

On the cardinality of infinite symmetric groups Matt Getzen





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A new proof is given that the symmetric group of any set X with three or more elements, finite or infinite, has cardinality strictly greater than that of X. Use of the axiom of choice is avoided throughout.

John Dawson and Paul Howard [1976] proved that the symmetric group of any set X with three or more elements, finite or infinite, has cardinality strictly greater than that of X. Significantly, their proof does not rely upon the axiom of choice. However, it does rely upon Cantor's theorem that the power set of any set X, finite or infinite, has cardinality strictly greater than that of X. We give a new proof of Dawson and Howard's result that relies upon neither the axiom of choice nor Cantor's theorem.

Recall that Sym(X) is the *symmetric group* of X, that is the set of all bijections between a set X and itself under function composition. More specifically, we call each bijection between a set and itself a *permutation*, each element that is mapped to itself by a permutation a *fixed point*, each pair of elements that are mapped to one another by a permutation a *transposition*, and each permutation that is its own inverse an *involution*.

The following results can easily be obtained and are listed without proof: (i) every fixed point in a permutation is also a fixed point in that permutation's inverse; (ii) every transposition in a permutation is also a transposition in that permutation's inverse; (iii) every permutation is an involution if and only if it is made up entirely of fixed points and transpositions; (iv) for all sets X, there exists an injection from X into Sym(X); and (v) in the case of all sets X with three or more elements, Sym(X) contains at least three involutions.

Theorem. For any set X with three or more elements, finite or infinite, Sym(X) has cardinality strictly greater than that of X.

Proof. We proceed by contradiction. Assume that there does exist a bijection \mathcal{F} from X to Sym(X), and construct the permutation \star in Sym(X) as follows:

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Let a, b, and c be three elements of X such that F(a), F(b), and F(c) are all involutions in Sym(X) with

$$\star(a) = b,$$

$$\star(b) = c,$$

$$\star(c) = a.$$

(2) For every other element *i* of *X* such that *F*(*i*) is an involution in Sym(*X*), but *i* is not equal to *a*, *b*, or *c*, we have

$$\star(i) = i.$$

(3) For each pair of permutations σ and μ in Sym(X) that are one another's inverses, and for each pair of elements s and m of X such that F(s) = σ and F(m) = μ, if σ transposes s and m then we have *(s) = s and *(m) = m, but if σ does not transpose s and m then we have *(s) = m and *(m) = s. In other words,

$$\star(s) = \begin{cases} s & \text{if } \sigma(s) = m \text{ and } \sigma(m) = s, \\ m & \text{if } \sigma(s) \neq m \text{ or } \sigma(m) \neq s, \end{cases}$$
$$\star(m) = \begin{cases} m & \text{if } \sigma(s) = m \text{ and } \sigma(m) = s, \\ s & \text{if } \sigma(s) \neq m \text{ or } \sigma(m) \neq s. \end{cases}$$

Note that \star is a permutation of X and therefore an element in Sym(X). Note also that \star is not an involution and therefore must have a distinct inverse, call it \star^{-1} . Thus, some element of X must be the preimage of \star under \mathcal{F} . Let *n* denote just such an element of X. Additionally, some element of X other than *n* must be the preimage of \star^{-1} under \mathcal{F} . Let *w* denote just such an element of X. That is, $\mathcal{F}(n) = \star$ and $\mathcal{F}(w) = \star^{-1}$. As \star and \star^{-1} are of the same general form as σ and μ above, it now follows that

$$\star(n) = \begin{cases} n & \text{if } \star(n) = w \text{ and } \star(w) = n, \\ w & \text{if } \star(n) \neq w \text{ or } \star(w) \neq n, \end{cases}$$
$$\star(w) = \begin{cases} w & \text{if } \star(n) = w \text{ and } \star(w) = n, \\ n & \text{if } \star(n) \neq w \text{ or } \star(w) \neq n. \end{cases}$$

In other words, assuming that the bijection \mathcal{F} does in fact exist, n and w will be transposed with one another in \star if and only if n and w are not transposed with one another in \star , a contradiction! Therefore no such bijection exists between X and Sym(X). Conversely, as we already know that there does exist an injection from X into Sym(X), we conclude that Sym(X) must have cardinality strictly greater than that of X.

750

Through showing that the power set of any set X, finite or infinite, has cardinality strictly greater than that of X, Georg Cantor revolutionized mathematics and inspired the field of set theory. It is interesting to wonder how different the world might have been if mathematicians' first forays into the higher realms of the infinite had been inspired not by power sets, but by symmetric groups.

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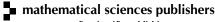
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A simplification of grid equivalence NANCY SCHERICH	721
A permutation test for three-dimensional rotation data DANIEL BERO AND MELISSA BINGHAM	735
Power values of the product of the Euler function and the sum of divisors function LUIS ELESBAN SANTOS CRUZ AND FLORIAN LUCA	745
On the cardinality of infinite symmetric groups MATT GETZEN	749
Adjacency matrices of zero-divisor graphs of integers modulo <i>n</i> MATTHEW YOUNG	753
Expected maximum vertex valence in pairs of polygonal triangulations TIMOTHY CHU AND SEAN CLEARY	763
Generalizations of Pappus' centroid theorem via Stokes' theorem COLE ADAMS, STEPHEN LOVETT AND MATTHEW MCMILLAN	771
A numerical investigation of level sets of extremal Sobolev functions STEFAN JUHNKE AND JESSE RATZKIN	787
Coalitions and cliques in the school choice problem SINAN AKSOY, ADAM AZZAM, CHAYA COPPERSMITH, JULIE GLASS, GIZEM KARAALI, XUEYING ZHAO AND XINJING ZHU	801
The chromatic polynomials of signed Petersen graphs MATTHIAS BECK, ERIKA MEZA, BRYAN NEVAREZ, ALANA SHINE AND MICHAEL YOUNG	825
Domino tilings of Aztec diamonds, Baxter permutations, and snow leopard permutations BENJAMIN CAFFREY, ERIC S. EGGE, GREGORY MICHEL, KAILEE RUBIN AND JONATHAN VER STEEGH	833
The Weibull distribution and Benford's law VICTORIA CUFF, ALLISON LEWIS AND STEVEN J. MILLER	859
Differentiation properties of the perimeter-to-area ratio for finitely many overlapped unit squares PAUL D. HUMKE, CAMERON MARCOTT, BJORN MELLEM AND COLE STIEGLER	875
On the Levi graph of point-line configurations JESSICA HAUSCHILD, JAZMIN ORTIZ AND OSCAR VEGA	893

