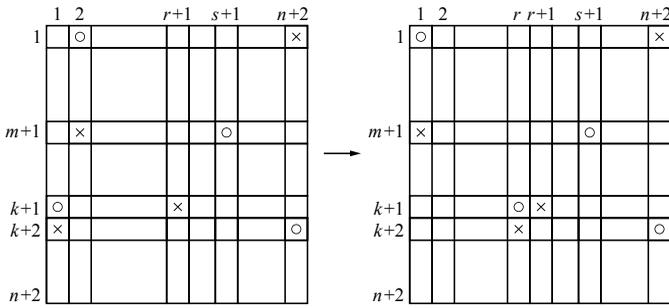
Definition 3.5. Starting from the ending position of the I_2 move, the move I_3 reduces the grid number by 1 and simplifies the bottom portion of the grid as shown in Figure 13.

Proposition 3.6. The I_3 move can be accomplished by a sequence of commutation and (de-)stabilization grid moves and preserves the grid equivalence class.

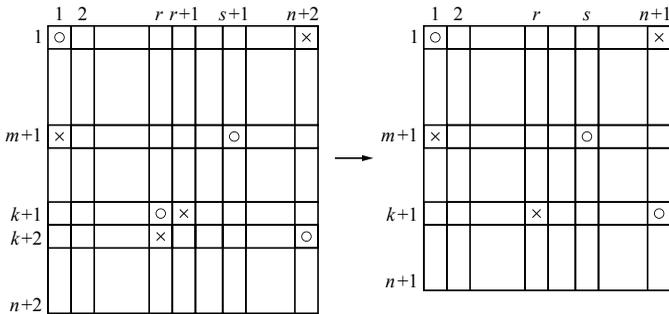
Proof. (1) Since the \times and \circ decorations in the $(n+2)$ -th row occupy the first and last grid squares, by Corollary 2.3 commuting this row preserves the knot type. So, commute the $(n+2)$ -th row upwards $(n+2) - (k+1) - 1$ times, until the \times and \circ decorations in the $(k+1)$ -th and $(k+2)$ -th rows in the first column are adjacent.



(2) Since the \times and \circ decorations in the first column are in adjacent grid squares, by Corollary 2.2 commuting this column preserves the knot type. So commute the first column to the right $r - 1$ times until the \times and \circ decorations in the $(k+1)$ -th row are adjacent.



(3) Remove the kink in the r -th column and the $(n+2)$ -th row, reducing the grid number to $n+1$.



Since all commutations preserved the knot type, the grid equivalence class was preserved and the grid number was reduced to $n + 1$. □

Fourth intermediate move I_4 .

Definition 3.7. Starting from the ending position of the I_3 move, the move I_4 mirrors the move I_3 and decreases the grid number to n as shown in Figure 14.

Proposition 3.8. The I_4 move can be accomplished by a sequence of commutation and (de-)stabilization grid moves that preserve the grid equivalence class.

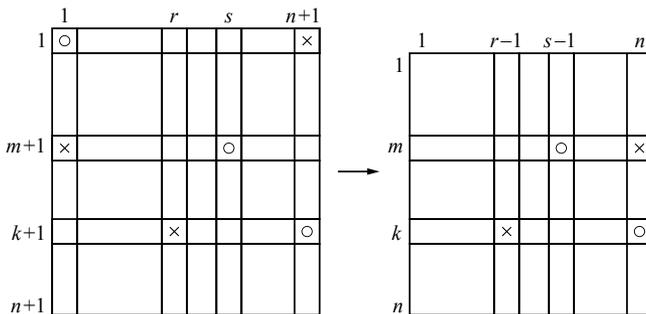
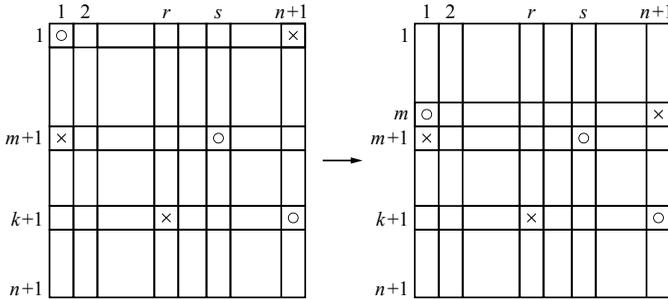
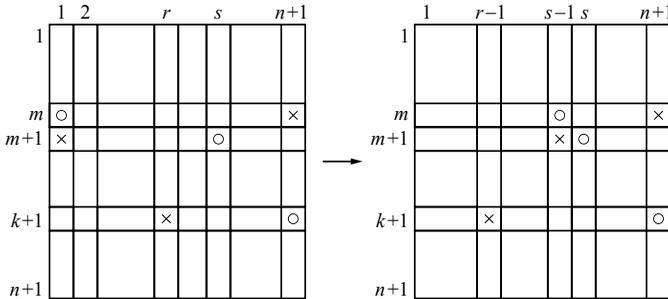


Figure 14. An illustration of the I_4 move.

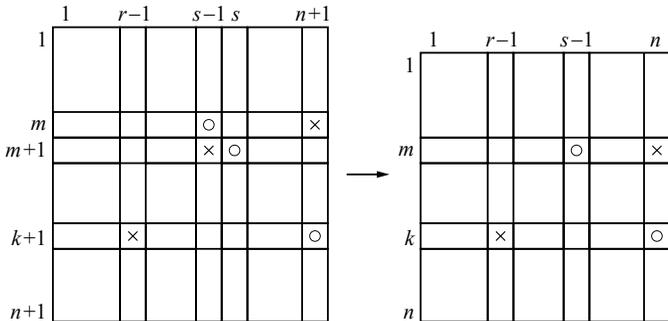
Proof. (1) Since the \times and \circ decorations in the first row occupy the first and last grid squares, by [Corollary 2.3](#) commuting this row preserves the knot type. So, commute the top row down $m - 1$ times, so that the \times and the \circ in the first column are adjacent.



(2) Since the \times and \circ decorations in the first column are in adjacent grid squares, by [Corollary 2.2](#) commuting this column preserves the knot type. So, commute the first column to the right s times until the \times and \circ decorations in the $(m+1)$ -th row are adjacent.



(3) Lastly, remove the kink in the $(s-1)$ -th column and $(m+1)$ -th row reducing the grid back to its original grid number n .



After the I_4 move, the grid number returns to the original value n , and the \times and the \circ in the last column are in the same relative row positions as before the

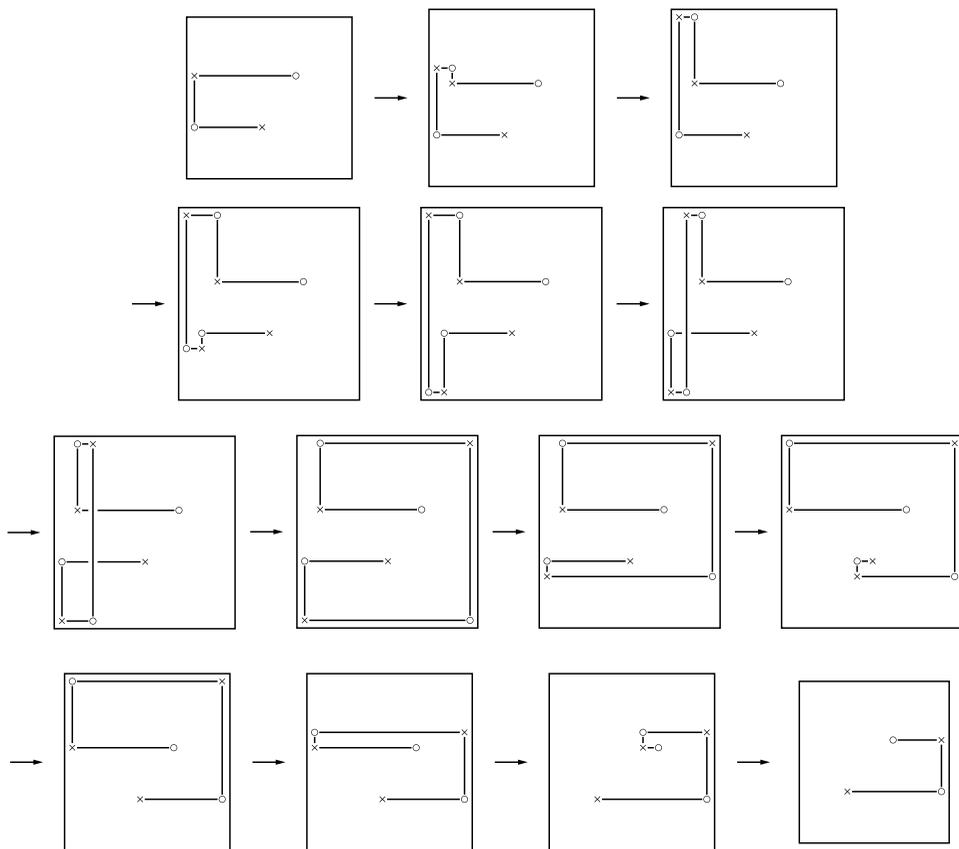


Figure 15. An illustration of the application of the intermediate grid moves used to produce a cyclic permutation grid move.

intermediate grid moves were applied. Since all commutations preserved the knot type, the grid equivalence class was preserved. \square

4. Proof of Theorem 1.2

Theorem 1.2. *There exists a sequence of commutation and (de-)stabilization grid moves that perform the cyclic permutation grid move.*

Proof. Given a grid diagram, apply the intermediate grid moves I_1 , I_2 , I_3 and I_4 sequentially. As shown by construction, this sequence of intermediate moves preserves the grid number and relative row position of the \times and \circ decorations in the permuted column. Thus this sequence of intermediate moves performs a column permutation with only commutations and (de-)stabilizations. Figure 15 is a stylized diagram following the strand of the knot through the sequential application of the

intermediate grid moves to explicate this construction. This process can be applied with an appropriate change of orientation to accomplish a cyclic permutation for a row or column in any direction. \square

Acknowledgements

This paper formed part of a VIGRE funded undergraduate research project at UCLA (summer 2010). I would like to thank faculty advisor Liam Watson for significant guidance, inspiration, editing and choice of topic.

References

- [Cromwell 1995] P. R. Cromwell, “Embedding knots and links in an open book, I: Basic properties”, *Topology Appl.* **64**:1 (1995), 37–58. [MR 96g:57006](#) [Zbl 0845.57004](#)
- [Dynnikov 2006] I. A. Dynnikov, “Arc-presentations of links: monotonic simplification”, *Fund. Math.* **190** (2006), 29–76. [MR 2007e:57006](#) [Zbl 1132.57006](#)
- [Manolescu et al. 2007] C. Manolescu, P. Ozsváth, Z. Szabó, and D. Thurston, “On combinatorial link Floer homology”, *Geom. Topol.* **11** (2007), 2339–2412. [MR 2009c:57053](#) [Zbl 1155.57030](#)
- [Manolescu et al. 2009] C. Manolescu, P. Ozsváth, and S. Sarkar, “A combinatorial description of knot Floer homology”, *Ann. of Math. (2)* **169**:2 (2009), 633–660. [MR 2009k:57047](#) [Zbl 1179.57022](#)
- [Ozsváth et al. 2008] P. Ozsváth, Z. Szabó, and D. Thurston, “Legendrian knots, transverse knots and combinatorial Floer homology”, *Geom. Topol.* **12**:2 (2008), 941–980. [MR 2009f:57051](#) [Zbl 1144.57012](#)
- [Reidemeister 1932] K. Reidemeister, *Knotentheorie*, Ergebnisse der Mathematik und ihrer Grenzgebiete **1**, Springer, Berlin, 1932. Reprinted in 1974. [MR 49 #9828](#) [Zbl 0005.12001](#)

Received: 2012-11-20

Revised: 2014-10-21

Accepted: 2015-01-16

nancy.scherich@gmail.com

Department of Mathematics, University of California, Santa Barbara, Santa Barbara, CA 93106, United States

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	La Trobe University, Australia P.Cerone@latrobe.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tbriell@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsdam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jpgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Pomomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sgupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nhritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION

Silvio Levy, Scientific Editor

Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2015 is US \$140/year for the electronic version, and \$190/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow[®] from Mathematical Sciences Publishers.

PUBLISHED BY



mathematical sciences publishers

nonprofit scientific publishing

<http://msp.org/>

© 2015 Mathematical Sciences Publishers

involve

2015

vol. 8

no. 5

A simplification of grid equivalence	721
NANCY SCHERICH	
A permutation test for three-dimensional rotation data	735
DANIEL BERO AND MELISSA BINGHAM	
Power values of the product of the Euler function and the sum of divisors function	745
LUIS ELESBAN SANTOS CRUZ AND FLORIAN LUCA	
On the cardinality of infinite symmetric groups	749
MATT GETZEN	
Adjacency matrices of zero-divisor graphs of integers modulo n	753
MATTHEW YOUNG	
Expected maximum vertex valence in pairs of polygonal triangulations	763
TIMOTHY CHU AND SEAN CLEARY	
Generalizations of Pappus' centroid theorem via Stokes' theorem	771
COLE ADAMS, STEPHEN LOVETT AND MATTHEW MCMILLAN	
A numerical investigation of level sets of extremal Sobolev functions	787
STEFAN JUHNKE AND JESSE RATZKIN	
Coalitions and cliques in the school choice problem	801
SINAN AKSOY, ADAM AZZAM, CHAYA COPPERSMITH, JULIE GLASS, GIZEM KARAALI, XUEYING ZHAO AND XINJING ZHU	
The chromatic polynomials of signed Petersen graphs	825
MATTHIAS BECK, ERIKA MEZA, BRYAN NEVAREZ, ALANA SHINE AND MICHAEL YOUNG	
Domino tilings of Aztec diamonds, Baxter permutations, and snow leopard permutations	833
BENJAMIN CAFFREY, ERIC S. EGGE, GREGORY MICHEL, KAILEE RUBIN AND JONATHAN VER STEEGH	
The Weibull distribution and Benford's law	859
VICTORIA CUFF, ALLISON LEWIS AND STEVEN J. MILLER	
Differentiation properties of the perimeter-to-area ratio for finitely many overlapped unit squares	875
PAUL D. HUMKE, CAMERON MARCOTT, BJORN MELLEM AND COLE STIEGLER	
On the Levi graph of point-line configurations	893
JESSICA HAUSCHILD, JAZMIN ORTIZ AND OSCAR VEGA	