involve
a journal of mathematics

On the cardinality of infinite symmetric groups

Matt Getzen

msp

2015 vol. 8, no. 5

INVOLVE 8:5 (2015)
dx.doi.org/10.2140/involve.2015.8.749

On the cardinality of infinite symmetric groups

Matt Getzen

(Communicated by Kenneth S. Berenhaut)

A new proof is given that the symmetric group of any set X with three or more elements, finite or infinite, has cardinality strictly greater than that of X. Use of the axiom of choice is avoided throughout.

John Dawson and Paul Howard [1976] proved that the symmetric group of any set X with three or more elements, finite or infinite, has cardinality strictly greater than that of X. Significantly, their proof does not rely upon the axiom of choice. However, it does rely upon Cantor's theorem that the power set of any set X, finite or infinite, has cardinality strictly greater than that of X. We give a new proof of Dawson and Howard's result that relies upon neither the axiom of choice nor Cantor's theorem.

Recall that $\mathrm{Sym}(X)$ is the *symmetric group* of X, that is the set of all bijections between a set X and itself under function composition. More specifically, we call each bijection between a set and itself a *permutation*, each element that is mapped to itself by a permutation a *fixed point*, each pair of elements that are mapped to one another by a permutation a *transposition*, and each permutation that is its own inverse an *involution*.

The following results can easily be obtained and are listed without proof: (i) every fixed point in a permutation is also a fixed point in that permutation's inverse; (ii) every transposition in a permutation is also a transposition in that permutation's inverse; (iii) every permutation is an involution if and only if it is made up entirely of fixed points and transpositions; (iv) for all sets X, there exists an injection from X into $\mathrm{Sym}(X)$; and (v) in the case of all sets X with three or more elements, $\mathrm{Sym}(X)$ contains at least three involutions.

Theorem. *For any set X with three or more elements, finite or infinite,* $\mathrm{Sym}(X)$ *has cardinality strictly greater than that of X.*

Proof. We proceed by contradiction. Assume that there does exist a bijection $\mathcal{F}$ from X to $\mathrm{Sym}(X)$, and construct the permutation $\star$ in $\mathrm{Sym}(X)$ as follows:

MSC2010: primary 03E99; secondary 20B30, 03E25.
Keywords: set theory, infinite symmetric groups, axiom of choice.

(1) Let a, b, and c be three elements of X such that $\mathcal{F}(a)$, $\mathcal{F}(b)$, and $\mathcal{F}(c)$ are all involutions in $\mathrm{Sym}(X)$ with

$$\star(a) = b,$$
$$\star(b) = c,$$
$$\star(c) = a.$$

(2) For every other element i of X such that $\mathcal{F}(i)$ is an involution in $\mathrm{Sym}(X)$, but i is not equal to a, b, or c, we have

$$\star(i) = i.$$

(3) For each pair of permutations σ and μ in $\mathrm{Sym}(X)$ that are one another's inverses, and for each pair of elements s and m of X such that $\mathcal{F}(s) = \sigma$ and $\mathcal{F}(m) = \mu$, if σ transposes s and m then we have $\star(s) = s$ and $\star(m) = m$, but if σ does not transpose s and m then we have $\star(s) = m$ and $\star(m) = s$. In other words,

$$\star(s) = \begin{cases} s & \text{if } \sigma(s) = m \text{ and } \sigma(m) = s, \\ m & \text{if } \sigma(s) \neq m \text{ or } \sigma(m) \neq s, \end{cases}$$

$$\star(m) = \begin{cases} m & \text{if } \sigma(s) = m \text{ and } \sigma(m) = s, \\ s & \text{if } \sigma(s) \neq m \text{ or } \sigma(m) \neq s. \end{cases}$$

Note that $\star$ is a permutation of X and therefore an element in $\mathrm{Sym}(X)$. Note also that $\star$ is not an involution and therefore must have a distinct inverse, call it $\star^{-1}$. Thus, some element of X must be the preimage of $\star$ under $\mathcal{F}$. Let n denote just such an element of X. Additionally, some element of X other than n must be the preimage of $\star^{-1}$ under $\mathcal{F}$. Let w denote just such an element of X. That is, $\mathcal{F}(n) = \star$ and $\mathcal{F}(w) = \star^{-1}$. As $\star$ and $\star^{-1}$ are of the same general form as σ and μ above, it now follows that

$$\star(n) = \begin{cases} n & \text{if } \star(n) = w \text{ and } \star(w) = n, \\ w & \text{if } \star(n) \neq w \text{ or } \star(w) \neq n, \end{cases}$$

$$\star(w) = \begin{cases} w & \text{if } \star(n) = w \text{ and } \star(w) = n, \\ n & \text{if } \star(n) \neq w \text{ or } \star(w) \neq n. \end{cases}$$

In other words, assuming that the bijection $\mathcal{F}$ does in fact exist, n and w will be transposed with one another in $\star$ if and only if n and w are not transposed with one another in $\star$, a contradiction! Therefore no such bijection exists between X and $\mathrm{Sym}(X)$. Conversely, as we already know that there does exist an injection from X into $\mathrm{Sym}(X)$, we conclude that $\mathrm{Sym}(X)$ must have cardinality strictly greater than that of X. $\square$

Through showing that the power set of any set X, finite or infinite, has cardinality strictly greater than that of X, Georg Cantor revolutionized mathematics and inspired the field of set theory. It is interesting to wonder how different the world might have been if mathematicians' first forays into the higher realms of the infinite had been inspired not by power sets, but by symmetric groups.

Acknowledgements

The author would like to thank John Dawson and Paul Howard for commenting on an earlier draft of this paper, and Rommy Marquez, Xizhong Zheng, Louis Friedler, Ned Wolff, and especially Carlos Ortiz for all their help and encouragement.

References

[Dawson and Howard 1976] J. W. Dawson, Jr. and P. E. Howard, "Factorials of infinite cardinals", *Fund. Math.* **93**:3 (1976), 185–195. MR 55 #7779 Zbl 0365.02050

Received: 2014-01-07 Accepted: 2014-07-12

mgetzen@arcadia.edu *Department of Mathematics & Computer Science, Arcadia University, 450 South Easton Road, Glenside, PA 19038, United States*

involve

msp.org/involve

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2015 is US $140/year for the electronic version, and $190/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2015 vol. 8 no. 5

1944-4176(2015)8:5;1-2