

involve

a journal of mathematics

Radio number for fourth power paths

Min-Lin Lo and Linda Victoria Alegria



Radio number for fourth power paths

Min-Lin Lo and Linda Victoria Alegria

(Communicated by Jerrold Griggs)

Let G be a connected graph. For any two vertices u and v , let $d(u, v)$ denote the distance between u and v in G . The maximum distance between any pair of vertices of G is called the diameter of G and denoted by $\text{diam}(G)$. A *radio labeling* (or multilevel distance labeling) of G is a function f that assigns to each vertex a label from the set $\{0, 1, 2, \dots\}$ such that the following holds for any vertices u and v : $|f(u) - f(v)| \geq \text{diam}(G) - d(u, v) + 1$. The *span* of f is defined as $\max_{u, v \in V(G)} \{|f(u) - f(v)|\}$. The *radio number* of G is the minimum span over all radio labelings of G . The *fourth power* of G is a graph constructed from G by adding edges between vertices of distance four or less apart in G . In this paper, we completely determine the radio number for the fourth power of any path, except when its order is congruent to 1 (mod 8).

1. Introduction

Motivated by the *channel assignment problem* [Hale 1980] of dividing the radio broadcasting spectrum among radio stations in such a way that the interference caused by their proximity is minimized, radio labeling was introduced by Chartrand et al. [2001] to model the problem of finding the optimal distribution of channels using the smallest necessary range of frequencies.

Let G be a connected graph. For any two vertices u and v of G , the *distance* between u and v is the length of a shortest u - v path in G and is denoted by $d_G(u, v)$ or simply $d(u, v)$ if the graph G under consideration is clear. The *diameter* of G , denoted by $\text{diam}(G)$, is the greatest distance between any two vertices of G . A *radio labeling* (or *multilevel distance labeling* [Liu 2008; Liu and Zhu 2005]) of a connected graph G is a function $f : V(G) \rightarrow \{0, 1, 2, 3, \dots\}$ with the property that

$$|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$$

for every two distinct vertices u and v of G . The *span* of f is defined as

$$\max_{u, v \in V(G)} \{|f(u) - f(v)|\}.$$

MSC2010: 05C78.

Keywords: channel assignment problem, multilevel distance labeling, radio number, radio labeling.

The radio number of G , denoted by $\text{rn}(G)$, is defined as

$$\min\{\text{span of } f : f \text{ is a radio labeling of } G\}.$$

A radio labeling for G with span equal to $\text{rn}(G)$ is called an *optimal radio labeling*.

Finding the radio number for a graph is an interesting yet challenging task. So far the value is known only for very limited families of graphs. The radio numbers for paths and cycles were investigated in [Chartrand et al. 2001; Chartrand, Erwin and Zhang 2005; Zhang 2002] and were completely solved by Liu and Zhu [2005]. The radio number for trees was investigated in [Liu 2008].

The r -th power of a graph G , denoted by G^r , is the graph constructed from G by adding edges between vertices of distance r or less apart in G . The radio number for the square of a path on n vertices, denoted by P_n^2 , was completely determined by Liu and Xie [2009], who also partially solved the problem for the square of a cycle on n vertices, denoted by C_n^2 [2004]. Motivated by [Liu and Xie 2009], Lo [2010] and Sooryanarayana et al. [2010] determined $\text{rn}(P_n^3)$.

This paper will follow the structure in [Liu and Xie 2009] closely to determine the radio number of the fourth power of paths (or simply, fourth power paths). It is our hope that this paper will be helpful for those readers who wish to pursue finding the radio number for P_n^5 , P_n^6 , and eventually P_n^r for any positive integer r .

Theorem 1. *Let P_n^4 be a fourth power path on n vertices where $n \geq 6$ and let $k = \text{diam}(P_n^4) = \lceil \frac{1}{4}(n-1) \rceil$. Then*

$$\text{rn}(P_n^4) = \begin{cases} 2k^2 + 1 & \text{if } n \equiv 0, 3, 6, \text{ or } 7 \pmod{8} \text{ or } n = 9, \\ 2k^2 + 2 & \text{if } n \equiv 4 \text{ or } 5 \pmod{8}, \\ 2k^2 & \text{if } n \equiv 2 \pmod{8}. \end{cases}$$

If $n \equiv 1 \pmod{8}$ and $n \geq 17$ (where n is of the form $8q + 1$), then

$$2k^2 + 2 \leq \text{rn}(P_{8q+1}^4) \leq 2k^2 + q.$$

2. General properties and notation

The diameter of P_n^4 is $\lceil \frac{1}{4}(n-1) \rceil$, based on the definition of P_n^4 . Figure 1 shows P_8^4 .

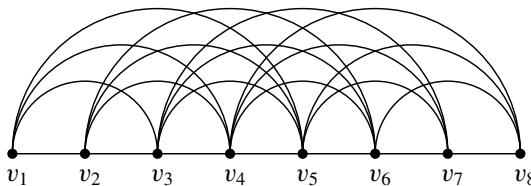


Figure 1. A fourth power path on 8 vertices, denoted by P_8^4 .

Proposition 2. *For any $u, v \in V(P_n^4)$, we have*

$$d(u, v) = \lceil \frac{1}{4}d_{P_n}(u, v) \rceil.$$

A center of P_n is defined as a “middle” vertex of P_n . An odd path P_{2m+1} has only one center v_{m+1} , while an even path P_{2m} has two centers v_m and v_{m+1} . For each vertex $u \in V(P_n)$, the level of u , denoted by $L(u)$ is the smallest distance in P_n from u to a center of P_n . If we denote the levels of a sequence of vertices A by $L(A)$, we have

$$\begin{aligned} n = 2m+1 &\Rightarrow L(v_1, v_2, \dots, v_{2m+1}) = (m, m-1, \dots, 2, 1, 0, 1, 2, \dots, m-1, m), \\ n = 2m &\Rightarrow L(v_1, v_2, \dots, v_{2m}) = (m-1, \dots, 2, 1, 0, 0, 1, 2, \dots, m-1). \end{aligned}$$

Define the *left-vertices* and *right-vertices* as follows:

If $n = 2m + 1$, then the left-vertices and right-vertices respectively are

$$\{v_1, v_2, \dots, v_m, v_{m+1}\} \quad \text{and} \quad \{v_{m+1}, v_{m+2}, \dots, v_{2m}, v_{2m+1}\}.$$

In this case, the center v_{m+1} is both a left-vertex and a right-vertex.

If $n = 2m$, then the left-vertices and right-vertices respectively are

$$\{v_1, v_2, \dots, v_m\} \quad \text{and} \quad \{v_{m+1}, v_{m+2}, \dots, v_{2m}\}.$$

If two vertices are both right-vertices or left-vertices, then we say that they are on the *same side*; otherwise, they are on *opposite sides*.

Lemma 3. *If n is odd, then for any $u, v \in V(P_n^4)$, we have*

$$d(u, v) = \begin{cases} \lceil \frac{1}{4}(L(u) + L(v)) \rceil & \text{if } u \text{ and } v \text{ are on opposite sides,} \\ \lceil \frac{1}{4}|L(u) - L(v)| \rceil & \text{if } u \text{ and } v \text{ are on the same side.} \end{cases}$$

If n is even, then for any $u, v \in V(P_n^4)$, we have

$$d(u, v) = \begin{cases} \lceil \frac{1}{4}(L(u) + L(v) + 1) \rceil & \text{if } u \text{ and } v \text{ are on opposite sides,} \\ \lceil \frac{1}{4}|L(u) - L(v)| \rceil & \text{if } u \text{ and } v \text{ are on the same side.} \end{cases}$$

In the proof of Lemma 7 below, the following proposition will be used frequently:

Proposition 4. *For any d_1, d_2 in \mathbb{N} , we have*

$$\left\lceil \frac{d_1+d_2}{r} \right\rceil = \begin{cases} \lceil d_1/r \rceil + \lceil d_2/r \rceil - 1 & \text{if } (d_1, d_2) \equiv (l, m) \pmod{r}, \text{ where } l \neq 0, m \neq 0, \\ & \text{and } 2 \leq (d_1+d_2) \pmod{r} \leq r, \\ \lceil d_1/r \rceil + \lceil d_2/r \rceil & \text{otherwise,} \end{cases}$$

$$\left\lceil \frac{d_1-d_2}{r} \right\rceil = \begin{cases} \lceil d_1/r \rceil - \lceil d_2/r \rceil + 1 & \text{if } (d_1, d_2) \equiv (0, m) \pmod{r}, \text{ where } m \neq 0, \\ & \text{or } (d_1, d_2) \equiv (l, m) \pmod{r}, \text{ where } l \neq 0, m \neq 0, \\ & \text{and } 1 \leq (d_1-d_2) \pmod{r} \leq (r-2), \\ \lceil d_1/r \rceil - \lceil d_2/r \rceil & \text{otherwise.} \end{cases}$$

It is important for the reader to understand the notation used in the labeling of P_n^4 so we will define a few terms and notation first.

Let $M, N \in \mathbb{N}$. We define a *block* (M, N) to be a pattern to follow when consecutively labeling a certain group of vertices in P_n^r . Take an (M, N) -block for example: The first vertex labeled, x_i , will have $L(x_i) \equiv M \pmod{r}$. The next vertex labeled, x_{i+1} , will have $L(x_{i+1}) \equiv N \pmod{r}$. The following vertex labeled, x_{i+2} , will have $L(x_{i+2}) \equiv M \pmod{r}$. Continue in this fashion until we end at a vertex of level congruent to $N \pmod{r}$. We may also choose to specify what side the vertex is on by writing (LM, RN) . This would mean that the first vertex labeled, x_i , would be a left-vertex with $L(x_i) \equiv M \pmod{r}$, and x_{i+1} would be a right-vertex with $L(x_{i+1}) \equiv N \pmod{r}$, so on and so forth.

We say that a *disconnection* occurs when $L(x_i) + L(x_{i+1})$ is not congruent to said specified value modulo r that maximizes the distance between two consecutively labeled vertices. This specific value changes depending upon the parity of n for P_n^4 .

A *labeling pattern* is a specific arrangement of blocks. Note that the same block may appear multiple times in a labeling pattern; however, the number of vertices in each “identical” block may be different. For any labeling pattern, P_n^4 will be said to have an “even” pairing if, for each (M, N) -block in the labeling pattern, the number of vertices with level congruent to $M \pmod{r}$ on one side equals the number of vertices with level congruent to $N \pmod{r}$ on the other side. Otherwise, P_n^4 will be said to have “extra” vertices.

3. Lower bound of $\text{rn}(P_n^4)$ when n is even

Lemma 5. *Let P_n^4 be a fourth power path on n vertices, where $n \geq 6$, and let $k = \text{diam}(P_n^4) = \lceil \frac{1}{4}(n - 1) \rceil$. If n is even, then*

$$\text{rn}(P_n^4) \geq \begin{cases} 2k^2 + 1 & \text{if } n \equiv 0 \text{ or } 6 \pmod{8}, \\ 2k^2 & \text{if } n \equiv 2 \pmod{8}, \\ 2k^2 + 2 & \text{if } n \equiv 4 \pmod{8}. \end{cases}$$

Proof. Let f be a radio labeling for P_n^4 . Rearrange $V(P_n^4) = \{x_1, x_2, \dots, x_n\}$ so that $0 = f(x_1) < f(x_2) < f(x_3) < \dots < f(x_n)$. Note that $f(x_n)$ is the span of f . By definition, $f(x_{i+1}) - f(x_i) \geq k + 1 - d(x_i, x_{i+1})$ for $1 \leq i \leq n - 1$. Summing up these $n - 1$ inequalities, we have

$$f(x_n) \geq (n - 1)(k + 1) - \sum_{i=1}^{n-1} d(x_i, x_{i+1}). \tag{3-1}$$

Thus to minimize $f(x_n)$, it suffices to maximize $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$. Since n is even,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \lceil \frac{1}{4}(L(x_i) + L(x_{i+1}) + 1) \rceil.$$

Observe, from the above inequality we have:

- (1) For each i , the equality for $d(x_i, x_{i+1}) \leq \lceil \frac{1}{4}(L(x_i) + L(x_{i+1}) + 1) \rceil$ holds when x_i and x_{i+1} are on opposite sides, or when they are on the same side but one of them is a center and the other vertex is of level not congruent to 0 (mod 4).
- (2) In the summation $\sum_{i=1}^{n-1} \lceil \frac{1}{4}(L(x_i) + L(x_{i+1}) + 1) \rceil$, each vertex of P_n^4 occurs exactly twice, except for x_1 and x_n , which both occur only once.

By direct calculation, we have

$$\lceil \frac{1}{4}(L(u) + L(v) + 1) \rceil = \begin{cases} \frac{1}{4}(L(u) + L(v) + 4) & \text{if } L(u) + L(v) \equiv 0 \pmod{4}, \\ \frac{1}{4}(L(u) + L(v) + 4) - \frac{1}{4} & \text{if } L(u) + L(v) \equiv 1 \pmod{4}, \\ \frac{1}{4}(L(u) + L(v) + 4) - \frac{2}{4} & \text{if } L(u) + L(v) \equiv 2 \pmod{4}, \\ \frac{1}{4}(L(u) + L(v) + 4) - \frac{3}{4} & \text{if } L(u) + L(v) \equiv 3 \pmod{4}. \end{cases}$$

Therefore,

$$\lceil \frac{1}{4}(L(x_i) + L(x_{i+1}) + 1) \rceil \leq \frac{1}{4}(L(x_i) + L(x_{i+1}) + 4),$$

and the equality holds only if $L(x_i) + L(x_{i+1}) \equiv 0 \pmod{4}$. Combining this with (1) above, there exist at most $n-4$ of the i such that $d(x_i, x_{i+1}) = \frac{1}{4}(L(x_i) + L(x_{i+1}) + 4)$; that is, there are at least three disconnections in the labeling. Note that when $L(x_i) + L(x_{i+1}) \equiv 1, 2,$ or $3 \pmod{4}$, we say that there is a disconnection between x_i and x_{i+1} of the *best type*, *second best type*, or the *worst type*, respectively. Moreover, among all the vertices, only the centers are of level zero. Hence, $L(x_1) + L(x_n) \geq 0 + 0 = 0$. We conclude that

$$\begin{aligned} \sum_{i=1}^{n-1} d(x_i, x_{i+1}) &\leq \left(\sum_{i=1}^{n-1} \frac{1}{4}(L(x_i) + L(x_{i+1}) + 4) \right) - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} \\ &= \frac{1}{4} \left(\left(2 \sum_{i=1}^n L(x_i) \right) - L(x_1) - L(x_n) \right) + (n-1) - \frac{3}{4} \\ &\leq \frac{1}{4} \left(\left(2 \sum_{i=1}^n L(x_i) \right) - 0 - 0 \right) + (n-1) - \frac{3}{4} \\ &= \frac{1}{2} (2(0 + 1 + 2 + \dots + (\frac{1}{2}n - 1))) + n - \frac{7}{4} \\ &= \frac{1}{8}n^2 + \frac{3}{4}n - \frac{7}{4}. \end{aligned}$$

By direct calculation for (3-1) and considering that $rn(P_n^4)$ is an integer, we have

$$rn(P_n^4) \geq \begin{cases} \lceil 2k^2 + \frac{3}{4} \rceil = 2k^2 + 1 & \text{if } n \equiv 0 \pmod{8} \text{ (i.e., } n = 4k \text{ and } k \text{ is even),} \\ \lceil 2k^2 - \frac{1}{4} \rceil = 2k^2 & \text{if } n \equiv 2 \pmod{8} \text{ (i.e., } n = 4k - 2 \text{ and } k \text{ is odd),} \\ \lceil 2k^2 + \frac{3}{4} \rceil = 2k^2 + 1 & \text{if } n \equiv 4 \pmod{8} \text{ (i.e., } n = 4k \text{ and } k \text{ is odd),} \\ \lceil 2k^2 - \frac{1}{4} \rceil = 2k^2 & \text{if } n \equiv 6 \pmod{8} \text{ (i.e., } n = 4k - 2 \text{ and } k \text{ is even).} \end{cases}$$

Further investigation for a sharper lower bound of $\text{rn}(P_n^4)$ when $n \equiv 4$ or $6 \pmod{8}$ is needed. There are three cases to consider based on the number of disconnections that occur in the labeling pattern.

Case 1: There are at least five disconnections. Then we have,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left(\sum_{i=1}^{n-1} \frac{1}{4}(L(x_i) + L(x_{i+1}) + 4) \right) - \frac{5}{4} \leq \frac{1}{8}n^2 + \frac{3}{4}n - \frac{9}{4}.$$

Hence, by direct calculation for (3-1) we have

$$\text{rn}(P_n^4) \geq \begin{cases} \lceil (2k^2 + \frac{3}{4}) + \frac{2}{4} \rceil = 2k^2 + 2 & \text{if } n \equiv 4 \pmod{8} \text{ (i.e., } n = 4k \text{ and } k \text{ is odd),} \\ \lceil (2k^2 - \frac{1}{4}) + \frac{2}{4} \rceil = 2k^2 + 1 & \text{if } n \equiv 6 \pmod{8} \text{ (i.e., } n = 4k - 2 \text{ and } k \text{ is even).} \end{cases}$$

Case 2: There are exactly four disconnections. This case will be broken down into two subcases based on $L(x_1) + L(x_n)$.

Case 2.1: $L(x_1) + L(x_n) \geq 1$. Therefore,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left(\sum_{i=1}^{n-1} \frac{1}{4}(L(x_i) + L(x_{i+1}) + 4) \right) - \frac{4}{4} \leq \frac{1}{8}n^2 + \frac{3}{4}n - \frac{9}{4}.$$

Case 2.2: $L(x_1) + L(x_n) = 0$.

Claim. *In this case, at least two of the disconnections that occur cannot be of the best type.*

Proof of claim. For $n \equiv 4$ or $6 \pmod{8}$, we have the following types of blocks as well as extra vertices (without loss of generality, we start each block with a left-vertex):

$$(L0, R0), \quad (L1, R3), \quad (L2, R2), \quad (L3, R1) \quad L1, \quad R1.$$

We wish to have exactly four disconnections and we also want $L(x_1) + L(x_n) = 0 + 0 = 0$ under this case. Therefore we must use two (L0, R0)-blocks. Thus our new blocks become (blocks are boxed for easy identification of disconnections that occur in the labeling pattern):

$$\boxed{(L0, R0)}, \quad \boxed{(L1, R3) - L1}, \quad \boxed{(L2, R2)}, \quad \boxed{R1 - (L3, R1)}, \quad \boxed{(L0, R0)}.$$

Since we want $L(x_1) + L(x_n) = 0 + 0 = 0$, our labeling pattern must start and end with the (L0, R0)-blocks. Special attention is given to the “end-1” vertices, namely, the first and the last vertices of the two block patterns $(L1, R3) - L1$ and $R1 - (L3, R1)$ from above. All disconnections in the labeling pattern will occur at these four end-1 vertices. The best type of disconnection would occur if an end-1 vertex was followed or preceded by a vertex whose level was congruent to $0 \pmod{4}$. However, there are only two such vertices available. Therefore, at least two of the four end-1 vertices cannot have disconnections of the best type. \square

By direct calculation, our claim, and the assumption that $L(x_1) + L(x_n) = 0$, we have,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left(\sum_{i=1}^{n-1} \frac{1}{4}(L(x_i) + L(x_{i+1}) + 4) \right) - \frac{6}{4} = \frac{1}{8}n^2 + \frac{3}{4}n - \frac{10}{4}.$$

Hence, by direct calculation for (3-1) for the two subcases, the same bounds as in the conclusion of Case 1 are obtained.

Case 3: There are exactly three disconnections.

Claim. *In this case, at least one of the disconnections in the labeling pattern will not be of the best type.*

Proof of claim. Similar to Case 2.2, to ensure that there are only three disconnections, our new blocks must be

$$\boxed{(L0, R0)}, \quad \boxed{(L1, R3) - L1}, \quad \boxed{(L2, R2)}, \quad \boxed{R1 - (L3, R1)}.$$

Thus, out of the three disconnections that occur, at least two of them will occur at the end-1 vertices. Furthermore, out of the disconnections that occur at the end-1 vertices, at least one of them will not be of the best type, unless two (L0, R0)-blocks are used, which would increase the number of disconnections. \square

By calculation, our claim, and noting that $L(x_1) + L(x_n) \geq 1$ under this case, we have,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left(\sum_{i=1}^{n-1} \frac{1}{4}(L(x_i) + L(x_{i+1}) + 4) \right) - \frac{4}{4} \leq \frac{1}{8}n^2 + \frac{3}{4}n - \frac{9}{4}.$$

Direct calculation for (3-1) in this case also leads to the same bounds as in the conclusion of Case 1. \square

4. Lower bound of $rn(P_n^4)$ when n is odd

Lemma 6. *Let P_n^4 be a fourth power path on n vertices, where $n \geq 6$, and let $k = \text{diam}(P_n^4) = \lceil \frac{1}{4}(n - 1) \rceil$. If n is odd, then*

$$rn(P_n^4) \geq \begin{cases} 2k^2 + 2 & \text{if } n \equiv 1 \pmod{8} \text{ and } n \geq 17 \text{ or } n \equiv 5 \pmod{8}, \\ 2k^2 + 1 & \text{if } n \equiv 3 \text{ or } 7 \pmod{8} \text{ or } n = 9. \end{cases}$$

Proof. We retain the same notation and employ the same method used in the proof of Lemma 5. Since n is odd,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \lceil \frac{1}{4}(L(x_i) + L(x_{i+1})) \rceil.$$

Observe, from the above inequality we have:

- (1) For each i , the equality for $d(x_i, x_{i+1}) \leq \lceil \frac{1}{4}(L(x_i) + L(x_{i+1})) \rceil$ holds only when x_i and x_{i+1} are on opposite sides, unless one of them is a center.
- (2) In the summation $\sum_{i=1}^{n-1} \lceil \frac{1}{4}(L(x_i) + L(x_{i+1})) \rceil$, each vertex of P_n^4 occurs exactly twice, except x_1 and x_n , which each occurs only once.

By direct calculation, we have

$$\lceil \frac{1}{4}(L(u) + L(v)) \rceil = \begin{cases} \frac{1}{4}(L(u) + L(v) + 3) - \frac{3}{4} & \text{if } L(u) + L(v) \equiv 0 \pmod{4}, \\ \frac{1}{4}(L(u) + L(v) + 3) & \text{if } L(u) + L(v) \equiv 1 \pmod{4}, \\ \frac{1}{4}(L(u) + L(v) + 3) - \frac{1}{4} & \text{if } L(u) + L(v) \equiv 2 \pmod{4}, \\ \frac{1}{4}(L(u) + L(v) + 3) - \frac{2}{4} & \text{if } L(u) + L(v) \equiv 3 \pmod{4}. \end{cases}$$

Therefore

$$\lceil \frac{1}{4}(L(x_i) + L(x_{i+1})) \rceil \leq \frac{1}{4}(L(x_i) + L(x_{i+1}) + 3),$$

and the equality holds only if $L(x_i) + L(x_{i+1}) \equiv 1 \pmod{4}$. Note that when $L(x_i) + L(x_{i+1}) \equiv 2, 3,$ or $0 \pmod{4}$, we say that there is a disconnection between x_i and x_{i+1} of the *best type*, *second best type*, or the *worst type*, respectively. Combining this with (1), there are two possible cases to consider based on the number of disconnections in the labeling pattern:

Case 1: There are at least three disconnections. In this case, since n is odd, there is only one center. Therefore, $L(x_1) + L(x_n) \geq 1$. Then,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left(\sum_{i=1}^{n-1} \frac{1}{4}(L(x_i) + L(x_{i+1}) + 3) \right) - \frac{3}{4} \leq \frac{1}{8}n^2 + \frac{3}{4}n - \frac{15}{8}.$$

By direct calculation for (3-1), we have

$$\text{rn}(P_n^4) \geq \begin{cases} 2k^2 + 1 & \text{if } n \equiv 1 \pmod{8} \text{ (i.e., } n = 4k + 1 \text{ and } k \text{ is even),} \\ \lceil 2k^2 + \frac{1}{2} \rceil = 2k^2 + 1 & \text{if } n \equiv 3 \pmod{8} \text{ (i.e., } n = 4k - 1 \text{ and } k \text{ is odd),} \\ 2k^2 + 1 & \text{if } n \equiv 5 \pmod{8} \text{ (i.e., } n = 4k + 1 \text{ and } k \text{ is odd),} \\ \lceil 2k^2 + \frac{1}{2} \rceil = 2k^2 + 1 & \text{if } n \equiv 7 \pmod{8} \text{ (i.e., } n = 4k - 1 \text{ and } k \text{ is even).} \end{cases}$$

Case 2: There are exactly two disconnections. In this case, neither x_1 nor x_n is the center (denoted by C).

Case 2.1: $n \equiv 1 \pmod{8}$. The labeling pattern must be a permutation of the boxed blocks

$$\boxed{(L0, R1) - C - (L1, R0)}, \quad \boxed{(L2, R3)}, \quad \boxed{(L3, R2)}.$$

Therefore, $L(x_1) + L(x_n) \geq 4$. By similar calculations to Case 1, we have

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left(\sum_{i=1}^{n-1} \frac{1}{4}(L(x_i) + L(x_{i+1}) + 3) \right) - \frac{2}{4} \leq \frac{1}{8}n^2 + \frac{3}{4}n - \frac{19}{8}.$$

By direct calculations, since $n = 4k + 1$ and k is even, we have

$$\text{rn}(P_n^4) \geq \lceil (2k^2 + 1) + \frac{2}{4} \rceil = 2k^2 + 2.$$

Case 2.2: $n \equiv 3, 5, \text{ or } 7 \pmod{8}$. Note that P_{8q+3}^4 and P_{8q+7}^4 both have an extra pair of vertices whose level is congruent to 1 (mod 4). Therefore, the labeling pattern must be a permutation of the boxed blocks

$$\boxed{\text{R1} - (\text{L0, R1}) - \text{C} - (\text{L1, R0}) - \text{L1}}, \quad \boxed{(\text{L2, R3})}, \quad \boxed{(\text{L3, R2})}.$$

Now, P_{8q+5}^4 has two extra pairs of vertices whose levels are congruent to 1 (mod 4) and 2 (mod 4). The labeling pattern must be a permutation of the boxed blocks

$$\boxed{\text{R1} - (\text{L0, R1}) - \text{C} - (\text{L1, R0}) - \text{L1}}, \quad \boxed{(\text{L2, R3}) - \text{L2}}, \quad \boxed{\text{R2} - (\text{L3, R2})}.$$

Therefore, for $n \equiv 3, 5, \text{ or } 7 \pmod{8}$, considering all possible permutations mentioned above, $L(x_1) + L(x_n) \geq 3$. Therefore,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left(\sum_{i=1}^{n-1} \frac{1}{4}(L(x_i) + L(x_{i+1}) + 3) \right) - \frac{2}{4} \leq \frac{1}{8}n^2 + \frac{3}{4}n - \frac{17}{8}.$$

Thus, by direct calculation we have,

$$\text{rn}(P_n^4) \geq \begin{cases} \lceil (2k^2 + \frac{1}{2}) + \frac{1}{4} \rceil = 2k^2 + 1 & \text{if } n \equiv 3 \pmod{8} \text{ (i.e., } n = 4k - 1 \text{ and } k \text{ is odd),} \\ \lceil (2k^2 + 1) + \frac{1}{4} \rceil = 2k^2 + 2 & \text{if } n \equiv 5 \pmod{8} \text{ (i.e., } n = 4k + 1 \text{ and } k \text{ is odd),} \\ \lceil (2k^2 + \frac{1}{2}) + \frac{1}{4} \rceil = 2k^2 + 1 & \text{if } n \equiv 7 \pmod{8} \text{ (i.e., } n = 4k - 1 \text{ and } k \text{ is even).} \end{cases}$$

Now assume $n \equiv 1 \pmod{8}$ and $n \geq 17$; that is, $n = 4k + 1$, k is even and $k \geq 4$. Assume to the contrary that $f(x_n) = 2k^2 + 1$. Then only Case 1 is possible and all of the following must hold:

- (1) $\{x_1, x_n\} = \{v_{2k+1}, v_{2k+2}\}$ or $\{v_{2k+1}, v_{2k}\}$. That is, $\{x_1, x_n\}$ is of the form $\{x_1, x_n\} = \{\text{center, a vertex right next to center}\}$.
- (2) $f(x_{i+1}) = f(x_i) + k + 1 - d(x_i, x_{i+1})$ for all i .
- (3) For all $i \geq 1$, the two vertices x_i and x_{i+1} are on opposites sides unless one of them is the center.
- (4) There exist three t -values, $1 \leq t \leq n - 1$, such that $L(x_t) + L(x_{t+1}) \equiv 2 \pmod{4}$ while $L(x_t) + L(x_{t+1}) \equiv 1 \pmod{4}$ for all other $i \neq t$.

By (1) and by symmetry, we can assume that $x_1 = v_{2k+1}$; i.e., x_1 is the center. Excluding the center, there are $\frac{1}{2}k$ vertices whose level is congruent to 0 (mod 4), 1 (mod 4), 2 (mod 4), and 3 (mod 4) on each side, respectively. Since x_n is of level one, by (2), (3), and (4) we have:

(5) The labeling pattern must be the arrangement of boxed blocks

$$\boxed{C - (1, 0)} - \boxed{(2, 3)} - \boxed{(3, 2)} - \boxed{(0, 1)}.$$

Claim. $\{v_1, v_n\} = \{x_{k+1}, x_{3k+2}\}$ (i.e., $\{v_1, v_n\}$ consists of the last vertex whose level is congruent to 0 (mod 4) in the (1, 0)-block and the first vertex whose level is congruent to 0 (mod 4) in the (0, 1)-block).

Proof of claim. Suppose $v_1 \notin \{x_{k+1}, x_{3k+2}\}$. Then v_1 is inside one of the (0, 1)- or (1, 0)-blocks, since $L(v_1) = 2k \equiv 0 \pmod{4}$. Let $v_1 = x_c$ for some c , where x_{c-1} and x_{c+1} are both vertices on the right side. Thus, $L(x_{c-1}) \equiv L(x_{c+1}) \equiv 1 \pmod{4}$. Let $L(x_{c-1}) = y$ and $L(x_{c+1}) = z$. By (2),

$$\begin{aligned} f(x_c) - f(x_{c-1}) &= \frac{1}{2}k + 1 - \lceil \frac{1}{4}y \rceil, \\ f(x_{c+1}) - f(x_c) &= \frac{1}{2}k + 1 - \lceil \frac{1}{4}z \rceil. \end{aligned}$$

Therefore,

$$f(x_{c+1}) - f(x_{c-1}) = k + 2 - \lceil \frac{1}{4}y \rceil - \lceil \frac{1}{4}z \rceil,$$

contradicting that

$$f(x_{c+1}) - f(x_{c-1}) \geq k + 1 - \lceil \frac{1}{4}|z - y| \rceil \quad (\text{as } y \equiv z \equiv 1 \pmod{4}, \text{ so } y, z \neq 0).$$

Therefore $v_1 \in \{x_{k+1}, x_{3k+2}\}$. Similarly, we can show that $v_n \in \{x_{k+1}, x_{3k+2}\}$. \square

By the claim, we may assume that $v_n = x_{k+1}$ and $v_1 = x_{3k+2}$ (the proof for the other case is symmetric). By (5), $L(x_k) = a \equiv 1 \pmod{4}$ and $L(x_{k+2}) = b \equiv 2 \pmod{4}$. By (2), (3), the fact that k is even, and our assumption that $L(x_{k+1}) = L(v_n) = L(v_{4k+1}) = 2k$, we have

$$\begin{aligned} f(x_{k+1}) - f(x_k) &= \frac{1}{2}k + 1 - \lceil \frac{1}{4}a \rceil, \\ f(x_{k+2}) - f(x_{k+1}) &= \frac{1}{2}k + 1 - \lceil \frac{1}{4}b \rceil, \end{aligned}$$

and so,

$$f(x_{k+2}) - f(x_k) = k + 2 - \lceil \frac{1}{4}a \rceil - \lceil \frac{1}{4}b \rceil.$$

By definition and by Lemma 3,

$$f(x_{k+2}) - f(x_k) \geq k + 1 - \lceil \frac{1}{4}|a - b| \rceil.$$

Therefore, a must equal 1. Thus $L(x_k) = 1$, which means x_k is the level-one vertex on the left side, since $x_{k+1} = v_n$ is a right-vertex. Thus $x_k = v_{2k}$. Similarly, we can show that x_{3k+3} is of level one and on the right side. Thus, $x_{3k+3} = v_{2k+2}$.

Now, x_n is a right-vertex since $x_{3k+2} = v_1$ is a left-vertex, and so $x_n = v_{2k+2}$. This implies that $x_n = v_{2k+2} = x_{3k+3}$ and therefore $k = 2$, contradicting the assumption $k \geq 4$. Therefore $\text{rn}(P_n^4) \geq 2k^2 + 2$ if $n \equiv 1 \pmod{8}$ and $n \geq 17$.

Similar techniques can be applied for the case $n \equiv 5 \pmod{8}$. Assume that $n \equiv 5 \pmod{8}$ and $n \geq 21$; that is, $n = 4k + 1$, k is odd, and $k \geq 5$. Assume to

the contrary that $f(x_n) = 2k^2 + 1$. Then only Case 1 is possible and the same requirements (1), (2), (3), and (4) for the case $n = 1 \pmod 8$ and $n \geq 17$ must hold.

By (1) and by symmetry, we can assume that $x_1 = v_{2k+1}$; i.e., x_1 is the center. Excluding the center, there are $\frac{1}{2}(k - 1)$ vertices whose level is congruent to 0 (mod 4), $\frac{1}{2}(k + 1)$ vertices whose level is congruent to 1 (mod 4), $\frac{1}{2}(k + 1)$ vertices whose level is congruent to 2 (mod 4), and $\frac{1}{4}(k - 1)$ vertices whose level is congruent to 3 (mod 4), on each side. By (1), (2), (3), and the second part of (4), the labeling pattern must be the arrangement of boxed blocks

$$\boxed{C - (1 - 0 - 1)} - \boxed{(2 - 3 - 2)} - \boxed{(2 - 3 - 2)} - \boxed{(1 - 0 - 1)}.$$

However, in this arrangement the three t -values for which $L(x_t) + L(x_{t+1})$ is not congruent to 1 (mod 4) are not all congruent to 2 (mod 4), which contradicts the first part of (4). Therefore, $\text{rn}(P_n^4) \geq 2k^2 + 2$. □

5. Upper bound and optimal radio labelings

To establish Theorem 1, it suffices to give radio labelings achieving the desired spans. To this end, we will use the next lemma, which provides us with an easy way to verify that a given labeling of P_n^r is indeed a radio labeling of P_n^r .

Lemma 7. *Let P_n^r be an r -th power path graph on n vertices, where $k = \text{diam}(P_n^r) = \lceil \frac{1}{r}(n - 1) \rceil$. Let $\{x_1, x_2, x_3, \dots, x_n\}$ be a permutation of $V(P_n^r)$ such that for any $1 \leq i \leq n - 2$,*

$$\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq \frac{1}{2}rk + 1$$

and $\max\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \not\equiv 1 \pmod r$ if k is even and the equality in the above holds. Let f be a function, $f : V(P_n^r) \rightarrow \{0, 1, 2, \dots\}$ with $f(x_1) = 0$ and $f(x_{i+1}) - f(x_i) = k + 1 - d(x_i, x_{i+1})$ for all $1 \leq i \leq n - 1$. Then f is a radio labeling for P_n^r .

Before we present the proof of Lemma 7, note that Proposition 4 will be used frequently throughout the proof of Lemma 7 below. The construction of this proof is adapted from [Liu and Xie 2009].

Proof. Let f be a function satisfying the assumption. It suffices to prove that $f(x_j) - f(x_i) \geq k + 1 - d(x_i, x_j)$ for any $j \geq i + 2$. For $i = 1, 2, \dots, n - 1$, set

$$f_i = f(x_{i+1}) - f(x_i).$$

For any $j \geq i + 2$, it follows that $f(x_j) - f(x_i) = f_i + f_{i+1} + f_{i+2} + \dots + f_{j-1}$. We divide the proof into three cases:

Case 1: $j = i + 2$. Assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$ (the proof for $d(x_i, x_{i+1}) \leq d(x_{i+1}, x_{i+2})$ is similar). Then,

$$d(x_{i+1}, x_{i+2}) \leq \left\lceil \frac{\frac{1}{2}rk + 1}{r} \right\rceil \leq \begin{cases} \frac{1}{2}(k + 2) & \text{if } k \text{ is even,} \\ \frac{1}{2}(k + 1) & \text{if } k \text{ is odd.} \end{cases}$$

Therefore, $d(x_{i+1}, x_{i+2}) \leq \frac{1}{2}(k + 2)$. It suffices to consider the following subcases:

Case 1.1: x_i is between x_{i+1} and x_{i+2} . Then $d(x_i, x_{i+1}) \leq d(x_{i+1}, x_{i+2})$. Since we assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$, we have $d(x_i, x_{i+1}) = d(x_{i+1}, x_{i+2}) \leq \frac{1}{2}(k + 2)$ and $d_{P_n}(x_i, x_{i+2}) \leq (r - 1)$, from which we have $d(x_i, x_{i+2}) = 1$. Hence,

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= k + 1 - d(x_i, x_{i+1}) + k + 1 - d(x_{i+1}, x_{i+2}) \\ &\geq k + 1 - d(x_i, x_{i+2}). \end{aligned}$$

Case 1.2: x_{i+1} is between x_i and x_{i+2} . This implies

$$d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) - 1.$$

Similar to the calculations above, we have $f(x_{i+2}) - f(x_i) \geq k + 1 - d(x_i, x_{i+2})$.

Case 1.3: x_{i+2} is between x_i and x_{i+1} . Assume k is odd or

$$\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq (\frac{1}{2}rk + 1) - 1,$$

then we have $d(x_{i+1}, x_{i+2}) \leq \frac{1}{2}(k + 1)$ and $d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})$. Hence, $f(x_{i+2}) - f(x_i) \geq k + 1 - d(x_i, x_{i+2})$. If k is even and

$$\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} = \frac{1}{2}rk + 1,$$

then by our assumption, it must be that $d_{P_n}(x_{i+1}, x_{i+2}) = \frac{1}{2}rk + 1 \equiv 1 \pmod{r}$ and $d_{P_n}(x_i, x_{i+1}) \not\equiv 1 \pmod{r}$. Thus we have,

$$d(x_i, x_{i+2}) = d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) + 1,$$

which implies

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= 2k + 2 - (d(x_i, x_{i+2}) + d(x_{i+1}, x_{i+2}) - 1) - d(x_{i+1}, x_{i+2}) \\ &\geq k + 1 - d(x_i, x_{i+2}). \end{aligned}$$

Case 2: $j = i + 3$.

Case 2.1: The sum of some pair of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, and $d(x_{i+2}, x_{i+3})$ is at most $k + 2$. Then,

$$\begin{aligned} f(x_{i+3}) - f(x_i) &\geq 3k + 3 - (k + 2) - k \\ &> k + 1 - d(x_i, x_{i+3}). \end{aligned}$$

Case 2.2: The sum of any pair of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, and $d(x_{i+2}, x_{i+3})$ is greater than $k + 2$. If we then assume that $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$ (the proof for $d(x_i, x_{i+1}) \leq d(x_{i+1}, x_{i+2})$ is similar), from the calculation in Case 1,

we have $d(x_{i+1}, x_{i+2}) \leq \frac{1}{2}(k + 2)$. By our hypothesis, it follows that $d(x_i, x_{i+1})$ and $d(x_{i+2}, x_{i+3})$ must both be greater than $\frac{1}{2}(k + 2)$. This result, together with $\text{diam}(P_n^r) = k$ and our assumption under this case, implies that x_i must appear before x_{i+2} , then x_{i+1} , then x_{i+3} , from left to right on the r -th power path (or x_{i+3} must appear before x_{i+1} , then x_{i+2} , then x_i). Therefore,

$$d(x_i, x_{i+3}) \geq d(x_i, x_{i+1}) + d(x_{i+2}, x_{i+3}) - d(x_{i+1}, x_{i+2}) - 1.$$

Therefore, we have

$$\begin{aligned} f(x_{i+3}) - f(x_i) &\geq 3k + 3 - d(x_i, x_{i+3}) - 2d(x_{i+1}, x_{i+2}) - 1 \\ &\geq k + 1 - d(x_i, x_{i+3}). \end{aligned}$$

Case 3: $j \geq i + 4$. Since

$$\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq \frac{1}{2}(k + 2)$$

and $f_i \geq k + 1 - d(x_i, x_{i+1})$ for any i , we have $\max\{f_i, f_{i+1}\} \geq \frac{1}{2}k$ for any $1 \leq i \leq n - 2$. Therefore,

$$\begin{aligned} f(x_j) - f(x_i) &\geq (f_i + f_{i+1}) + (f_{i+2} + f_{i+3}) \\ &\geq \left(\frac{1}{2}k + 1\right) + \left(\frac{1}{2}k + 1\right) > k + 1 - d(x_i, x_j). \quad \square \end{aligned}$$

When $\text{diam}(P_n^r)$ is odd, we have the following ‘‘looser’’ condition for checking that a given labeling is indeed a radio labeling:

Lemma 8. *Let P_n^r be an r -th power path graph on n vertices, where $k = \text{diam}(P_n^r) = \lceil \frac{1}{r}(n - 1) \rceil$ is odd. Let $\{x_1, x_2, x_3, \dots, x_n\}$ be a permutation of $V(P_n^r)$ such that for any $1 \leq i \leq n - 2$,*

$$\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq \frac{1}{2}r(k + 1).$$

Let f be a function, $f : V(P_n^r) \rightarrow \{0, 1, 2, \dots\}$ with $f(x_1) = 0$ and $f(x_{i+1}) - f(x_i) = k + 1 - d(x_i, x_{i+1})$ for all $1 \leq i \leq n - 1$. Then f is a radio labeling for P_n^r .

Proof. Assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$ (the proof for $d(x_i, x_{i+1}) \leq d(x_{i+1}, x_{i+2})$ is similar). Then

$$d(x_{i+1}, x_{i+2}) \leq \left\lceil \frac{\frac{1}{2}r(k + 1)}{r} \right\rceil = \frac{1}{2}k + 1 \leq \frac{1}{2}k + 2.$$

Note that this is the same conclusion we obtained in the beginning of the proof of Lemma 7. Therefore we can use exactly the same proof as above for the case when k is odd to prove this lemma. \square

For each radio labeling f of P_n^4 given in the following, we shall first define a permutation (line-up) of the vertices $V(P_n^4) = \{x_1, x_2, x_3, \dots, x_n\}$, then define f by $f(x_1) = 0$, and for all $1 \leq i \leq n - 1$, $f(x_{i+1}) - f(x_i) = k + 1 - d(x_i, x_{i+1})$.

Case 1: $\text{rn}(P_{8q+5}^4) \leq 2k^2 + 2$. Let $n = 8q + 5$ for some $q \in \mathbb{N}$. Then $k = \text{diam}(P_{8q+5}^4) = 2q + 1$. We give a radio labeling with span $2k^2 + 2$. The line-up of $V(P_n^4) = \{x_1, x_2, \dots, x_n\}$ is given by the arrows in the display below. That is, x_1 is the center, x_2 is the left-vertex of P_n^4 whose level is equal to $4q+1$, \dots , x_n is the right-vertex of P_n^4 whose level is equal to 2. The values above and below each arrow indicate the distances in P_n^4 and P_n , respectively, between consecutively labeled vertices.

$$\begin{aligned} & \overset{q+1}{\underset{4q+1}{\text{C}}} \xrightarrow{\text{L}(4q+1)} \xrightarrow{\frac{q+2}{4q+5}} \text{R4} \xrightarrow{\frac{q+1}{4q+1}} \text{L}(4q-3) \xrightarrow{\frac{q+2}{4q+5}} \dots \xrightarrow{\frac{q+1}{4q+1}} \text{L5} \xrightarrow{\frac{q+2}{4q+5}} \text{R}(4q) \xrightarrow{\frac{q+1}{4q+1}} \text{L1} \\ & \xrightarrow{\frac{q+1}{4q+2}} \text{R}(4q+1) \xrightarrow{\frac{q+2}{4q+5}} \text{L4} \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-3) \xrightarrow{\frac{q+2}{4q+5}} \text{L8} \xrightarrow{\frac{q+1}{4q+1}} \dots \xrightarrow{\frac{q+1}{4q+1}} \text{R5} \xrightarrow{\frac{q+2}{4q+5}} \text{L}(4q) \xrightarrow{\frac{q+1}{4q+1}} \text{R1} \\ & \xrightarrow{\frac{q+1}{4q+3}} \text{L}(4q+2) \xrightarrow{\frac{q+2}{4q+5}} \text{R3} \xrightarrow{\frac{q+1}{4q+1}} \text{L}(4q-2) \xrightarrow{\frac{q+2}{4q+5}} \text{R7} \xrightarrow{\frac{q+1}{4q+1}} \dots \xrightarrow{\frac{q+1}{4q+1}} \text{L6} \xrightarrow{\frac{q+2}{4q+5}} \text{R}(4q-1) \xrightarrow{\frac{q+1}{4q+1}} \text{L2} \\ & \xrightarrow{\frac{q+1}{4q+4}} \text{R}(4q+2) \xrightarrow{\frac{q+2}{4q+5}} \text{L3} \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-2) \xrightarrow{\frac{q+2}{4q+5}} \text{L7} \xrightarrow{\frac{q+1}{4q+1}} \dots \xrightarrow{\frac{q+1}{4q+1}} \text{R6} \xrightarrow{\frac{q+2}{4q+5}} \text{L}(4q-1) \xrightarrow{\frac{q+1}{4q+1}} \text{R2}. \end{aligned}$$

By Lemma 8, f is a radio labeling for P_{8q+5}^4 . Observe from the above display, there are two possible distances in P_{8q+5}^4 between consecutively labeled vertices, namely, $q + 1$ and $q + 2$, with the number of occurrences $4q + 4$ and $4q$, respectively. It follows by direct calculation that

$$f(x_{8q+5}) = (8q + 4)(k + 1) - \sum_{i=1}^{8q+4} d(x_i, x_{i+1}) = 2k^2 + 2.$$

Case 2: $\text{rn}(P_{8q+4}^4) \leq 2k^2 + 2$. Let $n = 8q + 4$ for some $q \in \mathbb{N}$. Then $k = \text{diam}(P_{8q+4}^4) = 2q + 1$. Let $G = P_{8q+5}^4$ and H be the subgraph of G induced by the vertices $\{v_1, v_2, \dots, v_{8q+4}\}$. Then $H \cong P_{8q+4}^4$, $\text{diam}(H) = \text{diam}(G) = 2q + 1$, and $d_G(u, v) = d_H(u, v)$ for every $u, v \in V(H)$. Let f be a radio labeling for G , then $f|_H$ is also a radio labeling for H . By Case 1, $\text{rn}(P_{8q+4}^4) \leq \text{rn}(P_{8q+5}^4) \leq 2k^2 + 2$.

Case 3: $\text{rn}(P_{8q+3}^4) \leq 2k^2 + 1$. Let $n = 8q + 3$ for some $q \in \mathbb{N}$. Then $k = \text{diam}(P_{8q+3}^4) = 2q + 1$. Similar to Case 1, we line up the vertices according to the display below.

$$\begin{aligned} & \overset{q+1}{\underset{4q+1}{\text{C}}} \xrightarrow{\text{L}(4q+1)} \xrightarrow{\frac{q+2}{4q+5}} \text{R4} \xrightarrow{\frac{q+1}{4q+1}} \text{L}(4q-3) \xrightarrow{\frac{q+2}{4q+5}} \dots \xrightarrow{\frac{q+1}{4q+1}} \text{L5} \xrightarrow{\frac{q+2}{4q+5}} \text{R}(4q) \xrightarrow{\frac{q+1}{4q+1}} \text{L1} \\ & \xrightarrow{\frac{q+1}{4q+2}} \text{R}(4q+1) \xrightarrow{\frac{q+2}{4q+5}} \text{L4} \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-3) \xrightarrow{\frac{q+2}{4q+5}} \text{L8} \xrightarrow{\frac{q+1}{4q+1}} \dots \xrightarrow{\frac{q+1}{4q+1}} \text{R5} \xrightarrow{\frac{q+2}{4q+5}} \text{L}(4q) \xrightarrow{\frac{q+1}{4q+1}} \text{R1} \\ & \xrightarrow{\frac{q}{4q-1}} \text{L}(4q-2) \xrightarrow{\frac{q+1}{4q+1}} \text{R3} \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-6) \xrightarrow{\frac{q+1}{4q+1}} \text{R7} \xrightarrow{\frac{q}{4q-3}} \dots \xrightarrow{\frac{q}{4q-3}} \text{L2} \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-1) \\ & \xrightarrow{\frac{q+1}{4q+2}} \text{L3} \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-2) \xrightarrow{\frac{q+2}{4q+5}} \text{L7} \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-6) \xrightarrow{\frac{q+2}{4q+5}} \dots \xrightarrow{\frac{q+2}{4q+5}} \text{L}(4q-1) \xrightarrow{\frac{q+1}{4q+1}} \text{R2}. \end{aligned}$$

By Lemma 7, f is a radio labeling for P_{8q+3}^4 . It follows by direct calculation that

$$f(x_{8q+3}) = (8q + 2)(k + 1) - \sum_{i=1}^{8q+2} d(x_i, x_{i+1}) = 2k^2 + 1.$$

Case 4: $\text{rn}(P_{8q+2}^4) \leq 2k^2$. Let $n = 8q + 2$ for some $q \in \mathbb{N}$. Then $k = \text{diam}(P_{8q+2}^4) = 2q + 1$. Similarly, we line up the vertices according to the display below.

$$\begin{aligned} & \text{R}0 \xrightarrow{\frac{q+1}{4q+1}} \text{L}(4q) \xrightarrow{\frac{q+2}{4q+5}} \text{R}4 \xrightarrow{\frac{q+1}{4q+1}} \text{L}(4q-4) \xrightarrow{\frac{q+2}{4q+5}} \dots \xrightarrow{\frac{q+1}{4q+1}} \text{L}4 \xrightarrow{\frac{q+2}{4q+5}} \text{R}(4q) \\ & \xrightarrow{\frac{q+1}{4q+2}} \text{L}1 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-1) \xrightarrow{\frac{q+2}{4q+5}} \text{L}5 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-5) \xrightarrow{\frac{q+2}{4q+5}} \dots \xrightarrow{\frac{q+2}{4q+5}} \text{L}(4q-3) \xrightarrow{\frac{q+1}{4q+1}} \text{R}3 \\ & \xrightarrow{\frac{q+1}{4q+2}} \text{L}(4q-2) \xrightarrow{\frac{q+1}{4q+1}} \text{R}2 \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-6) \xrightarrow{\frac{q+1}{4q+1}} \text{R}6 \xrightarrow{\frac{q}{4q-3}} \dots \xrightarrow{\frac{q}{4q-3}} \text{L}2 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-2) \\ & \xrightarrow{\frac{q+1}{4q+2}} \text{L}3 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-3) \xrightarrow{\frac{q+2}{4q+5}} \text{L}7 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-7) \xrightarrow{\frac{q+2}{4q+5}} \dots \xrightarrow{\frac{q+2}{4q+5}} \text{L}(4q-1) \xrightarrow{\frac{q+1}{4q+1}} \text{R}1 \xrightarrow{\frac{1}{2}} \text{L}0. \end{aligned}$$

By Lemma 7, f is a radio labeling for P_{8q+2}^4 . It follows by direct calculation that

$$f(x_{8q+2}) = (8q + 1)(k + 1) - \sum_{i=1}^{8q+1} d(x_i, x_{i+1}) = 2k^2.$$

Case 5: $\text{rn}(P_{8q+1}^4) \leq 2k^2 + q$. Let $n = 8q + 1$ for some $q \in \mathbb{N}$. Then $k = \text{diam}(P_{8q+1}^4) = 2q$. Similarly, we line up the vertices according to the display below.

$$\begin{aligned} & \text{C} \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-3) \xrightarrow{\frac{q+1}{4q+1}} \text{R}4 \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-7) \xrightarrow{\frac{q+1}{4q+1}} \dots \xrightarrow{\frac{q}{4q-3}} \text{L}1 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q) \\ & \xrightarrow{\frac{2q}{8q-2}} \text{L}(4q-2) \xrightarrow{\frac{q+1}{4q+1}} \text{R}3 \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-6) \xrightarrow{\frac{q+1}{4q+1}} \text{R}7 \xrightarrow{\frac{q}{4q-3}} \dots \xrightarrow{\frac{q}{4q-3}} \text{L}2 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-1) \\ & \xrightarrow{\frac{2q}{8q-2}} \text{L}(4q-1) \xrightarrow{\frac{q+1}{4q+1}} \text{R}2 \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-5) \xrightarrow{\frac{q+1}{4q+1}} \text{R}6 \xrightarrow{\frac{q}{4q-3}} \dots \xrightarrow{\frac{q}{4q-3}} \text{L}3 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-2) \\ & \xrightarrow{\frac{2q}{8q-2}} \text{L}(4q) \xrightarrow{\frac{q+1}{4q+1}} \text{R}1 \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-4) \xrightarrow{\frac{q+1}{4q+1}} \text{R}5 \xrightarrow{\frac{q}{4q-3}} \dots \xrightarrow{\frac{q}{4q-3}} \text{L}4 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-3). \end{aligned}$$

By Lemma 7, f is a radio labeling for P_{8q+1}^4 . It follows by direct calculation that

$$f(x_{8q+1}) = (8q)(k + 1) - \sum_{i=1}^{8q} d(x_i, x_{i+1}) = 2k^2 + q.$$

Case 6: $\text{rn}(P_{8q}^4) \leq 2k^2 + 1$. Let $n = 8q$ for some $q \in \mathbb{N}$. Then $k = \text{diam}(P_{8q}^4) = 2q$. Similarly, we line up the vertices according to the display below.

$$\begin{aligned} & \text{R}0 \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-4) \xrightarrow{\frac{q+1}{4q+1}} \text{R}4 \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-8) \xrightarrow{\frac{q+1}{4q+1}} \dots \xrightarrow{\frac{q}{4q-3}} \text{L}4 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-4) \\ & \xrightarrow{\frac{2q-1}{8q-6}} \text{L}(4q-3) \xrightarrow{\frac{q+1}{4q+1}} \text{R}3 \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-7) \xrightarrow{\frac{q+1}{4q+1}} \text{R}7 \xrightarrow{\frac{q}{4q-3}} \dots \xrightarrow{\frac{q}{4q-3}} \text{L}1 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-1) \\ & \xrightarrow{\frac{2q}{8q-2}} \text{L}(4q-2) \xrightarrow{\frac{q+1}{4q+1}} \text{R}2 \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-6) \xrightarrow{\frac{q+1}{4q+1}} \text{R}6 \xrightarrow{\frac{q}{4q-3}} \dots \xrightarrow{\frac{q}{4q-3}} \text{L}2 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-2) \\ & \xrightarrow{\frac{2q}{8q-2}} \text{L}(4q-1) \xrightarrow{\frac{q+1}{4q+1}} \text{R}1 \xrightarrow{\frac{q}{4q-3}} \text{L}(4q-5) \xrightarrow{\frac{q+1}{4q+1}} \text{R}5 \xrightarrow{\frac{q}{4q-3}} \dots \xrightarrow{\frac{q}{4q-3}} \text{L}3 \xrightarrow{\frac{q+1}{4q+1}} \text{R}(4q-3) \xrightarrow{\frac{q}{4q-2}} \text{L}0. \end{aligned}$$

By Lemma 7, f is a radio labeling for P_{8q}^4 . It follows by direct calculation that

$$f(x_{8q}) = (8q - 1)(k + 1) - \sum_{i=1}^{8q-1} d(x_i, x_{i+1}) = 2k^2 + 1.$$

Case 7: $\text{rn}(P_{8q-2}^4) \leq \text{rn}(P_{8q-1}^4) \leq 2k^2 + 1$. Since $k = \text{diam}(P_{8q-2}^4) = \text{diam}(P_{8q-1}^4) = \text{diam}(P_{8q}^4) = 2q$, using the same subgraph argument as in Case 2, we have that $\text{rn}(P_{8q-2}^4) \leq \text{rn}(P_{8q-1}^4) \leq \text{rn}(P_{8q}^4) \leq 2k^2 + 1$.

Cases 1–7, together with Lemmas 5 and 6, complete the proof of Theorem 1.

References

- [Chartrand, Erwin and Zhang 2005] G. Chartrand, D. Erwin, and P. Zhang, “A graph labeling problem suggested by FM channel restrictions”, *Bull. Inst. Combin. Appl.* **43** (2005), 43–57. MR 2005h:05175 Zbl 1066.05125
- [Chartrand et al. 2001] G. Chartrand, D. Erwin, F. Harary, and P. Zhang, “Radio labelings of graphs”, *Bull. Inst. Combin. Appl.* **33** (2001), 77–85. MR 2003d:05185 Zbl 0989.05102
- [Hale 1980] W. K. Hale, “Frequency assignment: theory and applications”, *Proc. IEEE* **68**:12 (1980), 1497–1514.
- [Liu 2008] D. D.-F. Liu, “Radio number for trees”, *Discrete Math.* **308**:7 (2008), 1153–1164. MR 2009h:05180 Zbl 1133.05090
- [Liu and Xie 2004] D. D.-F. Liu and M. Xie, “Radio number for square of cycles”, *Congr. Numer.* **169** (2004), 101–125. MR 2005m:05198 Zbl 1064.05089
- [Liu and Xie 2009] D. D.-F. Liu and M. Xie, “Radio number for square paths”, *Ars Combin.* **90** (2009), 307–319. MR 2010b:05141 Zbl 1224.05451
- [Liu and Zhu 2005] D. D.-F. Liu and X. Zhu, “Multilevel distance labelings for paths and cycles”, *SIAM J. Discrete Math.* **19**:3 (2005), 610–621. MR 2006i:05142 Zbl 1095.05033
- [Lo 2010] M.-L. Lo, “Radio number for cube paths”, unpublished manuscript, 2010.
- [Sooryanarayana et al. 2010] B. Sooryanarayana, M. Vishu Kumar, and K. Manjula, “Radio number of cube of a path”, *Int. J. Math. Comb.* **1** (2010), 5–29. MR 2662413 Zbl 1203.05136
- [Zhang 2002] P. Zhang, “Radio labelings of cycles”, *Ars Combin.* **65** (2002), 21–32. MR 2003i:05117 Zbl 1071.05573

Received: 2014-11-24 Revised: 2015-04-12 Accepted: 2015-04-12

mlo@csusb.edu

*Department of Mathematics, California State University,
San Bernardino, San Bernardino, CA 92407, United States*

linda.alegria05@gmail.com

*Department of Mathematics, California State University,
San Bernardino, San Bernardino, CA 92407, United States*

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	La Trobe University, Australia P.Cerone@latrobe.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsdam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION

Silvio Levy, Scientific Editor


Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2016 is US \$160/year for the electronic version, and \$215/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2016 Mathematical Sciences Publishers

involve

2016

vol. 9

no. 2

On the independence and domination numbers of replacement product graphs JAY CUMMINGS AND CHRISTINE A. KELLEY	181
An optional unrelated question RRT model JEONG S. SIHM, ANU CHHABRA AND SAT N. GUPTA	195
On counting limited outdegree grid digraphs and greatest increase grid digraphs JOSHUA CHESTER, LINNEA EDLIN, JONAH GALEOTA-SPRUNG, BRADLEY ISOM, ALEXANDER MOORE, VIRGINIA PERKINS, A. MALCOLM CAMPBELL, TODD T. ECKDAHL, LAURIE J. HEYER AND JEFFREY L. POET	211
Polygonal dissections and reversions of series ALISON SCHUETZ AND GWYN WHIELDON	223
Factor posets of frames and dual frames in finite dimensions KILEEN BERRY, MARTIN S. COPENHAVER, ERIC EVERT, YEON HYANG KIM, TROY KLINGLER, SIVARAM K. NARAYAN AND SON T. NGHIEM	237
A variation on the game SET DAVID CLARK, GEORGE FISK AND NURULLAH GOREN	249
The kernel of the matrix $[ij \pmod n]$ when n is prime MARIA I. BUENO, SUSANA FURTADO, JENNIFER KARKOSKA, KYANNE MAYFIELD, ROBERT SAMALIS AND ADAM TELATOVICH	265
Harnack's inequality for second order linear ordinary differential inequalities AHMED MOHAMMED AND HANNAH TURNER	281
The isoperimetric and Kazhdan constants associated to a Paley graph KEVIN CRAMER, MIKE KREBS, NICOLE SHABAZI, ANTHONY SHAHEEN AND EDWARD VOSKANIAN	293
Mutual estimates for the dyadic reverse Hölder and Muckenhoupt constants for the dyadically doubling weights OLEKSANDRA V. BEZNOVA AND TEMITOPE ODE	307
Radio number for fourth power paths MIN-LIN LO AND LINDA VICTORIA ALEGRIA	317
On closed graphs, II DAVID A. COX AND ANDREW ERSKINE	333
Klein links and related torus links ENRIQUE ALVARADO, STEVEN BERES, VESTA COUFAL, KAIA HLAVACEK, JOEL PEREIRA AND BRANDON REEVES	347



1944-4176(2016)9:2;1-3