involve
a journal of mathematics

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msp

2015 vol. 8, no. 5

INVOLVE 8:5 (2015)
dx.doi.org/10.2140/involve.2015.8.893

On the Levi graph of point-line configurations

Jessica Hauschild, Jazmin Ortiz and Oscar Vega

(Communicated by Joseph A. Gallian)

We prove that the well-covered dimension of the Levi graph of a point-line configuration with v points, b lines, r lines incident with each point, and every line containing k points is equal to 0, whenever $r > 2$.

1. Introduction

The concept of the well-covered space of a graph was first introduced by Caro, Ellingham, Ramey, and Yuster [Caro et al. 1998; Caro and Yuster 1999] as an effort to generalize the study of well-covered graphs. Brown and Nowakowski [2005] continued the study of this object and, among other things, provided several examples of graphs featuring odd behaviors regarding their well-covered spaces. One of these special situations occurs when the well-covered space of the graph is trivial, i.e., when the graph is *anti-well-covered*. In this work, we prove that almost all Levi graphs of configurations in the family of the so-called (v_r, b_k)-configurations (see Definition 3) are anti-well-covered.

We start our exposition by providing the following definitions and previously known results. Any introductory concepts we do not present here may be found in the books by Bondy and Murty [1976] and Grünbaum [2009].

We consider only simple and undirected graphs. A graph will be denoted by $G = (V(G), E(G))$, as is customary, where $V(G)$ is the set of vertices of the graph and $E(G)$ is the set of edges of the graph. We think of $E(G)$ as an irreflexive symmetric relation on $V(G)$. Two vertices of a graph are said to be *adjacent* if they are connected by an edge. An *independent* set of vertices is one in which no two vertices in the set are adjacent. If an independent set, M, of a graph G is not a proper subset of any other independent set of G, then M is a *maximal independent set* of G.

Definition 1. Let G be a graph and F a field.

(1) A function $f : V(G) \to F$ is said to be a *weighting* of G. If the sum of all weights is constant for all maximal independent sets of G, then the weighting is a *well-covered weighting* of G.

MSC2010: primary 05B30; secondary 51E05, 51E30.

Keywords: Levi graph, maximal independent sets, configurations.

(2) The F-vector space consisting of all well-covered weightings of G is called the well-covered space of G (relative to F).

(3) The dimension of this vector space is called the *well-covered dimension* of G, denoted $\mathrm{wcdim}(G, F)$.

Remark 1. For some graphs, the characteristic of the field F makes a difference when calculating the well-covered dimension (see [Birnbaum et al. 2014] and [Brown and Nowakowski 2005]). If $\mathrm{char}(F)$ does not cause a change in the well-covered dimension, then the well-covered dimension is denoted as $\mathrm{wcdim}(G)$.

In order to calculate the well-covered dimension of a graph, G, one would generally need to find all possible maximal independent sets of G. However, finding all maximal independent sets is not always an easy task, as this is a known NP-complete problem.

Despite the NP-complete nature of this problem, let us assume that we have found all possible maximal independent sets of G. We will denote these maximal independent sets as M_i for $i = 0, 1, \dots, k-1$. The well-covered weightings of G are determined by solving a system of linear equations that arise from considering all equations of the form $M_0 = M_i$ for $i = 1, \dots, k-1$. We replace this system with the equivalent homogeneous one via standard operations and create an associated matrix A_G. Observe that the dimension of the nullspace of A_G is equal to the dimension of the well-covered space of G. Thus,

$$\mathrm{wcdim}(G, F) = |V(G)| - \mathrm{rank}(A_G).$$

We now move onto another component of our work: configurations.

Definition 2. A (point-line) configuration is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ is set of points, $\mathcal{L}$ is a set of lines, and $\mathcal{I}$ is an incidence relation between $\mathcal{P}$ and $\mathcal{L}$, that has the following properties:

(1) Any two points are incident with at most one line.

(2) Any two lines are incident with at most one point.

Next, there is some notation for configurations that needs to be set, as well as specific parameters that need to be established for the main result of this work.

Definition 3. We define a (v_r, b_k)-configuration as a configuration such that

(1) $|\mathcal{P}| = v$, and $v \geq 4$.

(2) $|\mathcal{L}| = b$, and $b \geq 4$.

(3) There are exactly k points incident with each line, and $k \geq 2$.

(4) There are exactly r lines incident with each point, and $r \geq 2$.

When $v = b$ and $r = k$, the configuration will be denoted by (v_r).

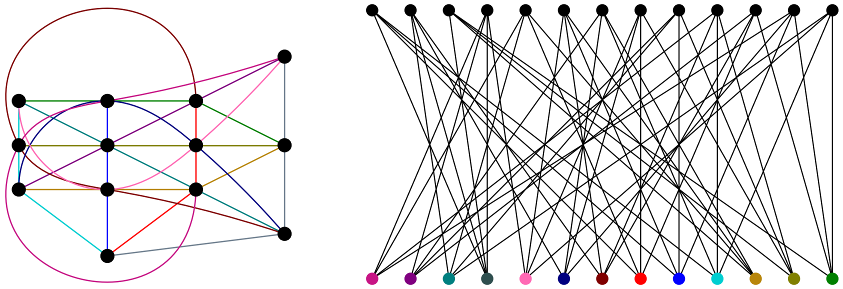

Figure 1. $(13_4) = PG(2,3)$ and $\mathrm{Levi}_{(13_4)}$.

Example 1. Several well-known geometric structures fall into the category of (v_r, b_k)-configurations. For instance:

(1) A projective plane of order q is a $(q^2+q+1_{(q+1)})$-configuration, where q is the power of a prime. See Figure 1 for a representation of $PG(2,3) = (13_4)$.

(2) The Pappus configuration is a (9_3)-configuration, and the Desargues configuration is a (10_3)-configuration.

(3) $PG(n,q)$ is a

$$\left(\frac{q^{n+1}-1}{q-1}_{(q+1)}, \frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}_{(q^2+q+1)}\right)\text{-configuration,}$$

where q is the power of a prime.

(4) A generalized quadrangle $G(s,t)$ is a $((1+s)(st+1)_{(1+s)}, (1+t)(st+1)_{(1+t)})$-configuration.

The reader is referred to the book by Batten [1997] for more information about these important geometric objects.

Finally, we define Levi graphs, which will connect configurations and graphs.

Definition 4. The Levi graph of a (v_r, b_k)-configuration $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the bipartite graph G with $V(G) = \mathcal{P} \cup \mathcal{L}$ and $E(G) = \mathcal{I}$. That is, $p \in \mathcal{P}$ is adjacent to $\ell \in \mathcal{L}$ if and only if $p\mathcal{I}\ell$. We will denote this graph $\mathrm{Levi}_{(v_r,b_k)}$.
Note that $\mathcal{P}$ and $\mathcal{L}$ are independent sets — the partite sets — in G.

Our main result, which will be proven in the following section, combines all of these objects as follows:

Theorem 1. *If r is a positive integer greater than* 2, *then* $\mathrm{wcdim}(\mathrm{Levi}_{(v_r,b_k)}) = 0$.

We would like to remark that Theorem 1 says is that almost all Levi graphs of (v_r, b_k)-configurations are anti-well-covered.

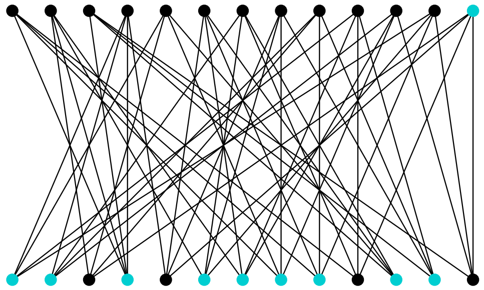

Figure 2. A maximal independent set M_P in $\text{Levi}_{(13_4)}$.

2. The well-covered dimension of $\text{Levi}_{(v_r,b_k)}$

We will prove Theorem 1 by first proving a technical lemma that introduces a family of maximal independent sets that will prove to be useful later on.

Lemma 1. *A Levi graph of a configuration* (v_r, b_k), *where* $r > 2$, *has at least* $v + b + 2$ *maximal independent sets.*

Proof. Let P be a fixed point in (v_r, b_k). We consider the set, M_P, of vertices of $\text{Levi}_{(v_r,b_k)}$ given by P and all the lines not incident to P. This is an independent set of $\text{Levi}_{(v_r,b_k)}$ because there is no incidence between vertices in the set. Moreover, note that if we included another point-vertex to M_P, then that vertex would be adjacent to one of the line-vertices in M_P (because of condition (2) in Definition 2, and the fact that $r > 2$). Also, if another line-vertex were to be added to M_P, then this line would have to be incident with P. It follows that M_P is a maximal independent set of $\text{Levi}_{(v_r,b_k)}$. See Figure 2 for an example.

Repeating this construction for all v points in (v_r, b_k), we get v distinct maximal independent sets of $\text{Levi}_{(v_r,b_k)}$.

We will now construct another b distinct maximal independent sets of $\text{Levi}_{(v_r,b_k)}$. We start by fixing a line ℓ in (v_r, b_k) and then any two distinct points $P_1, P_2 \in \ell$ (recall that $k \geq 2$). We consider the set, M_{P_1,P_2} of vertices of $\text{Levi}_{(v_r,b_k)}$ given by P_1, P_2 and all the lines not incident to either of these points. Note that this forms an independent set since adjacency in $\text{Levi}_{(v_r,b_k)}$ only occurs if incidence occurs in (v_r, b_k). If we try to add in another vertex-point to M_{P_1,P_2}, since $r > 2$, this point will be incident to one of the lines not through P_1 or P_2 and will therefore be adjacent to the vertex-lines in M_{P_1,P_2}. If we try to add another vertex-line to M_{P_1,P_2}, then this line will be incident to one or both of P_1 and P_2. Therefore, M_{P_1,P_2} is a maximal independent set of $\text{Levi}_{(v_r,b_k)}$. See Figure 3 for an example.

Repeating this construction for all b lines in (v_r, b_k) (it does not matter what pair of points one picks on any given line), we get b distinct maximal independent sets of $\text{Levi}_{(v_r,b_k)}$.

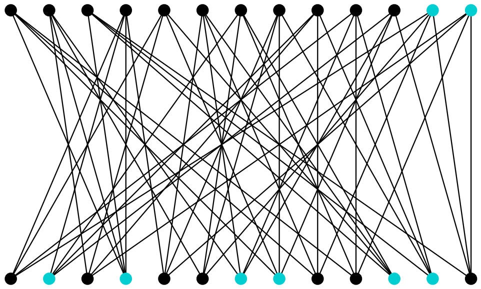

Figure 3. A maximal independent set M_{P_1,P_2} in $\mathrm{Levi}_{(13_4)}$.

Finally, note that the set of all point-vertices in $\mathrm{Levi}_{(v_r,b_k)}$ is a maximal independent set and the set of all line-vertices in $\mathrm{Levi}_{(v_r,b_k)}$ is as well. Hence, we have constructed $v+b+2$ distinct maximal independent sets in $\mathrm{Levi}_{(v_r,b_k)}$. ☐

Next, we proceed to prove our main result.

Proof of Theorem 1. We denote by F the field of scalars of the well-covered space of $G = \mathrm{Levi}_{(v_r,b_k)}$, where $r > 2$. Let A_G be the associated matrix of G, and note that A_G has $v+b$ columns. In order to prove that A_G has $v+b$ linearly independent rows we will consider the $v+b+2$ maximal independent sets in Lemma 1.

We create the first v rows of A_G by equating the weight of each of the maximal independent sets M_P to the weight of the maximal independent set consisting of all the lines of G. After subtracting, we obtain v equations of the form

$$f(P) - f(\ell_1) - f(\ell_2) - \cdots - f(\ell_r) = 0, \tag{1}$$

where each ℓ_i is incident with P. It follows that, after organizing the columns of A_G by putting point-vertices first and then line-vertices, the “first” v rows of A_G are

$$\begin{bmatrix} I_v & -C \end{bmatrix},$$

where C is the incidence matrix of $\mathrm{Levi}_{(v_r,b_k)}$.

In order to obtain the next b rows of A_G, we will consider maximal independent sets of the form $M_{P,Q}$. For any given line ℓ of (v_r, b_k), we choose (any) two points on it. We will denote these two points as P_1 and P_2. We then consider the maximal independent set M_{P_1,P_2} and equate its weight to the weight of the maximal independent set M_{P_1}. After subtracting, we obtain an equation of the form

$$f(P_2) - f(\ell_1) - f(\ell_2) - \cdots - f(\ell_r) + f(\ell) = 0, \tag{2}$$

where each ℓ_i is incident with P_2.

Note that subtracting (1) (with $P = P_2$) from (2) yields $f(\ell) = 0$. Since ℓ was arbitrary, we get $f(\ell) = 0$ for every line in (v_r, b_k). It follows that since subtracting equations is just a different way to describe row operations in A_G, we get that the "first" $v+b$ rows of A_G (after a few row operations) are

$$\begin{bmatrix} I_v & -C \\ \mathbf{0} & I_b \end{bmatrix}.$$

Note that addition and subtraction were the only two (row) operations needed to obtain the matrix above. Hence, the first $v+b$ rows of A_G do not change depending on the characteristic of F.

Since the determinant of the matrix above is nonzero, the rank of A_G is maximal, and thus $\operatorname{wcdim}(\mathrm{Levi}_{(v_r,b_k)}) = 0$. □

3. Possible generalizations

In this section, we study possible generalizations of Theorem 1. This will be done by providing a few results and by introducing objects to which this theorem could be extended. We begin by proving that Theorem 1 cannot be extended to configurations having exactly two lines being incident with every point. This will be done by an example that considers (v_2)-configurations.

We first notice that a (v_2)-configuration is a disjoint union of polygons/cycles. This is convenient because disjoint unions of graphs behave well with respect to the well-covered dimension. In fact, Lemma 5 in [Brown and Nowakowski 2005] says

$$\operatorname{wcdim}(G \cup H) = \operatorname{wcdim}(G) + \operatorname{wcdim}(H),$$

where $\cup$ stands for disjoint union.

Since we know that $\mathrm{Levi}_{C_n} = C_{2n}$, we get the following lemma.

Lemma 2. *Let $\mathcal{C}$ be a (v_2)-configuration. Then,*

$$\mathcal{C} = \bigcup_{i=1}^{t} C_{n_i},$$

where $n_i > 2$, for all $1 \le i \le t$. Moreover,

$$\operatorname{wcdim}(\mathrm{Levi}_{\mathcal{C}}) = \sum_{i=1}^{t} \operatorname{wcdim}(C_{2n_i}).$$

Finally, we notice that Theorem 5 in [Birnbaum et al. 2014] implies

$$\operatorname{wcdim}(C_{2n}) = \begin{cases} 2 & \text{if } n = 3, \\ 0 & \text{if } n \ge 4. \end{cases}$$

Next is an immediate corollary of that same theorem, together with our Lemma 2.

Corollary 1. *The well-covered dimension of* $\mathrm{Levi}_{\mathcal{C}}$ *is even for all* (v_2)*-configurations* $\mathcal{C}$*. Moreover, for every* $n \in \mathbb{N}$*, there is a* (v_2)*-configuration,* $\mathcal{C}_n$*, such that*

$$\mathrm{wcdim}(\mathrm{Levi}_{\mathcal{C}_n}) = 2n.$$

In particular, the sequence $\{\mathrm{wcdim}(\mathrm{Levi}_{\mathcal{C}_n})\}_{n=1}^{\infty}$ *is unbounded.*

We conclude that Theorem 1 cannot be expanded to the case $r = 2$. However, it is still an open problem to find the well-covered dimension of all Levi graphs of (v_2, b_k)-configurations.

Of course, the study of the well-covered dimension of Levi graphs of configurations not of the form (v_r, b_k) is also an interesting open problem.

Block designs are another family of objects that could be studied to attempt a generalization of Theorem 1. These objects can be much less "geometric" than (v_r, b_k)-configurations, given that they are obtained after relaxing items (3) and (4) in Definition 2. In order to be more precise, we provide the following definition.

Definition 5. Let $\lambda, t \geq 1$. A t-(v, k, λ)-design (or t-design), is an incidence structure of points and blocks with the following properties:

(1) There are v points.

(2) Each block is incident with k points.

(3) Any t points are incident with λ common blocks.

It is easy to see that a 1-(v, k, λ)-design is a (v_λ, b_k)-configuration, where $b = v\lambda/k$. Moreover, a 2-$(v, k, 1)$-design is a configuration in which every pair of points are "collinear". For $t > 1$ and $\lambda > 1$, the obvious definition of the Levi graph of a t-design would yield a multigraph. This apparent setback is not so much of a problem since having one edge or multiple edges between two vertices would mean the same thing when looking for maximal independent sets. We claim that the ideas used to prove Theorem 1 can be generalized to be applicable to block designs.

Finally, in this work, we studied the well-covered space of the Levi graph of a (v_r, b_k)-configuration. We propose, as an interesting open problem, the study of configurations via understanding the well-covered spaces of their collinearity graphs (in which points in a configuration are defined as vertices, and adjacency occurs if and only if the points are collinear). The third author is currently working on a particular case of this problem: generalized quadrangles.

Acknowledgments

The authors would like to thank the referee for his/her suggestions. We believe they greatly improved the exposition in this article. We gratefully acknowledge the NSF for their financial support (Grant #DMS-1156273), and the REU program at California State University, Fresno.

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Received: 2014-10-04 Revised: 2014-12-28 Accepted: 2015-01-02

jessica.hauschild@kwu.edu *Department of Math and Physics, Kansas Wesleyan University,*
100 East Claflin Avenue, Salina, KS 91711-5901, United States

jortiz@g.hmc.edu *Department of Mathematics, Harvey Mudd College, 301*
Platt Boulevard, Claremont, CA 91711-5901, United States

ovega@csufresno.edu *Department of Mathematics, California State University,*
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2015 vol. 8 no. 5

1944-4176(2015)8:5;1-2