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On the independence and domination numbers of replacement product graphs

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This paper examines invariants of the replacement product of two graphs in terms of the properties of the component graphs. In particular, we present results on the independence number, the domination number, and the total domination number of these graphs. The replacement product is a noncommutative graph operation that has been widely applied in many areas. One of its advantages over other graph products is its ability to produce sparse graphs. The results in this paper give insight into how to construct large, sparse graphs with optimal independence or domination numbers.

1. Introduction

It is natural to construct graphs from smaller component graphs, and as such, products of graphs have long been studied for both their theoretical interest and practical applicability. Standard products include the cartesian product, direct product, and strong product [Imrich and Klavžar 2000; Hammack et al. 2011]. As many modern applications require sparse graphs, newer products have been introduced. In particular, the replacement product is a noncommutative graph product of two regular component graphs that produces a regular graph whose degree depends only on the degree of the second component graph. Thus, the replacement product can be easily used to generate large, sparse graphs. In addition, it was shown that the expansion of the replacement product graph inherits the expansion properties of both component graphs [Reingold et al. 2002; Hoory et al. 2006]. The replacement product has been widely used in many areas including group theory, expander graphs, and graph-based coding schemes [Reingold et al. 2002; Hoory et al. 2006; Gamburd and Pak 2006; Kelley et al. 2008].

Invariants of graphs, including the independence and domination numbers of a graph, have also been widely studied. Many applications in computer science and

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engineering require graphs with large independence numbers or small domination numbers. For example, in [Shannon 1956] it is shown that the independence number characterizes the largest number of bits that can be communicated without error in a particular communication problem. Studying the invariants of product graphs based on the invariants of the component graphs is an interesting problem, and in fact has led to many long-standing open problems in graph theory. Notable examples include Vizing's conjecture on the domination number of cartesian product graphs and Hedetniemi's conjecture on the chromatic number of direct product graphs (see, e.g., [Brešar et al. 2012; Hammack et al. 2011]). In [Alon and Orlitsky 1995], the independence numbers of graphs constructed using the n-fold AND product and the n-fold OR product are determined with respect to communicating multiple bits per channel use in a repeated communication model, generalizing the result in [Shannon 1956]. Similar applications that studied large independence number and large chromatic number in graph products are given in [Alon and Lubetzky 2006; Witsenhausen 1976]. Domination numbers have also been heavily studied and generalized (see, e.g., [Haynes et al. 1998a; 1998b; Chelvam and Chellathurai 2011]). The importance of the independence and domination numbers in applications and the advantages of the replacement product provide the motivation to study these invariants in replacement product graphs.

In this paper, we investigate the independence number, the domination number, and the total domination number of replacement product graphs in terms of their component graphs. One of our main results, Theorem 3.4, expresses the independence number of the replacement product of G and H in terms of the independence number of the second component graph, H. We also derive lower and upper bounds on the domination and total domination numbers for replacement product graphs. Another main result, Theorem 4.14, gives an upper bound on the total domination number for the replacement product of G and H in terms of the number of edges in a certain spanning subgraph of G.

The paper is organized as follows. We introduce some preliminary definitions and notation in Section 2. In Section 3, we determine the independence number of replacement product graphs. In Section 4, we present lower and upper bounds on the domination number and the total domination number of replacement product graphs. In addition, we include examples of families of graphs that meet the bounds.

2. Preliminaries

This paper studies properties of the replacement product $G \circledR H$ of two graphs G and H. We will assume in this work that G and H are finite simple connected graphs. We first recall some basic terminology and notation that will be used in this paper. We will use $V(G)$ and $E(G)$ to denote the vertex set and edge set of a

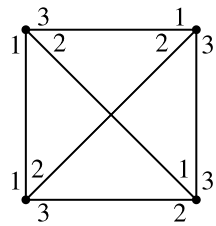

Figure 1. Rotation map.

graph G, respectively. Moreover, the minimum degree and the maximum degree of a vertex in G will be denoted by $\delta(G)$ and $\Delta(G)$, respectively. A *walk* is an alternating sequence of vertices and edges, beginning and ending with a vertex, where each vertex is incident to both the edge that precedes it and the edge that follows it in the sequence. The *length* of a walk is the number of edges in the walk. A *trail* is a walk where all edges are distinct. An *Eulerian trail* in G is a trail that contains each edge from G exactly once. A *closed Eulerian trail* is an Eulerian trail that begins and ends at the same vertex. A *path* is a walk where each vertex in the walk is distinct. A *Hamiltonian path* in G is a path that contains each vertex from G exactly once. The *distance* between vertices $u, v \in G$, denoted $d(u, v)$, is the length of the shortest path between vertices u and v. Finally, we will use $[n]$ to denote the set of integers $\{1, \dots, n\}$.

Definition 2.1. A *rotation map* on a graph G is a labeling of the edges of G where each edge gets two labels, one at each endpoint, and in addition, the edge labels around each vertex v in G are distinct and numbered using $1, 2, \dots, \deg(v)$.

For example, Figure 1 is an example of a rotation map on K_4.

We now introduce the replacement product of two graphs. This product is noncommutative and depends on the specific rotation map on the first component graph.

Definition 2.2. Let G be a b-regular graph with $|V(G)| = n$ and H be a k-regular graph with $|V(H)| = b$. Assign the vertices of G distinct labels in $[n]$ and assign the vertices of H distinct labels in $[b]$. Then given a rotation map on G, the replacement product $G \circledR H$ is a graph whose vertices are the ordered pairs (u, v) for $u \in [n]$ and $v \in [b]$. There is an edge between (u, v) and (w, l) in $G \circledR H$ if either (i) $u = w$ and there is an edge between vertex v and vertex l in H, or (ii) $u \neq w$ and there is an edge between u and w in G having label v at u and label l at w assigned by the rotation map on G.

The replacement product graph $G \circledR H$ as described in Definition 2.2 is a $(k+1)$-regular graph with nb vertices. Note that the degree of regularity of the product graph depends only on the degree of regularity of the second component graph H.

The graph $G \circledR H$ may be more easily seen in the following way. First, each vertex of G is replaced by a copy of the graph H; such a copy will be referred to as a *cloud*

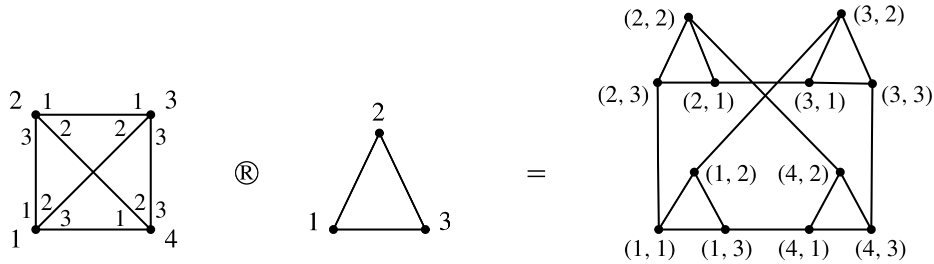

Figure 2. The replacement product $K_4 \circledR K_3$ with the specified rotation map on K_4.

and the cloud that replaces vertex i will be called the i-th cloud. Specifically, the i-th cloud is the subgraph induced by the set of vertices $\{(i, w) \in G \circledR H \mid w \in [b]\}$. Next, given any pair of distinct $i, j \in [n]$, there is exactly one edge between clouds i and j in $G \circledR H$ if and only if $(i, j) \in E(G)$. The vertices in the clouds that are connected by such an edge are determined by the rotation map on G. We will refer to edges that go between clouds as *intercloud edges*, and edges within clouds as *cloud edges*.

See Figure 2 for an example of the replacement graph product $K_4 \circledR K_3$.

3. Independence number of replacement product graphs

In this section we determine the independence number for replacement product graphs based on the independence number of the second component graph.

Definition 3.1. Let G be a graph. An *independent set* of G is a subset $S \subseteq V(G)$ such that no pair of distinct elements in S is adjacent. The *independence number* of G, denoted $\alpha(G)$, is the size of a largest independent set.

Due to the dependence of the replacement product on the rotation map, we introduce the following definition.

Definition 3.2. For graphs G and H, the *maximized independence number*, denoted $\hat{\alpha}(G \circledR H)$, is defined as the maximum possible independence number of $G \circledR H$ over all rotation maps m on G. That is,

$$\hat{\alpha}(G \circledR H) = \max_m \{\alpha(G^{(m)} \circledR H)\} = \max_m \{\max_S |S|\},$$

where $G^{(m)}$ is the graph G with rotation map m and S is an independent set of $G^{(m)} \circledR H$.

Note that in practice, one can often choose the rotation map for the replacement product graph. Indeed, this is typically done randomly. When the independence number of the graph is of interest, Definition 3.2 characterizes the best possible value one can obtain. The main result in this section shows explicitly how to design a rotation map that attains $\hat{\alpha}(G \circledR H)$. We first state the following known result, and include its proof for convenience.

Lemma 3.3 [Haynes et al. 1998b]. *For a k-regular graph G,*

$$\alpha(G) \leq \frac{|V(G)|}{2}.$$

Proof. Let S be an independent set of G with $|S| = m$. We now bound the total number of edges incident with S in G. Each of the $(|V(G)| - m)$ vertices in $V(G) - S$ may be adjacent to at most k members of S. So the total number of edges in G from these vertices to S is at most $(|V(G)| - m)k$. Each vertex in S has degree k, and by the independence of S, the m vertices in S are pairwise nonadjacent. Thus, the total number of edges incident with S is mk. Therefore,

$$mk \leq (|V(G)| - m)k,$$

from which we obtain

$$m \leq \frac{|V(G)|}{2}.$$

Since the statement holds for any independent set, it holds for a maximally sized independent set. □

Next we present the main result of this section, which determines the maximized independence number for a replacement product graph.

Theorem 3.4. *Let G be a b-regular graph with $|V(G)| = n$ and H a k-regular graph with $|V(H)| = b$. Then*

$$\hat{\alpha}(G \circledR H) = \alpha(H)|V(G)|.$$

Proof. First, it is easy to see that $\hat{\alpha}(G \circledR H) \leq \alpha(H)|V(G)|$, since otherwise, for some choice of a rotation map, there would exist a maximal independent set $\mathcal{S}$ of $G \circledR H$ such that some cloud contains more than $\alpha(H)$ members of $\mathcal{S}$. Since each cloud is an isomorphic copy of H, this gives a contradiction.

Now we show the reverse inequality by designing a specific rotation map that meets the bound. Label the vertices of G using $1, 2, \ldots, n$. Let I' be an independent set of H. By Lemma 3.3, $|I'| \leq b/2$. Label the vertices in each copy of H in $G \circledR H$ using the numbers $\{1, \ldots, b\}$ such that the vertices in I' receive the even numbers $2, 4, \ldots, 2|I'|$. Let (i, j) be the vertex in cloud i with label j.

We will show that there exists a rotation map on G with the property that for every vertex $(i, j) \in G \circledR H$, if j is even and (i, j) is adjacent to some (k, l) where $i \neq k$, then l must be odd. From this we will conclude that

$$I := \big\{(i, j) \in G \circledR H \mid j \in \{2, 4, \ldots, 2|I'|\}\big\}$$

is an independent set of size $\alpha(H)|V(G)|$.

We introduce the following algorithm which will be used to generate such a rotation map.

(1) Assign to each vertex $v \in V(G)$ a number T_v and a set S_v with initial values $T_v = 0$ and $S_v = [b]$. Set $\mathcal{V} := V(G)$.

(2) Choose a vertex $v \in \mathcal{V}$.

(3) Choose an unlabeled edge e incident with v. If there is an even number in S_v, then choose any such even number $a \in S_v$ and label the endpoint of e at v using a. Then set $T_v := T_v + 1$ and $S_v := S_v - a$. Otherwise label the endpoint of e at v using any odd number $a \in S_v$ and set $T_v := T_v - 1$ and $S_v := S_v - a$. Let u be the other vertex incident to e, and label the endpoint of e at u using any odd number $c \in S_u$. Then set $T_u = T_u - 1$ and $S_u := S_u - c$.

(4) If there is an unlabeled edge at u, set $v := u$ and go to Step 3.

(5) Let $\mathcal{U} = \{u \in V(G) \mid S_u = \varnothing\}$. Set $\mathcal{V} := \mathcal{V} - (\mathcal{U} \cap \mathcal{V})$. If $\mathcal{V} = \varnothing$, stop. Otherwise, go to Step 2.

Observe that T_v counts the number of even-labeled edges at v minus the number of odd-labeled edges at v. During the algorithm, T_v is never less than -1 because in Step 3, whenever a vertex receives an odd label at an edge, either another edge at that vertex receives an even label at the next step of the algorithm, or the vertex has all its edges labeled from 1 to b. Moreover, note that each edge receives its two endpoint labels consecutively. Thus, the vertex u in Step 3 always exists.

We now show that any rotation map generated by this algorithm satisfies the desired property by considering the parity of b, the regularity of G.

Case 1: Suppose b is even. Then there exists a closed Eulerian trail T in G. Then in Step 3 of the algorithm, instead of arbitrarily choosing the next edge to take after reaching a vertex, we choose an edge in order according to T. When the algorithm stops, $T_v = 0$ for all vertices $v \in V(G)$. Thus, every edge has one even label and one odd label at its endpoints, ensuring that the resulting rotation map on G has the asserted property.

Case 2: Suppose b is odd. For any vertex v at any stage of the algorithm, $T_v = -1$ if $S_v = \varnothing$, and when $S_v \neq \varnothing$, we have $T_v = -1$, 0 or 1. Therefore, since $[b]$ contains one more odd number than even number, when $S_u \neq \varnothing$, there is always an odd number to select for the edge in Step 3, and as a result, no edge will receive two even labels during the algorithm. Thus, the resulting rotation map on G has the asserted property.

Finally, let $G \circledR H$ be the replacement product graph in which the rotation map on G was obtained as described above. Then by construction, an edge from (i, j) to (k, ℓ) in $G \circledR H$ for j even and $i \neq k$ must have ℓ odd. Therefore

$$I := \{(i, j) \in G \circledR H \mid j \in \{2, 4, \dots, 2|I'|\}\}$$

is an independent set and has size

$$|I| = |I'||V(G)| = \alpha(H)|V(G)|.$$

Thus, $\hat{\alpha}(G \circledR H) \geq \alpha(H)|V(G)|$, proving the equality. □

4. Domination numbers of replacement product graphs

In this section we present lower and upper bounds on two main types of domination numbers: the domination number and the total domination number. For more background on these parameters, see [Haynes et al. 1998a; 1998a].

Domination number.

Definition 4.1. A *dominating set* of a graph G is a subset $D \subseteq V(G)$ such that for every $v \in G \setminus D$, v is adjacent to some $v' \in D$. The *domination number* of G, denoted $\gamma(G)$, is the size of a smallest dominating set.

The domination number of a graph has been a parameter of great interest in applications such as communication and transportation networks. Again, due to the dependence of the replacement product on the rotation map, we introduce the following definition.

Definition 4.2. For graphs G and H, the *minimized domination number*, denoted $\hat{\gamma}(G \circledR H)$, is defined as the minimum possible domination number of $G \circledR H$ over all rotation maps m on G. That is,

$$\hat{\gamma}(G \circledR H) = \min_m\{\gamma(G^{(m)} \circledR H)\} = \min_m\{\min_S |S|\},$$

where $G^{(m)}$ is the graph G with rotation map m and S is a dominating set of $G^{(m)} \circledR H$.

We now give a lower bound on the domination number of a replacement product graph $G \circledR H$ in terms of the domination number of the second component graph, H.

Proposition 4.3. *Let G be a b-regular graph with $|V(G)| = n$ and H a k-regular graph with $|V(H)| = b$. Then*

$$\frac{n\gamma(H)}{2} \leq \hat{\gamma}(G \circledR H).$$

Moreover, if (i) $k = b - 2$, (ii) *n is even, and* (iii) *G contains a Hamiltonian cycle, then the bound is tight.*

Proof. Let G have any rotation map and let D be a dominating set of $G \circledR H$. Every vertex (i, j) in $G \circledR H$ is in one cloud (namely, the i-th cloud) and is adjacent to exactly one vertex in a different cloud. Note that there are at least $\gamma(H)$ elements of D in the vertex set of each cloud and its neighborhood, since otherwise there is a copy of H dominated by a vertex set of size strictly smaller than $\gamma(H)$. Thus, there are at least $\gamma(H)$ vertices in D dominating each cloud. Since there are n clouds and each vertex dominates vertices in two clouds, there must be at least $n\gamma(H)/2$ vertices in D. Thus, for any rotation map on G, we have $\gamma(G \circledR H) \geq n\gamma(H)/2$.

Now assume that the three additional properties (i)–(iii) hold as well. We will design a specific rotation map on G so that the lower bound is met. First, label the vertices of G in order from 1 to n according to a chosen Hamiltonian cycle $\mathcal{C}$. Then choose two nonadjacent vertices in H and label them 1 and 2, and label the rest of $V(H)$ using $3, \dots, b$. Since $k = b - 2$, each vertex is nonadjacent to exactly one other vertex, and the pair form a smallest dominating set in H. In particular, $\gamma(H) = 2$.

Now we construct our rotation map on G. For each $i \in [n-1]$, label the edge $(i, i+1) \in E(G)$ with a 1 at vertex i and a 2 at vertex $i+1$, and label edge $(n, 1)$ with a 1 at vertex n and a 2 at vertex 1. Then complete the rotation map in any way. Note that in $\mathcal{C}$, every edge is labeled by a 1 at one endpoint and a 2 at the other endpoint.

Now consider the product $G \circledR H$. We claim the set

$$D = \{(i, 1) \in V(G \circledR H) \mid i \in [n]\}$$

forms a dominating set of $G \circledR H$. For each i, every vertex in cloud i except vertex $(i, 2)$ is dominated by vertex $(i, 1)$, and vertex $(i, 2)$ is dominated by vertex $(i+1, 1)$. So D is a dominating set with size $|D| = n$. Therefore,

$$\hat{\gamma}(G \circledR H) \geq n = \frac{n\gamma(H)}{2}. \qquad \square$$

The next example gives a sequence of replacement product graphs that meet the bound in Proposition 4.3.

Example 4.4. Let m be any even integer and let K_n denote the complete graph on n vertices. Define $H_m := K_m - \mathcal{M}_1$ and $G_m := K_{m+2} - \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are perfect matchings of K_m and K_{m+2}, respectively. Then $k = m - 2 = b - 2$, $n = m + 2$ is even, and G contains a Hamiltonian cycle. So

$$\{G_m \circledR H_m\}_{m \in 2\mathbb{Z}}$$

is a sequence of replacement product graphs meeting the bound in Proposition 4.3. More generally, in order to have a pair of graphs G, H that satisfy the three conditions, H must be isomorphic to H_m for some m, since no other regular graph has the property that $k = b - 2$. However, G can be any graph of the form $C_{m+2k} \cup F$, where $k \in \mathbb{N}$, C_n denotes a cycle on n vertices, and F is an $(m-2)$-regular graph with $V(F) = V(C_{m+2k})$ and $E(F) \cap E(C_{m+2k}) = \varnothing$. $\square$

We have shown a lower bound on the minimized domination number of replacement product graphs, and a sequence of graphs that meet that bound. We now focus on deriving an upper bound on this parameter. For this, we will use the notion of the k-independence number of a graph, defined next.

Definition 4.5. For $k \in [|V(G)|-1]$, a *k-independent set* of a graph G is a subset $S \subseteq V(G)$ such that S is an independent set and for every $v \in V(G)-S$, we have v is adjacent to at most k members from S. The largest cardinality of a *k-independent set* will be called the *k-independence number* and will be denoted $\alpha_k(G)$.

This parameter is related to the more familiar 2-packing number of a graph as defined below.

Definition 4.6. A *2-packing set* of a graph G is a subset $S \subseteq V(G)$ such that S is an independent set and for any pair of distinct $u, v \in S$, we have $d(u, v) \geq 3$, i.e., u and v have disjoint neighborhoods. Define the *2-packing number* of G, denoted $P_2(G)$, to be the largest cardinality of a 2-packing set of G.

The 2-packing number of G was introduced in [Meir and Moon 1975] and is a generalization of the independence number of G. Note that from the above definitions, the $(|V(G)|-1)$-independence number of a graph G is simply the independence number of G, and the 1-independence number of G is the 2-packing number of G.

We are now ready to present the upper bound on the minimized domination number.

Proposition 4.7. *Let G be a b-regular graph with $|V(G)| = n$ and H a k-regular graph with $|V(H)| = b$. Then*

$$\hat{\gamma}(G \circledR H) \leq (n - \alpha_{\gamma(H)}(G))\,\gamma(H).$$

Proof. There exist $\alpha_{\gamma(H)}(G)$ vertices in G that form a $\gamma(H)$-independent set S. Choose such a set and label these vertices $1, 2, \ldots, \alpha_{\gamma(H)}(G)$. Label the rest of the vertices of G using $\alpha_{\gamma(H)}(G)+1, \ldots, n$. Choose a dominating set D' of H of size $\gamma(H)$ and label these vertices $1, 2, \ldots, \gamma(H)$. Label the rest of the vertices in H using $\gamma(H)+1, \ldots, b$.

We now create a rotation map on G. Pick any $i \in [n] - [\alpha_{\gamma(H)}(G)]$ and let v_i be the vertex in G with label i. Label the edges at v_i by starting with the edges adjacent to members of S. Since S is a $\gamma(H)$-independent set, we can ensure that these edges get labels from the set $[\gamma(H)]$. Once this labeling has been done for every vertex from $V(G) - S$, complete the rotation map on G in any way.

We will show that the set

$$D := \bigl\{(i, j) \in V(G \circledR H) \bigm| i \in \{\alpha_{\gamma(H)}(G)+1, \alpha_{\gamma(H)}(G)+2, \ldots, n\},\ j \in [\gamma(H)]\bigr\}.$$

is a dominating set in $G \circledR H$ with this rotation map.

Note that for each $i \in [n] - [\alpha_{\gamma(H)}(G)]$, cloud i is dominated by D, since every such cloud contains a copy of the dominating set D'. Furthermore, for each $i \in [\alpha_{\gamma(H)}(G)]$, every vertex in cloud i is adjacent via its intercloud edges to some cloud with label $j \in [n] - [\alpha_{\gamma(H)}(G)]$.

Moreover, by construction of the rotation map on G, we see that each vertex in cloud j, for $j \in [n] - [\alpha_{\gamma(H)}(G)]$, is adjacent to a vertex (a, b), for some $a \in [\alpha_{\gamma(H)}(G)]$ and $b \in [\gamma(H)]$, and hence an element of D. Thus, D is a dominating set and has size

$$|D| = \big(|V(G)| - \alpha_{\gamma(H)}(G)\big)\,\gamma(H),$$

giving the desired bound. □

The next example gives a sequence of graphs meeting the upper bound in Proposition 4.7.

Example 4.8. Let $G = K_{n+1}$ and $H = K_n$ for any $n \in \mathbb{N}$. Then $\gamma(H) = 1$ and $\alpha_1(G) = 1$. Then, given any rotation map on G, let D be the set of n vertices adjacent to a vertex in cloud 1 via an intercloud edge. Then D is a dominating set with size

$$|D| = n = ((n+1)-1)\cdot 1 = \big(|V(G)| - \alpha_{\gamma(H)}(G)\big)\,\gamma(H). \qquad \square$$

Total domination number. In this subsection we consider a related parameter, the total domination number of a graph, that has also been heavily studied in similar applications.

Definition 4.9. A *total dominating set* of a graph G is a subset $D \subseteq V(G)$ such that for every $v \in G$, v is adjacent to some $v' \in D$. The *total domination number* of G, denoted $\gamma_t(G)$, is the size of a smallest total dominating set.

Note that unlike in a dominating set of G, in a total dominating set of G a vertex does not dominate itself. Again, due to the dependence of the replacement product on the rotation map, we introduce the following definition.

Definition 4.10. For graphs G and H, the *minimized total domination number*, denoted $\hat{\gamma}_t(G ® H)$, is defined as the minimum possible total domination number of $G ® H$ over all rotation maps m on G. That is,

$$\hat{\gamma}_t(G ® H) = \min_m\{\gamma_t(G^{(m)} ® H)\} = \min_m\{\min_S |S|\},$$

where $G^{(m)}$ is the graph G with rotation map m and S is a total dominating set of $G^{(m)} ® H$.

In the rest of this section, we obtain lower and upper bounds on the total domination number of replacement product graphs. First, we state the following known result whose proof is straightforward.

Lemma 4.11 [Haynes et al. 1998b]. *If G is a k-regular graph then*

$$\gamma_t(G) \geq \frac{|V(G)|}{k}.$$

We next present a lower bound on the minimized total domination number of a replacement product graph, and the proof uses the notion of a k-factor. Recall that a *k-factor* of a graph G is a k-regular spanning subgraph of G.

Proposition 4.12. *Let G be a b-regular graph with $|V(G)| = n$ and H a k-regular graph with $|V(H)| = b$. Then,*

$$\hat{\gamma}_t(G \circledR H) \geq \frac{|V(G \circledR H)|}{k+1} = \frac{|V(G)||V(H)|}{k+1}.$$

Moreover, when G and H have the additional properties that (i) $b = \gamma(H)(k+1)$ *and* (ii) *G contains a $\gamma(H)$-factor, equality holds.*

Proof. The first statement follows from the $(k+1)$-regularity of $G \circledR H$ and Lemma 4.11. Assume that the additional properties (i) and (ii) hold for G and H. Let D' be a smallest dominating set in H, and let D be the set

$$D = \{(i, j) \in V(G \circledR H) \mid i \in [n], j \in D'\}.$$

Let G' be a $\gamma(H)$-factor of G. Design a rotation map on G by first labeling the edges of the subgraph G' at each vertex of G using the numbers $1, 2, \dots, \gamma(H)$, and label the remaining edges at each vertex using $\gamma(H)+1, \dots, b$. Label the vertices in H by using the numbers $1, 2, \dots, \gamma(H)$ for those in D' and the numbers $\gamma(H)+1, \dots, b$ for those not in D'.

Now consider the replacement product $G \circledR H$ with this rotation map, and as before let (i, j) denote the vertex in cloud i with label j. Consider an arbitrary intercloud edge, say from (i, j) to (m, l), where $i \neq m$. Then we see that, by construction, $j \in \{1, 2, \dots, \gamma(H)\}$ if and only if $l \in \{1, 2, \dots, \gamma(H)\}$. Moreover, since $(i, j) \in D$ if and only if $j \in \{1, 2, \dots, \gamma(H)\}$, and every vertex is incident to exactly one intercloud edge, this also implies that every $v \in D$ is adjacent via an intercloud edge to some other $v' \in D$. This guarantees that D is not only a dominating set, but is in fact a total dominating set. Finally,

$$|D| = |V(G)|\gamma(H) = |V(G)|\frac{b}{k+1} = \frac{|V(G)||V(H)|}{k+1}. \qquad \square$$

In the next example, we construct a sequence of pairs G, H such that $G \circledR H$ meets the bound in Proposition 4.12 by showing that G and H satisfy the additional properties in the proposition.

Example 4.13. Let $G = K_{4m+1}$, the complete graph on $n = 4m+1$ vertices. It is a known result from the theory of degree sequences that for any positive even integer m, the graph K_{4m+1} contains an m-factor [Chen 1988], and therefore G satisfies condition (ii). We now design H to be a 3-regular graph with $b = 4m$ vertices. Begin with m disjoint copies of $K_4 - e$, where e is any edge of K_4. Give each copy a distinct label from $[m]$. For each i, label the two vertices in the i-th copy that have degree two $(i, 1)$ and $(i, 2)$. Then connect vertices $(i, 1)$ to $(i+1, 2)$ for each $i \in [m-1]$ and connect $(m, 1)$ to $(1, 2)$. This yields the graph H. Since H is 3-regular,

$$\gamma(H) \geq \frac{b}{k+1} = \frac{4m}{4} = m.$$

Observe that each of the m copies of $K_4 - e$ can be dominated by exactly one vertex. Hence, $\gamma(H) = m$, satisfying the additional condition (i). Therefore, $\hat{\gamma}_t(G \circledR H)$ meets the bound. □

Our final result gives an upper bound on the minimized total domination number of replacement product graphs.

Theorem 4.14. *Let G be a b-regular graph with $|V(G)| = n$ and H a k-regular graph with $|V(H)| = b$. Let G' be a spanning subgraph of G for which $|E(G')|$ is minimal given that $\delta(G') \geq \gamma(H)$. Then*

$$\hat{\gamma}_t(G \circledR H) \leq 2|E(G')|.$$

Proof. First note that G' always exists since

$$\delta(G) = |V(H)| \geq \gamma(H).$$

This also shows that $|E(G)|$ is an upper bound for $|E(G')|$. Now let D' be a smallest dominating set in H and let D be the set

$$D = \{(i, j) \in V(G \circledR H) \mid i \in [n], j \in D'\}$$

in $G \circledR H$. Design a rotation map on G by first labeling the edges of the subgraph G' at each vertex v of G using the numbers $1, 2, \ldots, \deg_{G'}(v)$, and label the remaining edges at each vertex v using $\deg_{G'}(v) + 1, \ldots, b$. Label the vertices in H by using the numbers $1, 2, \ldots, \gamma(H)$ for those in D' and the numbers $\gamma(H) + 1, \ldots, b$ for those not in D'. Last, for each $v \in V(G)$, if $\deg_{G'}(v) > \gamma(H)$, then add the vertices $(L_v, \gamma(H) + 1), (L_v, \gamma(H) + 2), \ldots, (L_v, \deg_{G'}(v))$ to D, where L_v denotes the vertex label of v in G.

Now consider the product $G \circledR H$ with this rotation map on G. By construction of the rotation map, every $v \in D$ is adjacent to a vertex $v' \in D$ via an intercloud edge. This shows that D is a total dominating set. Finally,

$$\begin{aligned}
|D| &= \gamma(H)|V(G)| + \sum_{v \in V(G')} (\deg_{G'}(v) - \gamma(H)) \\
&= \gamma(H)|V(G)| + 2|E(G')| - |V(G')|\gamma(H) \\
&= \gamma(H)\big(|V(G)| - |V(G')|\big) + 2|E(G')| \\
&= 2|E(G')|.
\end{aligned}$$

Therefore with the specified rotation map,

$$\gamma_t(G \circledR H) \leq 2|E(G')|,$$

which implies that

$$\hat{\gamma}_t(G \circledR H) \leq 2|E(G')|. \qquad \square$$

The following example illustrates that the bound in Theorem 4.14 is sharp.

Example 4.15. Let G be a b-regular graph on n vertices that contains a 1-factor, and let $H = K_b$, where $b \geq 2$. Let G' be a 1-factor of G. Note that $\delta(G') = 1 \geq 1 = \gamma(H)$. Since a 1-factor has the fewest number of edges of any spanning subgraph of G, the graph G' satisfies the condition in Theorem 4.14. Fix a rotation map on G and let S be the set of all vertices in $G \circledR H$ that are incident to the intercloud edges corresponding to $E(G')$. Then each $v \in S$ is adjacent to some other vertex $v' \in S$, where $v' \neq v$. Moreover, since $E(G')$ is a 1-factor of G, there exists exactly one vertex from S in each cloud of $G \circledR H$. Since each cloud is a complete graph, every vertex in $G \circledR H$ is dominated by S. Therefore, S is a total dominating set and has size $|S| = 2|E(G')|$.

We now show that this is the smallest such total dominating set of $G \circledR H$. Assume that D is a smallest total dominating set of $G \circledR H$ for some arbitrary rotation map on G. Assume further that there exists a cloud $\mathcal{C}$ that does not contain a member of D. Then $\mathcal{C}$ must be dominated by b vertices in D, say $s_1, \ldots, s_b$, via intercloud edges. Moreover, each s_i is contained in a different cloud, say s_i is from cloud $\mathcal{C}_i$. Since each s_i must be adjacent to some other vertex in D using a cloud edge, there must be other members of D, say $t_1, \ldots, t_b$ such that $t_i \in \mathcal{C}_i$ for each i.

Thus, with the assumption that there exists such a cloud $\mathcal{C}$, we can deduce that at least b clouds must contain at least two vertices in D. Moreover, the vertices $t_1, \ldots, t_b$ from D collectively only dominate b additional vertices from $G \circledR H$ which were not already dominated by $s_1, \ldots, s_b$. So for each cloud that does not contain a member of D, b additional vertices are needed and at most one additional cloud can be dominated. Further note that if a cloud is not completely dominated via intercloud edges then there must exist a member of D within that cloud. Thus, since $b \geq 2$, we see that for D to be minimally sized, there cannot be a cloud that does not have a member of D contained within it. Therefore $|D| \geq n = 2|E(G')|$, which implies our conclusion. $\square$

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