

# involve

a journal of mathematics

A permutation test for three-dimensional  
rotation data

Daniel Bero and Melissa Bingham





# A permutation test for three-dimensional rotation data

Daniel Bero and Melissa Bingham

(Communicated by Mary C. Meyer)

Statistical inference procedures that require no distributional assumptions make up the area of nonparametric statistics. The permutation test is a common nonparametric test that can be used to compare measures of center for two data sets, but it is yet to be explored for three-dimensional rotation data. A permutation test for such data is developed and the statistical power of this test is considered under various scenarios. The test is then used in an application comparing movement around joints in the foot and ankle for humans, chimpanzees, and baboons.

## 1. Introduction

Data in the form of three-dimensional rotations are common in the study of human motion. As skeletal mammals move, the orientation of various joints can be tracked by using infrared emitting diodes attached to bones on opposite ends of the joint. Each joint orientation can be represented mathematically as a  $3 \times 3$  orthogonal rotation matrix. Of interest here is comparing movement around various joints in the ankle and foot for humans, chimpanzees, and baboons by comparing the central rotation of each joint for the various species.

While other works have considered comparing sets of three-dimensional rotation data, they rely on distributional assumptions [Rancourt et al. 2000; Hendriks and Landsman 1998]. Further, existing work for studying three-dimensional rotations is often in terms of manifold considerations. As such, it is often inaccessible to practitioners outside the area. Our aim here is development of methodology for comparing central rotations that is both nonparametric and does not rely on special manifold theory, so that it can be used more broadly. The permutation test is a commonly used nonparametric test, but it has yet to be implemented for three-dimensional rotation data. We develop such a test in Section 2, explore the statistical power of the test in Section 3, and apply the test to joint data in Section 4.

---

*MSC2010:* 62G09, 62G10.

*Keywords:* nonparametric statistics, permutation test, directional data.

This research was supported by NSF grant DMS-1104409.

## 2. Development of a three-dimensional permutation test

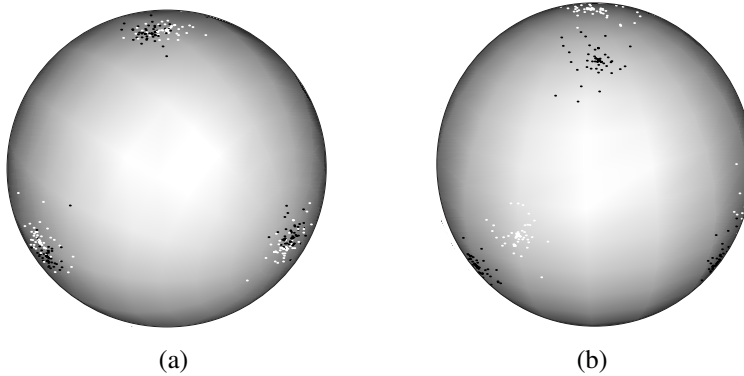
The permutation test is widely used in nonparametric statistics for determining if two data sets are different in some way (e.g., comparing means, variances, shapes). The most common example of a permutation test in one dimension is comparing population means for data sets  $A$  and  $B$  by using the difference in sample means,  $\bar{x}_A - \bar{x}_B$ , as a test statistic. To perform the permutation test, data sets  $A$  and  $B$  are combined and permuted so that data points are randomly reassigned to either  $A$  or  $B$ . The permuted test statistic is then calculated from this permuted data and this process is repeated a large number of times. If the means of the populations from which  $A$  and  $B$  come do in fact differ, then we expect the observed test statistic  $\bar{x}_A - \bar{x}_B$  to be more extreme than the permuted test statistics. For this reason, the  $p$ -value for a permutation test is defined to be the proportion of times that the permuted test statistic is more extreme than the observed test statistic. See [Higgins 2004] for more details on permutation tests.

To translate the idea of the permutation test to three-dimensional rotation data, we first need to define a sensible test statistic that could be used for comparing two central rotations. For each set of three-dimensional rotations, we begin by finding a measure of center as follows. Compute  $\bar{\mathbf{O}} = 1/n \sum_{i=1}^n \mathbf{O}_i$  for  $\mathbf{O}_1, \dots, \mathbf{O}_n \in \text{SO}(3)$ , where  $\text{SO}(3)$  represents the set of all  $3 \times 3$  orthogonal rotation matrices. Next, find the matrix  $\mathbf{T} = \mathbf{V}\mathbf{W}$ , where  $\bar{\mathbf{O}} = \mathbf{V}\mathbf{\Sigma}\mathbf{W}$  is the singular value decomposition of  $\bar{\mathbf{O}}$ . Using these components from the singular value decomposition is necessary since  $\bar{\mathbf{O}}$  may not be an element of  $\text{SO}(3)$ , but  $\mathbf{T}$  is. This is a commonly used measure of center [León et al. 2006; Bingham et al. 2009; Khatri and Mardia 1977], which we refer to as the “mean” rotation.

Once we have found the mean rotation for each of our two data sets, a natural test statistic is the difference between these mean rotations. One way of quantifying the difference between two three-dimensional rotations is by using angles. A misorientation angle is defined as the angle needed to rotate from one three-dimensional rotation to another via a spin about some axis. For  $\mathbf{O}, \mathbf{P} \in \text{SO}(3)$ , the misorientation angle between  $\mathbf{O}$  and  $\mathbf{P}$  is

$$\text{mis}(\mathbf{O}, \mathbf{P}) = \arccos\left(\frac{\text{tr}(\mathbf{O}'\mathbf{P}) - 1}{2}\right), \quad (1)$$

where  $\text{tr}$  is the trace of a matrix and  $\mathbf{O}'$  is the transpose of  $\mathbf{O}$ . We use the misorientation angle between our two mean rotations as the test statistic for the three-dimensional permutation test of  $H_0$ : There is no difference between the population mean rotations versus  $H_a$ : There is a difference between the population mean rotations. The steps of the permutation test are given below and R code for implementing this test is provided in the Appendix.



**Figure 1.** Plots of two simulated three-dimensional rotation data sets (each with  $n = 50$ ) with mean rotations that (a) are not significantly different and (b) are significantly different.

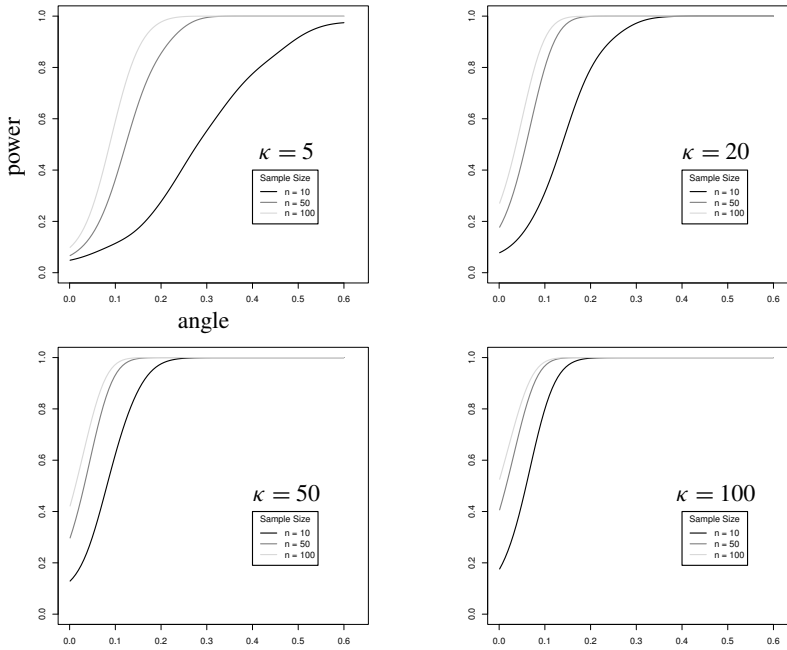
- (1) Calculate the mean rotation for each data set and then find the misorientation angle between these means. This serves as the observed test statistic,  $\theta_{\text{obs}}$ .
- (2) Permute the data a large number (say 10,000) of times, storing the misorientation angle between the permuted mean rotations,  $\theta_{\text{perm}}$ , each time.
- (3) Let the  $p$ -value be the fraction of times that the permuted misorientation angle is greater than the observed misorientation angle; that is,

$$p\text{-value} = \frac{\# \text{ of times } \theta_{\text{perm}} > \theta_{\text{obs}}}{\# \text{ of permutations}}.$$

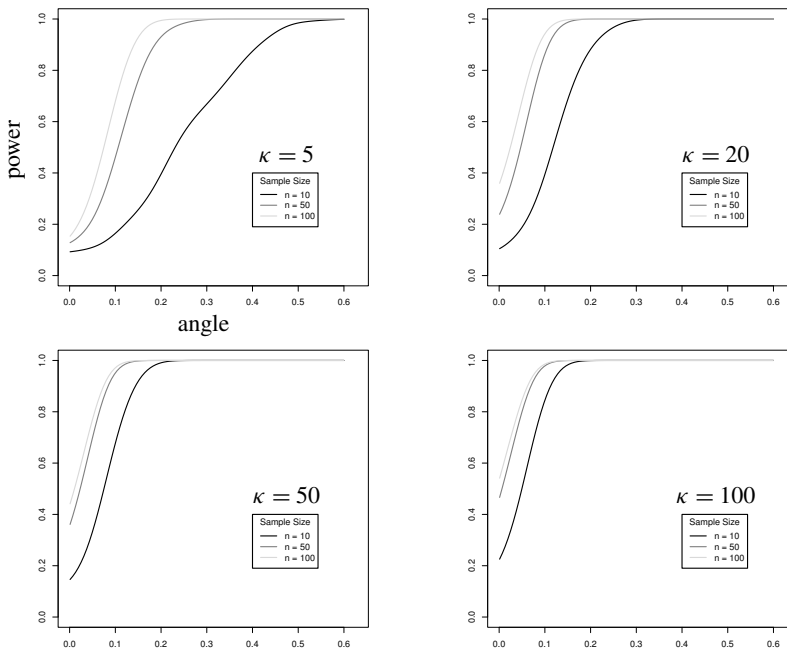
The three-dimensional permutation test outlined above is briefly illustrated in two different examples. Figure 1 shows three-dimensional data sets plotted as points on the sphere, with one observation represented by three points that would correspond to three orthogonal axes. In Figure 1(a), the two simulated data sets (in white and black, each of size 50) show considerable overlap. Under the permutation test, these data sets resulted in a test statistic of 0.0546 and a  $p$ -value of 0.3101. In Figure 1(b), the simulated data sets are more separated. These data sets gave a test statistic of 0.6102 and a  $p$ -value of 0, indicating a significant difference in the population mean rotations. These examples suggest that the  $p$ -value decreases as expected when the data sets have mean rotations that increase in distance.

### 3. Power: a simulation study

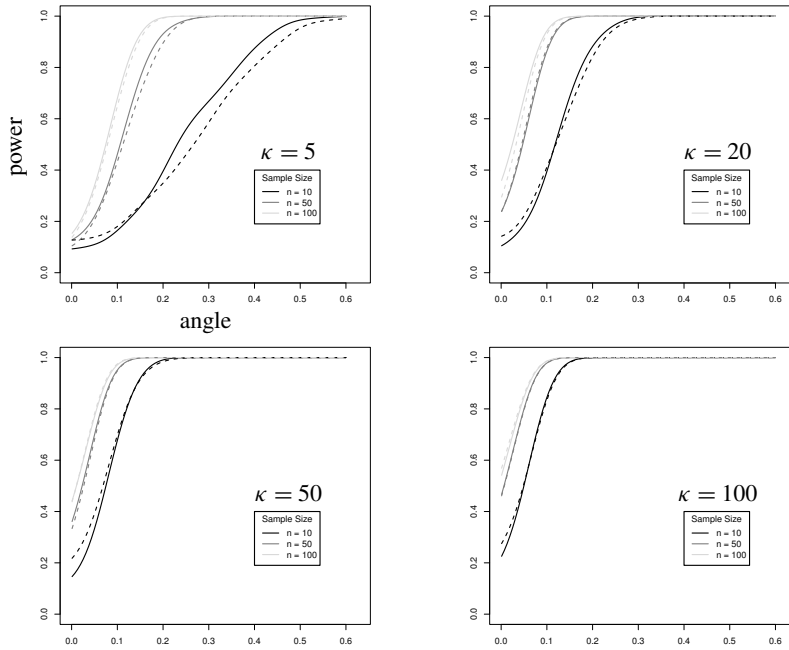
To examine the effectiveness of the three-dimensional permutation test developed in Section 2, we perform a simulation study to investigate statistical power. Power is the probability of correctly rejecting a false null hypothesis. We simulate data sets with centers that differ by a known misorientation angle,  $\phi$ , (i.e., there is a



**Figure 2.** Plots of power versus misorientation angle for the von Mises version of the UARS distributions with  $\kappa = 5, 20, 50, 100$ .



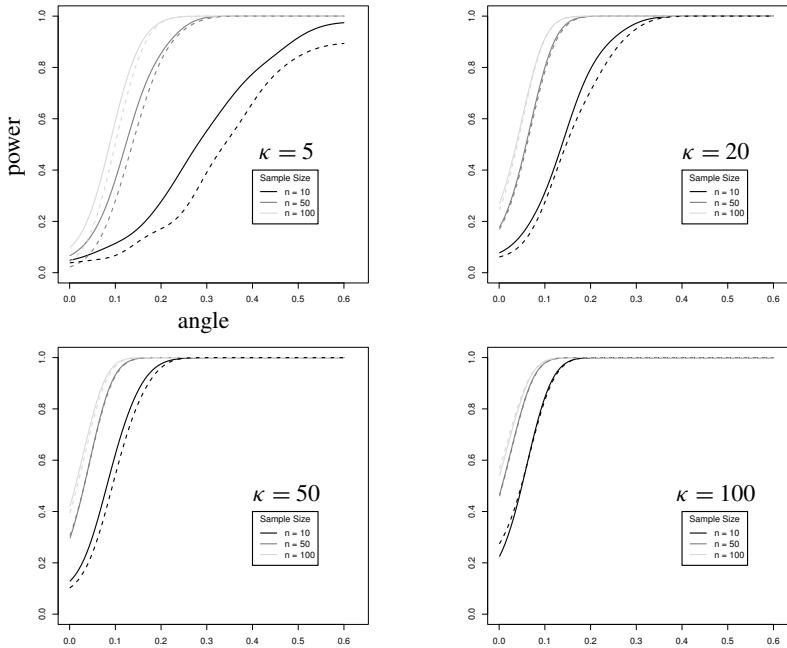
**Figure 3.** Plots of power versus misorientation angle for the symmetric matrix von Mises–Fisher distribution with  $\kappa = 5, 20, 50, 100$ .



**Figure 4.** Plots of power versus misorientation angle for the vM-F distribution with solid lines representing the permutation test and dashed lines representing the parametric approach for  $\kappa = 5, 20, 50, 100$ .

difference between the population mean rotations and the null is false) from both the von Mises version of the uniform axis-random spin (vM-UARS) distribution [Bingham et al. 2009] and the symmetric version of the matrix von Mises–Fisher (vM-F) distribution [Khatri and Mardia 1977]. A vM-UARS or vM-F distribution can be specified by a central rotation  $S \in SO(3)$  and a spread parameter  $\kappa \in (0, \infty)$ , where  $\kappa$  is best termed as a concentration parameter since larger values of  $\kappa$  indicate rotations that are less spread about the center at  $S$ . Two samples, each of size  $n$ , are generated from vM-UARS( $S_1, \kappa$ ) and vM-UARS( $S_2, \kappa$ ) distributions, where  $\phi = \text{mis}(S_1, S_2)$  as in (1). We consider  $\kappa$  values of 5, 20, 50, and 100, set  $n$  at 10, 50, and 100, and let the misorientation angle,  $\phi$ , vary between 0 and  $\pi/5$ . The same is done for the vM-F distribution.

For each combination of  $\kappa, n$ , and  $\phi$ , the permutation test was conducted 1,000 times with 1,000 permutations per test. The power was then found as the proportion of times (out of 1,000) that the test correctly rejected the null hypothesis of equal means. Plots of the power against the misorientation angle,  $\phi$ , for the various choices of  $n$  and  $\kappa$  are provided in Figure 2 for the vM-UARS distribution and in Figure 3 for the vM-F distribution. It can be seen from all plots that as sample size increases,

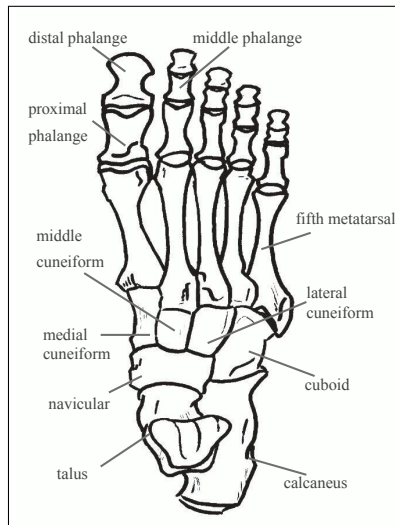


**Figure 5.** Plots of power versus misorientation angle for the vM-UARS distribution with solid lines representing the permutation test and dashed lines representing the parametric approach for  $\kappa = 5, 20, 50, 100$ .

the power of the test increases. In addition, as the concentration parameter,  $\kappa$ , increases (i.e., data sets become more clustered around their mean rotation), the power increases. Finally, as the misorientation angle increases and the true centers become farther apart, the power increases. This mimics properties of power for traditional hypothesis tests for differences in means (for nonrotational data), giving evidence that the three-dimensional permutation test performs as desired.

The power of the three-dimensional permutation test was also compared to that of the parametric approach presented in [Rancourt et al. 2000], which requires the observations be distributed according to the matrix von Mises–Fisher distribution. The plots in Figure 4 show power versus misorientation angle for the various choices of  $n$  and  $\kappa$  using the matrix von Mises–Fisher distribution. The solid lines represent power for the permutation test, with the dashed lines representing power for the parametric approach. We see that the power of the permutation test is comparable to the power of parametric approach in all cases. The permutation test was also compared to the parametric approach for the vM-UARS distribution, with power plots given in Figure 5. We see that the permutation test outperforms the parametric approach in terms of resulting in a larger power, with this fact more visible when





**Figure 6.** Bones in the ankle and foot (image taken from <http://www.ceuarmy.com/BSFAFpdf.pdf>).

we have smaller sample sizes or data that is more spread (small  $\kappa$ ). Thus, the three-dimensional permutation test is comparable to the parametric approach when the assumptions of the parametric test are met, and it performs better than the parametric approach when the assumptions are not met.

#### 4. Application to ankle joint rotation data

Now that we have verified that the three-dimensional permutation test performs as expected with regard to power, we apply the test to ankle/foot joint rotation data collected by Prof. Thomas Greiner of the Department of Physical Therapy at the University of Wisconsin-La Crosse. Data was collected from humans, baboons, and chimps during circumduction, which is the movement characterized by the foot being placed flat on the floor and the leg rotating in a circular motion around it. Infrared emitting diodes attached to bones on each side of a joint give the orientation of each bone as the movement occurs. If the orientation of the first bone is represented as  $F$  and the orientation of second bone is represented as  $G$ , then the resulting orientation of the joint is defined as  $F'G$ . Because markers may not have been placed identically on all subjects, the orientations of all joints under consideration were measured with the tibia-talus joint as the reference to allow for comparison of species. Joints considered were the cuboid-calcaneus, navicular-cuboid, navicular-talus, talus - calcaneus, and fifth metatarsal-cuboid. (See Figure 6 for a diagram of the bones in the foot and ankle region.) Orientations were collected for six human subjects, four chimpanzee subjects, and seven baboon subjects, and the base alignment matrix

corresponding to the primary rotational axis (see [Ball and Greiner 2012]) was used in the three-dimensional permutation test to compare species.

Species were compared pairwise (human versus chimpanzee, human versus baboon, and chimpanzee versus baboon) for each of the joints mentioned above, and each test was done using 1,000 permutations. Out of all tests, there were four significant differences found. There was significant evidence to suggest that the orientation of the navicular-talus joint differs between the humans and chimpanzees ( $p$ -value = 0.001) and humans and baboons ( $p$ -value  $\approx 0$ ). The orientation of the talus-calcaneus joint was found to be significantly different between humans and chimpanzees ( $p$ -value = 0.019) and humans and baboons ( $p$ -value = 0.001). Therefore, it appears that movement for humans differs from baboons and chimps when considering two specific joints.

## 5. Conclusion

The analysis of joint rotation data provided here is just one of many applications that the three-dimensional permutation test could be used for. Given the abundance of three-dimensional rotation data in the study of human motion, as well as in the other fields like materials science, having methodology for comparing measures of center for three-dimensional data is important. The three-dimensional permutation test developed here provides that methodology without the need for any distributional assumptions on where the data sets come from. It also does not require any theory on special manifolds, making the three-dimensional permutation test an important addition to the field of statistics, as well as to practitioners who collect data in this form.

## Appendix

The following gives an R function called `PermTest` for performing the three-dimensional permutation test on data sets  $A$  (of size  $n_A$ ) and  $B$  (of size  $n_B$ ). The argument  $A$  must be an array of dimension  $3 \times 3 \times n_A$  and  $B$  must be an array of dimension  $3 \times 3 \times n_B$ . The argument `nspec` specifies the number of times the data should be permuted. The function `PermTest` outputs the test statistic (misorientation angle between the two sample mean rotations) and  $p$ -value.

```
PermTest=function(A,B,nspec){
  ##Loads functions needed for test
  trace=function(M){sum(diag(M))}
  Mis.Ang=function(C,D){acos((trace(t(C)%*%D)-1)/2)}

  ##Finds mean matrices for both sets of data
  na=dim(A)[3]
  Abar=matrix(rep(0,9),nrow=3)
  for(i in 1:na){Abar=Abar+A[,,i]}
```

```

Abar=Abar/na
M.A=svd(Abar)$u%*%t(svd(Abar)$v)
nb=dim(B)[3]
Bbar=matrix(rep(0,9),nrow=3)
for(i in 1:nb){Bbar=Bbar+B[,i]}
Bbar=Bbar/nb
M.B=svd(Bbar)$u%*%t(svd(Bbar)$v)

##Finds the test statistic
Test.Stat=Mis.Ang(M.A,M.B)

##Puts data into one array
T=array(c(A,B),dim=c(3,3,(na+nb)))

##Performs the permutation test
nsim=nspec
ang=rep(0,nsim)
for(i in 1:nsim){
  samp=sample(1:(na+nb))
  O=T[, ,samp[1:na]]
  P=T[, ,samp[(na+1):(na+nb)]]
  Obar=matrix(rep(0,9),nrow=3)
  for(j in 1:na){Obar=Obar+O[, ,j]}
  Obar=Obar/na
  M.O=svd(Obar)$u%*%t(svd(Obar)$v)
  Pbar=matrix(rep(0,9),nrow=3)
  for(k in 1:nb){Pbar=Pbar+P[, ,k]}
  Pbar=Pbar/nb
  M.P=svd(Pbar)$u%*%t(svd(Pbar)$v)
  ang[i]=Mis.Ang(M.O,M.P)
}
p.value=sum(ang>Test.Stat)/nsim
list(Test.Statistic=Test.Stat,P.Value=p.value)
}

```

## References

- [Ball and Greiner 2012] K. A. Ball and T. M. Greiner, "A procedure to refine joint kinematic assessments: functional alignment", *Comput. Methods Biomech. Biomed. Eng.* **15**:5 (2012), 487–500.
- [Bingham et al. 2009] M. A. Bingham, D. J. Nordman, and S. B. Vardeman, "Modeling and inference for measured crystal orientations and a tractable class of symmetric distributions for rotations in three dimensions", *J. Amer. Statist. Assoc.* **104**:488 (2009), 1385–1397. MR 2011a:62189 Zbl 1205.62215
- [Hendriks and Landsman 1998] H. Hendriks and Z. Landsman, "Mean location and sample mean location on manifolds: asymptotics, tests, confidence regions", *J. Multivariate Anal.* **67**:2 (1998), 227–243. MR 2000a:62125 Zbl 0941.62069

[Higgins 2004] J. J. Higgins, *Introduction to modern nonparametric statistics*, Brooks/Cole, Pacific Grove, CA, 2004.

[Khatri and Mardia 1977] C. G. Khatri and K. V. Mardia, “The von Mises–Fisher matrix distribution in orientation statistics”, *J. Roy. Statist. Soc. (B) Stat. Methodol.* **39**:1 (1977), 95–106. MR 58 #13506 Zbl 0356.62044

[León et al. 2006] C. A. León, J.-C. Massé, and L.-P. Rivest, “A statistical model for random rotations”, *J. Multivariate Anal.* **97**:2 (2006), 412–430. MR 2234030 Zbl 1085.62066

[Rancourt et al. 2000] D. Rancourt, L.-P. Rivest, and J. Asselin, “Using orientation statistics to investigate variations in human kinematics”, *J. Roy. Statist. Soc. (C) Appl. Stat.* **49**:1 (2000), 81–94. MR 1817876 Zbl 0974.62107

Received: 2013-06-03

Revised: 2014-07-22

Accepted: 2014-09-07

dbero@iastate.edu

*Colony Brands, Monroe, WI 53566, United States*

mbingham@uwlax.edu

*Mathematics Department, University of Wisconsin-La Crosse,  
1725 State Street, La Crosse, WI 54601, United States*

## MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

## BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	La Trobe University, Australia P.Cerone@latrobe.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsdam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

## PRODUCTION

Silvio Levy, Scientific Editor


Cover: Alex Scorpan

See inside back cover or [msp.org/involve](http://msp.org/involve) for submission instructions. The subscription price for 2015 is US \$140/year for the electronic version, and \$190/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2015 Mathematical Sciences Publishers

# involve

2015

vol. 8

no. 5

A simplification of grid equivalence NANCY SCHERICH	721
A permutation test for three-dimensional rotation data DANIEL BERO AND MELISSA BINGHAM	735
Power values of the product of the Euler function and the sum of divisors function LUIS ELESBAN SANTOS CRUZ AND FLORIAN LUCA	745
On the cardinality of infinite symmetric groups MATT GETZEN	749
Adjacency matrices of zero-divisor graphs of integers modulo $n$ MATTHEW YOUNG	753
Expected maximum vertex valence in pairs of polygonal triangulations TIMOTHY CHU AND SEAN CLEARY	763
Generalizations of Pappus' centroid theorem via Stokes' theorem COLE ADAMS, STEPHEN LOVETT AND MATTHEW McMILLAN	771
A numerical investigation of level sets of extremal Sobolev functions STEFAN JUHNKE AND JESSE RATZKIN	787
Coalitions and cliques in the school choice problem SINAN AKSOY, ADAM AZZAM, CHAYA COPPERSMITH, JULIE GLASS, GIZEM KARAALI, XUEYING ZHAO AND XINJING ZHU	801
The chromatic polynomials of signed Petersen graphs MATTHIAS BECK, ERIKA MEZA, BRYAN NEVAREZ, ALANA SHINE AND MICHAEL YOUNG	825
Domino tilings of Aztec diamonds, Baxter permutations, and snow leopard permutations BENJAMIN CAFFREY, ERIC S. EGGE, GREGORY MICHEL, KAILEE RUBIN AND JONATHAN VER STEEGH	833
The Weibull distribution and Benford's law VICTORIA CUFF, ALLISON LEWIS AND STEVEN J. MILLER	859
Differentiation properties of the perimeter-to-area ratio for finitely many overlapped unit squares PAUL D. HUMKE, CAMERON MARCOTT, BJORN MELLEM AND COLE STIEGLER	875
On the Levi graph of point-line configurations JESSICA HAUSCHILD, JAZMIN ORTIZ AND OSCAR VEGA	893



1944-4176(2015)8:5;1-2