Power values of the product of the Euler function and the sum of divisors function

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We find examples of positive integers $n$ such that $\phi(n^3)\sigma(n^3)$ is a perfect square.

1. Introduction

The Euler function $\phi(n)$ counts the number of positive integers $m \leq n$ which are coprime to $n$, the sum of divisors function $\sigma(n)$ is equal to the sum of the positive proper divisors of $n$, and both of these functions have fascinated mathematicians for centuries. A lot of effort has been spent trying to find positive integers $n$ such that $\phi(n)$ and $\sigma(n)$ have nice arithmetic properties.

It is easy to make $\phi(n)$ a square. Just take $n = 2^{2k+1}$ for some $k \geq 0$. Exactly half of all integers $m \leq 2^{2k+1}$ are odd, and hence, coprime to $n$. Thus, $\phi(2^{2k+1}) = 2^k$ is a perfect square. The situation for the sum of divisors function is harder. A nice presentation of this problem is in [Beukers et al. 2012]. Following that reference, we look at the factorizations

\[
\begin{align*}
\sigma(2) &= 3, & \sigma(11) &= 2^2 \times 3, \\
\sigma(3) &= 2^2, & \sigma(13) &= 2 \times 7, \\
\sigma(5) &= 2 \times 3, & \sigma(17) &= 2 \times 3^2, \\
\sigma(7) &= 2^3, & \sigma(19) &= 2^2 \times 5.
\end{align*}
\]

There are many ways to multiply together some of the above numbers to get a perfect square. First let us notice that 13 and 19 are useless because $\sigma(13) = 2 \times 7$ and $\sigma(19) = 2^2 \times 5$, and neither 7 nor 5 ever appear again on the right-hand side of the above equations. Throw out 13 and 19 and group squares on the right-hand sides in the following way, where $\Box$ represents a perfect square:

\[
\begin{align*}
\sigma(2) &= 3, & \sigma(3) &= \Box, & \sigma(5) &= 2 \times 3, & \sigma(7) &= 2 \Box, & \sigma(11) &= 3 \Box, & \sigma(17) &= 2 \Box.
\end{align*}
\]

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Note that all six inputs are prime numbers and all outputs have prime factorizations consisting of only 2 and 3. Let the primes 2, 3, 5, 7, 11, 17 correspond to the vectors $v_1, v_2, v_3, v_4, v_5, v_6$ in the six-dimensional vector space $\mathbb{F}_2^6$, where $v_i$ has $i$-th component equal to 1 and all others equal to 0 for $i = 1, \ldots, 6$. In $\mathbb{F}_2^2$ we let $w_1$ and $w_2$ be the vectors $(1, 0)^\top$ and $(0, 1)^\top$ and think of them as corresponding to the primes 2 and 3 respectively. We define a linear map from $\mathbb{F}_2^6 \mapsto \mathbb{F}_2^2$ whose matrix is

$$T = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}. $$

This matrix has rank 2, so it has $2^4 = 16$ vectors in its nullspace, and any of these vectors gives us a solution. For example, the vector $(1, 1, 1, 1, 0, 0)^\top$, which is in Null$(T)$, gives us the solution $n = 2 \times 3 \times 5 \times 7$, having $\sigma(n) = 2^6 \times 3^2$.

In [Beukers et al. 2012], the equation $\sigma(n^k) = m^l$ in positive integers $n$ and $m$ was studied for some exponents $k > 1$ and $l > 1$. On page 377, they conjecture that $\sigma(n^k) = m^l$ has only finitely many solutions if $k > 3$ and $l > 1$ are given. Here, we propose the following counterconjecture.

**Conjecture 1.** For every $k > 1$ and $l > 1$, there are infinitely many $n$ such that $\sigma(n^k) = m^l$ for some positive integer $m$.

To give some evidence, we propose a different conjecture. Let $P(n)$ denote the largest prime factor of the integer $n$, with the convention that $P(0) = P(\pm 1) = 1$.

**Conjecture 2.** Let $f(x) \in \mathbb{Z}[x]$ be a polynomial such that $f(0) \neq 0$. For every $\varepsilon > 0$, there exists $c := c(\varepsilon)$ and $x_0 := x_0(\varepsilon)$ such that

$$\# \{ p \leq x : P(f(p)) < x^\varepsilon \} > cx/\log x \quad \text{for all} \quad x > x_0. \quad (1)$$

The substance of the above conjecture is the following. It is well known that the numbers $n$ such that $P(n) < n^\varepsilon$ form a positive-density subset of $\mathbb{N}$. It is conjectured that the primes $p$ such that $P(p - 1) < p^\varepsilon$ form a positive-density subset of all primes. This is not known for small values of $\varepsilon > 0$. So, we venture even further and replace $p - 1$ by any fixed polynomial $f(p)$ such that $f(0) \neq 0$ (in order to make sure that $p$ does not show up as a natural divisor of $f(p)$) and conjecture that, in fact, the set of primes $p$ such that $P(f(p)) < p^\varepsilon$ is of positive density. This is known if all roots of $f(x)$ are rational, with some $\varepsilon < 1$ (like $\varepsilon = 1 - 1/2d$, where $d$ is the degree of $f(x)$), but it is not known for any $\varepsilon < 1$ once $f(x)$ has an irreducible factor of degree at least 2. The quantity $x/\log x$ in the right-hand side of (1) arises from the prime number theorem, which asserts that, asymptotically, the function $\pi(x) = \# \{ p \leq x \}$ equals $x/\log x$ as $x \to \infty$.

Let us see how Conjecture 1 would follow from Conjecture 2. Let $k \geq 2$, $f(x) = (x^{k+1} - 1)/(x - 1)$ and suppose first that $l = 2$. Let $x$ be large, put $\varepsilon = 1/2$
and let \( p_1, \ldots, p_t \) be such that \( P(f(p_i)) < x^{1/2} \). Let \( s = \pi(x^{1/2}) \). Then we can write

\[
    f(p_i) = w_i^l, \quad i = 1, \ldots, t,
\]

where the \( w_i \) are square-free numbers with \( P(w_i) \leq x^{1/2} \). As before, we can identify the \( w_i \) with vectors in \( \mathbb{F}_2^s \) obtained by putting 1 or 0 in the \( j \)-th component according to whether the \( j \)-th prime divides \( w_i \) or not. In this way, we get a linear application from \( \mathbb{F}_2^t \) to \( \mathbb{F}_2^s \) whose nullspace has dimension at least \( t - s \), where

\[
    t - s > c \frac{x}{\log x} - \pi(x^{1/2}) > c \frac{x}{\log x} - x^{1/2},
\]

and this last function certainly tends to infinity with \( x \). This is when \( l = 2 \). Assume now that \( l > 2 \). Then we write

\[
    f(p_i) = w_i u_i^l \quad \text{for all} \quad i = 1, \ldots, t,
\]

where the \( w_i \) are \( l \)-th power free and \( P(w_i) \leq x^{1/2} \). We attach to each \( w_i \) an element \( \mathbf{w}_i \) in the group \( (\mathbb{Z}/l\mathbb{Z})^s \) where in the \( j \)-th component we put the exponent of the \( j \)-th prime number in the factorization of \( w_i \). Note that \( \mathbb{Z}/l\mathbb{Z} \) is not a field unless \( l \) is a prime, and even if \( l \) is a prime, we only can multiply distinct primes \( p_i \) in attempts to create \( n \) such that \( \sigma(n^k) = m^l \). Thus, we are only allowed to take sums of distinct \( \mathbf{w}_i \) and get 0. There is a theorem (see [van Emde Boas and Kruyswijk 1967] and [Olson 1969, Theorem 1]) that says that if we have at least \( s(l - 1) \) such distinct elements \( \mathbf{w}_i \), we can find some of them whose sum is 0. Thus, we can create at least \( \lfloor t/(s(l - 1)) \rfloor \) distinct (in fact, even disjoint) subsets of the \( \mathbf{w}_i \) for \( i = 1, \ldots, t \) simply by finding some 0-sum among the first \( s(l - 1) \) of them, another 0-sum among the next \( s(l - 1) \) of them and so on. Since

\[
    \frac{t}{s(l - 1)} > \frac{c \sqrt{x}}{(l - 1) \log x},
\]

and the right-hand side is a function that tends to infinity with \( x \), we get Conjecture 1.

We can ask similar questions simultaneously for \( \phi(n) \) and \( \sigma(n) \), like making them simultaneously squares, or cubes, etc. This has already been treated in [Freiberg 2012]. There it is shown that the number of \( n \leq x \) such that both \( \phi(n) \) and \( \sigma(n) \) are perfect powers of an exponent \( l \) is less than \( c_1 l x^{1/L}/(\log x)^{l+2} \), where \( c_1 > 0 \) is some positive constant. Square values of the product \( \phi(n)\sigma(n) \) have been investigated in [Broughan et al. 2013]. In the next section, we present some computational examples of \( n \) such that \( \phi(n^3)\sigma(n^3) = \square \).

### 2. Computational examples

We wanted to find a positive integer \( n \) such that \( \phi(n^3)\sigma(n^3) = \square \). For a prime \( p \), we have \( \phi(p^3)\sigma(p^3) = p^2(p^4 - 1) \). So, we wrote \( p^4 - 1 = w_p \square \), where \( w_p \) is square-free for all \( p \leq 1000 \). Then we searched for a subset \( S \) of cardinality \( t \) such
that the set of prime factors appearing in the factorizations of \( w_p \) for \( p \in S \) has cardinality \( s < t \). We found the subset

\[
\{2, 3, 5, 7, 13, 17, 23, 31, 41, 43, 47, 73, 83, 191, 239, 307, 443, 499, 829\},
\]

with \( t = 21 \) and \( s = 17 \). Thus, this set gives us \( 2^{21-17} = 16 \) solutions. We wrote down the \( \{0, 1\} \) matrix with 17 rows and 21 columns, which ends up having rank 17 over \( \mathbb{F}_2 \). The largest solution in the nullspace of this matrix is

\[
n = 3 \times 7 \times 11 \times 13 \times 17 \times 23 \times 43 \times 47 \times 83 \times 239 \times 443 \times 499 \times 829,
\]

for which \( \phi(n^3)\sigma(n^3) = m^2 \), where

\[
m = 2^{30} \times 3^7 \times 5^{10} \times 7^2 \times 11^4 \times 17^3 \times 23 \times 29 \times 37 \times 41 \times 53 \times 61 \times 83 \times 157.
\]

Despite our efforts, we could not find an integer \( n > 1 \) such that \( \sigma(n^5) = \square \), and we leave finding such an example as a challenge to the reader.

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elesluis@gmail.com

florian.luca@wits.ac.za
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