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Power values of the product of the Euler function and the sum of divisors function

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We find examples of positive integers n such that $\phi(n^3)\sigma(n^3)$ is a perfect square.

1. Introduction

The Euler function $\phi(n)$ counts the number of positive integers $m \le n$ which are coprime to n, the sum of divisors function $\sigma(n)$ is equal to the sum of the positive proper divisors of n, and both of these functions have fascinated mathematicians for centuries. A lot of effort has been spent trying to find positive integers n such that $\phi(n)$ and $\sigma(n)$ have nice arithmetic properties.

It is easy to make $\phi(n)$ a square. Just take $n = 2^{2k+1}$ for some $k \ge 0$. Exactly half of all integers $m \le 2^{2k+1}$ are odd, and hence, coprime to n. Thus, $\phi(2^{2k+1}) = 2^{2k}$ is a perfect square. The situation for the sum of divisors function is harder. A nice presentation of this problem is in [Beukers et al. 2012]. Following that reference, we look at the factorizations

$$\begin{aligned} \sigma(2) &= 3, & \sigma(11) &= 2^2 \times 3, \\ \sigma(3) &= 2^2, & \sigma(13) &= 2 \times 7, \\ \sigma(5) &= 2 \times 3, & \sigma(17) &= 2 \times 3^2, \\ \sigma(7) &= 2^3, & \sigma(19) &= 2^2 \times 5. \end{aligned}$$

There are many ways to multiply together some of the above numbers to get a perfect square. First let us notice that 13 and 19 are useless because $\sigma(13) = 2 \times 7$ and $\sigma(19) = 2^2 \times 5$, and neither 7 nor 5 ever appear again on the right-hand side of the above equations. Throw out 13 and 19 and group squares on the right-hand sides in the following way, where $\Box$ represents a perfect square:

$$\sigma(2) = 3, \quad \sigma(3) = \Box, \quad \sigma(5) = 2 \times 3, \quad \sigma(7) = 2\Box, \quad \sigma(11) = 3\Box, \quad \sigma(17) = 2\Box.$$

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Santos Cruz worked on this paper during a summer project under of the supervision of Luca.

Note that all six inputs are prime numbers and all outputs have prime factorizations consisting of only 2 and 3. Let the primes 2, 3, 5, 7, 11, 17 correspond to the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6$ in the six-dimensional vector space $\mathbb{F}_2^6$, where $\boldsymbol{v}_i$ has i-th component equal to 1 and all others equal to 0 for $i = 1, \ldots, 6$. In $\mathbb{F}_2^2$ we let $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ be the vectors $(1, 0)^\top$ and $(0, 1)^\top$ and think of them as corresponding to the primes 2 and 3 respectively. We define a linear map from $\mathbb{F}_2^6 \mapsto \mathbb{F}_2^2$ whose matrix is

$$T = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

This matrix has rank 2, so it has $2^4 = 16$ vectors in its nullspace, and any of these vectors gives us a solution. For example, the vector $(1, 1, 1, 1, 0, 0)^\top$, which is in $\mathrm{Null}(T)$, gives us the solution $n = 2 \times 3 \times 5 \times 7$, having $\sigma(n) = 2^6 \times 3^2$.

In [Beukers et al. 2012], the equation $\sigma(n^k) = m^l$ in positive integers n and m was studied for some exponents $k > 1$ and $l > 1$. On page 377, they conjecture that $\sigma(n^k) = m^l$ has only finitely many solutions if $k > 3$ and $l > 1$ are given. Here, we propose the following counterconjecture.

Conjecture 1. *For every $k > 1$ and $l > 1$, there are infinitely many n such that $\sigma(n^k) = m^l$ for some positive integer m.*

To give some evidence, we propose a different conjecture. Let $P(n)$ denote the largest prime factor of the integer n, with the convention that $P(0) = P(\pm 1) = 1$.

Conjecture 2. *Let $f(x) \in \mathbb{Z}[x]$ be a polynomial such that $f(0) \neq 0$. For every $\varepsilon > 0$, there exists $c := c(\varepsilon)$ and $x_0 := x_0(\varepsilon)$ such that*

$$\#\{p \le x : P(f(p)) < x^\varepsilon\} > cx/\log x \quad \textit{for all} \quad x > x_0. \tag{1}$$

The substance of the above conjecture is the following. It is well known that the numbers n such that $P(n) < n^\varepsilon$ form a positive-density subset of $\mathbb{N}$. It is conjectured that the primes p such that $P(p-1) < p^\varepsilon$ form a positive-density subset of all primes. This is not known for small values of $\varepsilon > 0$. So, we venture even further and replace $p - 1$ by any fixed polynomial $f(p)$ such that $f(0) \neq 0$ (in order to make sure that p does not show up as a natural divisor of $f(p)$) and conjecture that, in fact, the set of primes p such that $P(f(p)) < p^\varepsilon$ is of positive density. This is known if all roots of $f(x)$ are rational, with some $\varepsilon < 1$ (like $\varepsilon = 1 - 1/2d$, where d is the degree of $f(x)$), but it is not known for any $\varepsilon < 1$ once $f(x)$ has an irreducible factor of degree at least 2. The quantity $x/\log x$ in the right-hand side of (1) arises from the prime number theorem, which asserts that, asymptotically, the function $\pi(x) = \#\{p \le x\}$ equals $x/\log x$ as $x \to \infty$.

Let us see how Conjecture 1 would follow from Conjecture 2. Let $k \ge 2$, $f(x) = (x^{k+1} - 1)/(x - 1)$ and suppose first that $l = 2$. Let x be large, put $\varepsilon = 1/2$

and let $p_1, \dots, p_t$ be such that $P(f(p_i)) < x^{1/2}$. Let $s = \pi(x^{1/2})$. Then we can write

$$f(p_i) = w_i \square, \quad i = 1, \dots, t,$$

where the w_i are square-free numbers with $P(w_i) \le x^{1/2}$. As before, we can identify the w_i with vectors in $\mathbb{F}_2^s$ obtained by putting 1 or 0 in the j-th component according to whether the j-th prime divides w_i or not. In this way, we get a linear application from $\mathbb{F}_2^t$ to $\mathbb{F}_2^s$ whose nullspace has dimension at least $t - s$, where

$$t - s > c\frac{x}{\log x} - \pi(x^{1/2}) > c\frac{x}{\log x} - x^{1/2},$$

and this last function certainly tends to infinity with x. This is when $l = 2$. Assume now that $l > 2$. Then we write

$$f(p_i) = w_i u_i^l \quad \text{for all} \quad i = 1, \dots, t,$$

where the w_i are l-th power free and $P(w_i) \le x^{1/2}$. We attach to each w_i an element $\boldsymbol{w}_i$ in the group $(\mathbb{Z}/l\mathbb{Z})^s$ where in the j-th component we put the exponent of the j-th prime number in the factorization of w_i. Note that $\mathbb{Z}/l\mathbb{Z}$ is not a field unless l is a prime, and even if l is a prime, we only can multiply *distinct* primes p_i in attempts to create n such that $\sigma(n^k) = m^l$. Thus, we are only allowed to take sums of distinct $\boldsymbol{w}_i$ and get 0. There is a theorem (see [van Emde Boas and Kruyswijk 1967] and [Olson 1969, Theorem 1]) that says that if we have at least $s(l-1)$ such distinct elements $\boldsymbol{w}_i$, we can find some of them whose sum is 0. Thus, we can create at least $\lfloor t/(s(l-1)) \rfloor$ distinct (in fact, even disjoint) subsets of the $\boldsymbol{w}_i$ for $i = 1, \dots, t$ simply by finding some 0-sum among the first $s(l-1)$ of them, another 0-sum among the next $s(l-1)$ of them and so on. Since

$$\frac{t}{s(l-1)} > \frac{c}{(l-1)} \frac{\sqrt{x}}{\log x},$$

and the right-hand side is a function that tends to infinity with x, we get Conjecture 1.

We can ask similar questions simultaneously for $\phi(n)$ and $\sigma(n)$, like making them simultaneously squares, or cubes, etc. This has already been treated in [Freiberg 2012]. There it is shown that the number of $n \le x$ such that both $\phi(n)$ and $\sigma(n)$ are perfect powers of an exponent l is less than $c_1 l x^{1/l}/(\log x)^{l+2}$, where $c_1 > 0$ is some positive constant. Square values of the product $\phi(n)\sigma(n)$ have been investigated in [Broughan et al. 2013]. In the next section, we present some computational examples of n such that $\phi(n^3)\sigma(n^3) = \square$.

2. Computational examples

We wanted to find a positive integer n such that $\phi(n^3)\sigma(n^3) = \square$. For a prime p, we have $\phi(p^3)\sigma(p^3) = p^2(p^4 - 1)$. So, we wrote $p^4 - 1 = w_p \square$, where w_p is square-free for all $p \le 1000$. Then we searched for a subset $\mathcal{S}$ of cardinality t such that the set of prime factors appearing in the factorizations of w_p for $p \in S$ has cardinality $s < t$. We found the subset

$$\{2, 3, 5, 7, 13, 17, 23, 31, 41, 43, 47, 73, 83, 191, 239, 307, 443, 499, 829\},$$

with $t = 21$ and $s = 17$. Thus, this set gives us $2^{21-17} = 16$ solutions. We wrote down the $\{0, 1\}$ matrix with 17 rows and 21 columns, which ends up having rank 17 over $\mathbb{F}_2$. The largest solution in the nullspace of this matrix is

$$n = 3 \times 7 \times 11 \times 13 \times \times 17 \times 23 \times 43 \times 47 \times 83 \times 239 \times 443 \times 499 \times 829,$$

for which $\phi(n^3)\sigma(n^3) = m^2$, where

$$m = 2^{30} \times 3^7 \times 5^{10} \times 7^2 \times 11 \times 13^4 \times 17^3 \times 23 \times 29 \times 37 \times 41 \times 53 \times 61 \times 83 \times 157.$$

Despite our efforts, we could not find an integer $n > 1$ such that $\sigma(n^5) = \square$, and we leave finding such an example as a challenge to the reader.

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