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Radio number for fourth power paths

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Let G be a connected graph. For any two vertices u and v, let $d(u, v)$ denote the distance between u and v in G. The maximum distance between any pair of vertices of G is called the diameter of G and denoted by $\operatorname{diam}(G)$. A *radio labeling* (or multilevel distance labeling) of G is a function f that assigns to each vertex a label from the set $\{0, 1, 2, \dots\}$ such that the following holds for any vertices u and v: $|f(u) - f(v)| \geq \operatorname{diam}(G) - d(u, v) + 1$. The *span* of f is defined as $\max_{u,v \in V(G)}\{|f(u) - f(v)|\}$. The *radio number* of G is the minimum span over all radio labelings of G. The *fourth power* of G is a graph constructed from G by adding edges between vertices of distance four or less apart in G. In this paper, we completely determine the radio number for the fourth power of any path, except when its order is congruent to 1 (mod 8).

1. Introduction

Motivated by the *channel assignment problem* [Hale 1980] of dividing the radio broadcasting spectrum among radio stations in such a way that the interference caused by their proximity is minimized, radio labeling was introduced by Chartrand et al. [2001] to model the problem of finding the optimal distribution of channels using the smallest necessary range of frequencies.

Let G be a connected graph. For any two vertices u and v of G, the *distance* between u and v is the length of a shortest u-v path in G and is denoted by $d_G(u, v)$ or simply $d(u, v)$ if the graph G under consideration is clear. The *diameter* of G, denoted by $\operatorname{diam}(G)$, is the greatest distance between any two vertices of G. A *radio labeling* (or *multilevel distance labeling* [Liu 2008; Liu and Zhu 2005]) of a connected graph G is a function $f : V(G) \to \{0, 1, 2, 3, \dots\}$ with the property that

$$|f(u) - f(v)| \geq \operatorname{diam}(G) + 1 - d(u, v)$$

for every two distinct vertices u and v of G. The *span* of f is defined as

$$\max_{u,v \in V(G)}\{|f(u) - f(v)|\}.$$

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The radio number of G, denoted by $\mathrm{rn}(G)$, is defined as

$$\min\{\text{span of } f : f \text{ is a radio labeling of } G\}.$$

A radio labeling for G with span equal to $\mathrm{rn}(G)$ is called an *optimal radio labeling*.

Finding the radio number for a graph is an interesting yet challenging task. So far the value is known only for very limited families of graphs. The radio numbers for paths and cycles were investigated in [Chartrand et al. 2001; Chartrand, Erwin and Zhang 2005; Zhang 2002]and were completely solved by Liu and Zhu [2005]. The radio number for trees was investigated in [Liu 2008].

The r-th power of a graph G, denoted by G^r, is the graph constructed from G by adding edges between vertices of distance r or less apart in G. The radio number for the square of a path on n vertices, denoted by P_n^2, was completely determined by Liu and Xie [2009], who also partially solved the problem for the square of a cycle on n vertices, denoted by C_n^2 [2004]. Motivated by [Liu and Xie 2009], Lo [2010] and Sooryanarayana et al. [2010] determined $\mathrm{rn}(P_n^3)$.

This paper will follow the structure in [Liu and Xie 2009] closely to determine the radio number of the fourth power of paths (or simply, fourth power paths). It is our hope that this paper will be helpful for those readers who wish to pursue finding the radio number for P_n^5, P_n^6, and eventually P_n^r for any positive integer r.

Theorem 1. *Let P_n^4 be a fourth power path on n vertices where $n \geq 6$ and let $k = \mathrm{diam}(P_n^4) = \left\lceil \frac{1}{4}(n-1) \right\rceil$. Then*

$$\mathrm{rn}(P_n^4) = \begin{cases} 2k^2+1 & \text{if } n \equiv 0, 3, 6, \text{ or } 7 \pmod 8 \text{ or } n = 9, \\ 2k^2+2 & \text{if } n \equiv 4 \text{ or } 5 \pmod 8, \\ 2k^2 & \text{if } n \equiv 2 \pmod 8. \end{cases}$$

If $n \equiv 1 \pmod 8$ and $n \geq 17$ (where n is of the form $8q+1$), then

$$2k^2 + 2 \leq \mathrm{rn}(P_{8q+1}^4) \leq 2k^2 + q.$$

2. General properties and notation

The diameter of P_n^4 is $\left\lceil \frac{1}{4}(n-1) \right\rceil$, based on the definition of P_n^4. Figure 1 shows P_8^4.

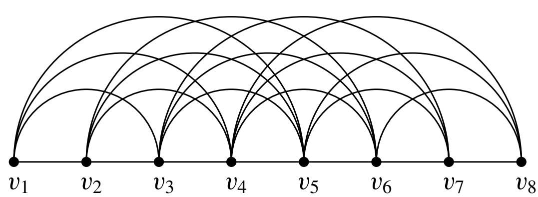

Figure 1. A fourth power path on 8 vertices, denoted by P_8^4.

Proposition 2. *For any $u, v \in V(P_n^4)$, we have*

$$d(u,v) = \left\lceil \tfrac{1}{4} d_{P_n}(u,v) \right\rceil.$$

A *center* of P_n is defined as a "middle" vertex of P_n. An odd path P_{2m+1} has only one center v_{m+1}, while an even path P_{2m} has two centers v_m and v_{m+1}. For each vertex $u \in V(P_n)$, the *level* of u, denoted by $L(u)$ is the smallest distance in P_n from u to a center of P_n. If we denote the levels of a sequence of vertices A by $L(A)$, we have

$$n = 2m+1 \quad \Rightarrow \quad L(v_1, v_2, \ldots, v_{2m+1}) = (m, m-1, \ldots, 2, 1, 0, 1, 2, \ldots, m-1, m),$$
$$n = 2m \quad \Rightarrow \quad L(v_1, v_2, \ldots, v_{2m}) = (m-1, \ldots, 2, 1, 0, 0, 1, 2, \ldots, m-1).$$

Define the *left-vertices* and *right-vertices* as follows:

If $n = 2m+1$, then the left-vertices and right-vertices respectively are

$$\{v_1, v_2, \ldots, v_m, v_{m+1}\} \quad \text{and} \quad \{v_{m+1}, v_{m+2}, \ldots, v_{2m}, v_{2m+1}\}.$$

In this case, the center v_{m+1} is both a left-vertex and a right-vertex.

If $n = 2m$, then the left-vertices and right-vertices respectively are

$$\{v_1, v_2, \ldots, v_m\} \quad \text{and} \quad \{v_{m+1}, v_{m+2}, \ldots, v_{2m}\}.$$

If two vertices are both right-vertices or left-vertices, then we say that they are on the *same side*; otherwise, they are on *opposite sides*.

Lemma 3. *If n is odd, then for any $u, v \in V(P_n^4)$, we have*

$$d(u,v) = \begin{cases} \left\lceil \tfrac{1}{4}(L(u) + L(v)) \right\rceil & \text{if } u \text{ and } v \text{ are on opposite sides,} \\ \left\lceil \tfrac{1}{4}|L(u) - L(v)| \right\rceil & \text{if } u \text{ and } v \text{ are on the same side.} \end{cases}$$

If n is even, then for any $u, v \in V(P_n^4)$, we have

$$d(u,v) = \begin{cases} \left\lceil \tfrac{1}{4}(L(u) + L(v) + 1) \right\rceil & \text{if } u \text{ and } v \text{ are on opposite sides,} \\ \left\lceil \tfrac{1}{4}|L(u) - L(v)| \right\rceil & \text{if } u \text{ and } v \text{ are on the same side.} \end{cases}$$

In the proof of Lemma 7 below, the following proposition will be used frequently:

Proposition 4. *For any d_1, d_2 in $\mathbb{N}$, we have*

$$\left\lceil \frac{d_1+d_2}{r} \right\rceil = \begin{cases} \lceil d_1/r \rceil + \lceil d_2/r \rceil - 1 & \text{if } (d_1,d_2) \equiv (l,m) \pmod r, \text{ where } l \neq 0, m \neq 0, \\ & \text{and } 2 \leq (d_1+d_2) \pmod r \leq r, \\ \lceil d_1/r \rceil + \lceil d_2/r \rceil & \text{otherwise,} \end{cases}$$

$$\left\lceil \frac{d_1-d_2}{r} \right\rceil = \begin{cases} \lceil d_1/r \rceil - \lceil d_2/r \rceil + 1 & \text{if } (d_1,d_2) \equiv (0,m) \pmod r, \text{ where } m \neq 0, \\ & \text{or } (d_1,d_2) \equiv (l,m) \pmod r, \text{ where } l \neq 0, m \neq 0, \\ & \text{and } 1 \leq (d_1-d_2) \pmod r \leq (r-2), \\ \lceil d_1/r \rceil - \lceil d_2/r \rceil & \text{otherwise.} \end{cases}$$

It is important for the reader to understand the notation used in the labeling of P_n^4 so we will define a few terms and notation first.

Let $M, N \in \mathbb{N}$. We define a *block* (M, N) to be a pattern to follow when consecutively labeling a certain group of vertices in P_n^r. Take an (M, N)-block for example: The first vertex labeled, x_i, will have $L(x_i) \equiv M \pmod r$. The next vertex labeled, x_{i+1}, will have $L(x_{i+1}) \equiv N \pmod r$. The following vertex labeled, x_{i+2}, will have $L(x_{i+2}) \equiv M \pmod r$. Continue in this fashion until we end at a vertex of level congruent to $N \pmod r$. We may also choose to specify what side the vertex is on by writing $(\mathrm{L}M, \mathrm{R}N)$. This would mean that the first vertex labeled, x_i, would be a left-vertex with $L(x_i) \equiv M \pmod r$, and x_{i+1} would be a right-vertex with $L(x_{i+1}) \equiv N \pmod r$, so on and so forth.

We say that a *disconnection* occurs when $L(x_i)+L(x_{i+1})$ is not congruent to said specified value modulo r that maximizes the distance between two consecutively labeled vertices. This specific value changes depending upon the parity of n for P_n^4.

A *labeling pattern* is a specific arrangement of blocks. Note that the same block may appear multiple times in a labeling pattern; however, the number of vertices in each "identical" block may be different. For any labeling pattern, P_n^4 will be said to have an "even" pairing if, for each (M, N)-block in the labeling pattern, the number of vertices with level congruent to $M \pmod r$ on one side equals the number of vertices with level congruent to $N \pmod r$ on the other side. Otherwise, P_n^4 will be said to have "extra" vertices.

3. Lower bound of $\mathrm{rn}(P_n^4)$ when n is even

Lemma 5. *Let P_n^4 be a fourth power path on n vertices, where $n \geq 6$, and let $k = \mathrm{diam}(P_n^4) = \lceil \frac{1}{4}(n-1) \rceil$. If n is even, then*

$$\mathrm{rn}(P_n^4) \geq \begin{cases} 2k^2+1 & \text{if } n \equiv 0 \text{ or } 6 \pmod 8, \\ 2k^2 & \text{if } n \equiv 2 \pmod 8, \\ 2k^2+2 & \text{if } n \equiv 4 \pmod 8. \end{cases}$$

Proof. Let f be a radio labeling for P_n^4. Rearrange $V(P_n^4) = \{x_1, x_2, \ldots, x_n\}$ so that $0 = f(x_1) < f(x_2) < f(x_3) < \cdots < f(x_n)$. Note that $f(x_n)$ is the span of f. By definition, $f(x_{i+1}) - f(x_i) \geq k+1-d(x_i, x_{i+1})$ for $1 \leq i \leq n-1$. Summing up these $n-1$ inequalities, we have

$$f(x_n) \geq (n-1)(k+1) - \sum_{i=1}^{n-1} d(x_i, x_{i+1}). \tag{3-1}$$

Thus to minimize $f(x_n)$, it suffices to maximize $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$. Since n is even,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \left\lceil \tfrac{1}{4}(L(x_i)+L(x_{i+1})+1) \right\rceil.$$

Observe, from the above inequality we have:

(1) For each i, the equality for $d(x_i, x_{i+1}) \le \lceil \frac{1}{4}(L(x_i) + L(x_{i+1}) + 1) \rceil$ holds when x_i and x_{i+1} are on opposite sides, or when they are on the same side but one of them is a center and the other vertex is of level not congruent to $0 \pmod 4$.

(2) In the summation $\sum_{i=1}^{n-1} \lceil \frac{1}{4}(L(x_i) + L(x_{i+1}) + 1) \rceil$, each vertex of P_n^4 occurs exactly twice, except for x_1 and x_n, which both occur only once.

By direct calculation, we have

$$\left\lceil \tfrac{1}{4}(L(u)+L(v)+1) \right\rceil = \begin{cases} \frac{1}{4}(L(u)+L(v)+4) & \text{if } L(u)+L(v) \equiv 0 \pmod 4, \\ \frac{1}{4}(L(u)+L(v)+4) - \frac{1}{4} & \text{if } L(u)+L(v) \equiv 1 \pmod 4, \\ \frac{1}{4}(L(u)+L(v)+4) - \frac{2}{4} & \text{if } L(u)+L(v) \equiv 2 \pmod 4, \\ \frac{1}{4}(L(u)+L(v)+4) - \frac{3}{4} & \text{if } L(u)+L(v) \equiv 3 \pmod 4. \end{cases}$$

Therefore,

$$\left\lceil \tfrac{1}{4}(L(x_i) + L(x_{i+1}) + 1) \right\rceil \le \tfrac{1}{4}(L(x_i) + L(x_{i+1}) + 4),$$

and the equality holds only if $L(x_i) + L(x_{i+1}) \equiv 0 \pmod 4$. Combining this with (1) above, there exist at most $n-4$ of the i such that $d(x_i, x_{i+1}) = \frac{1}{4}(L(x_i)+L(x_{i+1})+4)$; that is, there are at least three disconnections in the labeling. Note that when $L(x_i) + L(x_{i+1}) \equiv 1, 2,$ or $3 \pmod 4$, we say that there is a disconnection between x_i and x_{i+1} of the *best type*, *second best type*, or the *worst type*, respectively. Moreover, among all the vertices, only the centers are of level zero. Hence, $L(x_1) + L(x_n) \ge 0 + 0 = 0$. We conclude that

$$\begin{aligned} \sum_{i=1}^{n-1} d(x_i, x_{i+1}) &\le \left(\sum_{i=1}^{n-1} \tfrac{1}{4}(L(x_i) + L(x_{i+1}) + 4) \right) - \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} \\ &= \tfrac{1}{4}\left(\left(2 \sum_{i=1}^{n} L(x_i) \right) - L(x_1) - L(x_n) \right) + (n-1) - \tfrac{3}{4} \\ &\le \tfrac{1}{4}\left(\left(2 \sum_{i=1}^{n} L(x_i) \right) - 0 - 0 \right) + (n-1) - \tfrac{3}{4} \\ &= \tfrac{1}{2}\left(2\left(0 + 1 + 2 + \cdots + \left(\tfrac{1}{2}n - 1\right)\right)\right) + n - \tfrac{7}{4} \\ &= \tfrac{1}{8}n^2 + \tfrac{3}{4}n - \tfrac{7}{4}. \end{aligned}$$

By direct calculation for (3-1) and considering that $\mathrm{rn}(P_n^4)$ is an integer, we have

$$\mathrm{rn}(P_n^4) \ge \begin{cases} \lceil 2k^2 + \frac{3}{4} \rceil = 2k^2+1 & \text{if } n \equiv 0 \pmod 8 \text{ (i.e., } n = 4k \text{ and } k \text{ is even),} \\ \lceil 2k^2 - \frac{1}{4} \rceil = 2k^2 & \text{if } n \equiv 2 \pmod 8 \text{ (i.e., } n = 4k-2 \text{ and } k \text{ is odd),} \\ \lceil 2k^2 + \frac{3}{4} \rceil = 2k^2+1 & \text{if } n \equiv 4 \pmod 8 \text{ (i.e., } n = 4k \text{ and } k \text{ is odd),} \\ \lceil 2k^2 - \frac{1}{4} \rceil = 2k^2 & \text{if } n \equiv 6 \pmod 8 \text{ (i.e., } n = 4k-2 \text{ and } k \text{ is even).} \end{cases}$$

Further investigation for a sharper lower bound of $\mathrm{rn}(P_n^4)$ when $n \equiv 4$ or $6 \pmod 8$ is needed. There are three cases to consider based on the number of disconnections that occur in the labeling pattern.

Case 1: There are at least five disconnections. Then we have,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \le \left(\sum_{i=1}^{n-1} \tfrac{1}{4}(L(x_i) + L(x_{i+1}) + 4) \right) - \tfrac{\mathbf{5}}{4} \le \tfrac{1}{8}n^2 + \tfrac{3}{4}n - \tfrac{9}{4}.$$

Hence, by direct calculation for (3-1) we have

$$\mathrm{rn}(P_n^4) \ge \begin{cases} \left\lceil \left(2k^2 + \frac{3}{4}\right) + \frac{\mathbf{2}}{\mathbf{4}} \right\rceil = 2k^2+2 & \text{if } n \equiv 4 \pmod 8 \text{ (i.e., } n = 4k \text{ and } k \text{ is odd)}, \\ \left\lceil \left(2k^2 - \frac{1}{4}\right) + \frac{\mathbf{2}}{\mathbf{4}} \right\rceil = 2k^2+1 & \text{if } n \equiv 6 \pmod 8 \text{ (i.e., } n = 4k-2 \text{ and } k \text{ is even)}. \end{cases}$$

Case 2: There are exactly four disconnections. This case will be broken down into two subcases based on $L(x_1) + L(x_n)$.

Case 2.1: $L(x_1) + L(x_n) \ge 1$. Therefore,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \le \left(\sum_{i=1}^{n-1} \tfrac{1}{4}(L(x_i) + L(x_{i+1}) + 4) \right) - \tfrac{\mathbf{4}}{4} \le \tfrac{1}{8}n^2 + \tfrac{3}{4}n - \tfrac{9}{4}.$$

Case 2.2: $L(x_1) + L(x_n) = 0$.

Claim. *In this case, at least two of the disconnections that occur cannot be of the best type.*

Proof of claim. For $n \equiv 4$ or $6 \pmod 8$, we have the following types of blocks as well as extra vertices (without loss of generality, we start each block with a left-vertex):

$$(\mathrm{L0}, \mathrm{R0}), \quad (\mathrm{L1}, \mathrm{R3}), \quad (\mathrm{L2}, \mathrm{R2}), \quad (\mathrm{L3}, \mathrm{R1}) \quad \mathrm{L1}, \quad \mathrm{R1}.$$

We wish to have exactly four disconnections and we also want $L(x_1) + L(x_n) = 0 + 0 = 0$ under this case. Therefore we must use two (L0, R0)-blocks. Thus our new blocks become (blocks are boxed for easy identification of disconnections that occur in the labeling pattern):

$$\boxed{(\mathrm{L0}, \mathrm{R0})}, \quad \boxed{(\mathrm{L1}, \mathrm{R3}) - \mathrm{L1}}, \quad \boxed{(\mathrm{L2}, \mathrm{R2})}, \quad \boxed{\mathrm{R1} - (\mathrm{L3}, \mathrm{R1})}, \quad \boxed{(\mathrm{L0}, \mathrm{R0})}.$$

Since we want $L(x_1) + L(x_n) = 0 + 0 = 0$, our labeling pattern must start and end with the (L0, R0)-blocks. Special attention is given to the "end-1" vertices, namely, the first and the last vertices of the two block patterns (L1, R3) − L1 and R1 − (L3, R1) from above. All disconnections in the labeling pattern will occur at these four end-1 vertices. The best type of disconnection would occur if an end-1 vertex was followed or preceded by a vertex whose level was congruent to $0 \pmod 4$. However, there are only two such vertices available. Therefore, at least two of the four end-1 vertices cannot have disconnections of the best type. □

By direct calculation, our claim, and the assumption that $L(x_1)+L(x_n)=0$, we have,

$$\sum_{i=1}^{n-1} d(x_i,x_{i+1}) \le \left(\sum_{i=1}^{n-1}\tfrac{1}{4}(L(x_i)+L(x_{i+1})+4)\right)-\tfrac{\mathbf{6}}{4}=\tfrac{1}{8}n^2+\tfrac{3}{4}n-\tfrac{10}{4}.$$

Hence, by direct calculation for (3-1) for the two subcases, the same bounds as in the conclusion of Case 1 are obtained.

Case 3: There are exactly three disconnections.

Claim. *In this case, at least one of the disconnections in the labeling pattern will not be of the best type.*

Proof of claim. Similar to Case 2.2, to ensure that there are only three disconnections, our new blocks must be

$$\boxed{(\mathrm{L0},\mathrm{R0})},\quad \boxed{(\mathrm{L1},\mathrm{R3})-\mathrm{L1}},\quad \boxed{(\mathrm{L2},\mathrm{R2})},\quad \boxed{\mathrm{R1}-(\mathrm{L3},\mathrm{R1})}.$$

Thus, out of the three disconnections that occur, at least two of them will occur at the end-1 vertices. Furthermore, out of the disconnections that occur at the end-1 vertices, at least one of them will not be of the best type, unless two (L0, R0)-blocks are used, which would increase the number of disconnections. □

By calculation, our claim, and noting that $L(x_1)+L(x_n)\ge 1$ under this case, we have,

$$\sum_{i=1}^{n-1} d(x_i,x_{i+1}) \le \left(\sum_{i=1}^{n-1}\tfrac{1}{4}(L(x_i)+L(x_{i+1})+4)\right)-\tfrac{\mathbf{4}}{4}\le\tfrac{1}{8}n^2+\tfrac{3}{4}n-\tfrac{9}{4}.$$

Direct calculation for (3-1) in this case also leads to the same bounds as in the conclusion of Case 1. □

4. Lower bound of $\mathrm{rn}(P_n^4)$ when n is odd

Lemma 6. *Let P_n^4 be a fourth power path on n vertices, where $n\ge 6$, and let $k=\mathrm{diam}(P_n^4)=\lceil\frac{1}{4}(n-1)\rceil$. If n is odd, then*

$$\mathrm{rn}(P_n^4)\ge\begin{cases}2k^2+2 & \text{if } n\equiv 1 \pmod 8 \text{ and } n\ge 17 \text{ or } n\equiv 5 \pmod 8,\\ 2k^2+1 & \text{if } n\equiv 3 \text{ or } 7 \pmod 8 \text{ or } n=9.\end{cases}$$

Proof. We retain the same notation and employ the same method used in the proof of Lemma 5. Since n is odd,

$$\sum_{i=1}^{n-1} d(x_i,x_{i+1}) \le \sum_{i=1}^{n-1}\left\lceil\tfrac{1}{4}(L(x_i)+L(x_{i+1}))\right\rceil.$$

Observe, from the above inequality we have:

(1) For each i, the equality for $d(x_i, x_{i+1}) \le \lceil \frac{1}{4}(L(x_i) + L(x_{i+1})) \rceil$ holds only when x_i and x_{i+1} are on opposite sides, unless one of them is a center.

(2) In the summation $\sum_{i=1}^{n-1} \lceil \frac{1}{4}(L(x_i)+L(x_{i+1})) \rceil$, each vertex of P_n^4 occurs exactly twice, except x_1 and x_n, which each occurs only once.

By direct calculation, we have

$$\left\lceil \tfrac{1}{4}(L(u)+L(v)) \right\rceil = \begin{cases} \frac{1}{4}(L(u)+L(v)+3) - \frac{3}{4} & \text{if } L(u)+L(v) \equiv 0 \pmod 4, \\ \frac{1}{4}(L(u)+L(v)+3) & \text{if } L(u)+L(v) \equiv 1 \pmod 4, \\ \frac{1}{4}(L(u)+L(v)+3) - \frac{1}{4} & \text{if } L(u)+L(v) \equiv 2 \pmod 4, \\ \frac{1}{4}(L(u)+L(v)+3) - \frac{2}{4} & \text{if } L(u)+L(v) \equiv 3 \pmod 4. \end{cases}$$

Therefore

$$\left\lceil \tfrac{1}{4}(L(x_i)+L(x_{i+1})) \right\rceil \le \tfrac{1}{4}(L(x_i)+L(x_{i+1})+3),$$

and the equality holds only if $L(x_i) + L(x_{i+1}) \equiv 1 \pmod 4$. Note that when $L(x_i)+L(x_{i+1}) \equiv 2, 3$, or $0 \pmod 4$, we say that there is a disconnection between x_i and x_{i+1} of the *best type*, *second best type*, or the *worst type*, respectively. Combining this with (1), there are two possible cases to consider based on the number of disconnections in the labeling pattern:

Case 1: There are at least three disconnections. In this case, since n is odd, there is only one center. Therefore, $L(x_1)+L(x_n) \ge 1$. Then,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \le \left(\sum_{i=1}^{n-1} \tfrac{1}{4}(L(x_i)+L(x_{i+1})+3) \right) - \tfrac{\mathbf{3}}{\mathbf{4}} \le \tfrac{1}{8}n^2 + \tfrac{3}{4}n - \tfrac{15}{8}.$$

By direct calculation for (3-1), we have

$$\operatorname{rn}(P_n^4) \ge \begin{cases} 2k^2+1 & \text{if } n \equiv 1 \pmod 8 \ \text{(i.e., } n=4k+1 \text{ and } k \text{ is even)}, \\ \lceil 2k^2+\frac{1}{2} \rceil = 2k^2+1 & \text{if } n \equiv 3 \pmod 8 \ \text{(i.e., } n=4k-1 \text{ and } k \text{ is odd)}, \\ 2k^2+1 & \text{if } n \equiv 5 \pmod 8 \ \text{(i.e., } n=4k+1 \text{ and } k \text{ is odd)}, \\ \lceil 2k^2+\frac{1}{2} \rceil = 2k^2+1 & \text{if } n \equiv 7 \pmod 8 \ \text{(i.e., } n=4k-1 \text{ and } k \text{ is even)}. \end{cases}$$

Case 2: There are exactly two disconnections. In this case, neither x_1 nor x_n is the center (denoted by C).

Case 2.1: $n \equiv 1 \pmod 8$. The labeling pattern must be a permutation of the boxed blocks

$$\boxed{\text{(L0, R1)} - \text{C} - \text{(L1, R0)}}, \quad \boxed{\text{(L2,R3)}}, \quad \boxed{\text{(L3,R2)}}.$$

Therefore, $L(x_1)+L(x_n) \ge 4$. By similar calculations to Case 1, we have

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \le \left(\sum_{i=1}^{n-1} \tfrac{1}{4}(L(x_i)+L(x_{i+1})+3) \right) - \tfrac{\mathbf{2}}{\mathbf{4}} \le \tfrac{1}{8}n^2 + \tfrac{3}{4}n - \tfrac{19}{8}.$$

By direct calculations, since $n = 4k + 1$ and k is even, we have

$$\mathrm{rn}(P_n^4) \geq \left\lceil (2k^2+1) + \tfrac{2}{4} \right\rceil = 2k^2 + 2.$$

Case 2.2: $n \equiv 3, 5,$ or $7 \pmod 8$. Note that P_{8q+3}^4 and P_{8q+7}^4 both have an extra pair of vertices whose level is congruent to $1 \pmod 4$. Therefore, the labeling pattern must be a permutation of the boxed blocks

$$\boxed{\mathrm{R1} - (\mathrm{L0}, \mathrm{R1}) - \mathrm{C} - (\mathrm{L1}, \mathrm{R0}) - \mathrm{L1}}, \quad \boxed{(\mathrm{L2}, \mathrm{R3})}, \quad \boxed{(\mathrm{L3}, \mathrm{R2})}.$$

Now, P_{8q+5}^4 has two extra pairs of vertices whose levels are congruent to $1 \pmod 4$ and $2 \pmod 4$. The labeling pattern must be a permutation of the boxed blocks

$$\boxed{\mathrm{R1} - (\mathrm{L0}, \mathrm{R1}) - \mathrm{C} - (\mathrm{L1}, \mathrm{R0}) - \mathrm{L1}}, \quad \boxed{(\mathrm{L2}, \mathrm{R3}) - \mathrm{L2}}, \quad \boxed{\mathrm{R2} - (\mathrm{L3}, \mathrm{R2})}.$$

Therefore, for $n \equiv 3, 5,$ or $7 \pmod 8$, considering all possible permutations mentioned above, $L(x_1) + L(x_n) \geq 3$. Therefore,

$$\sum_{i=1}^{n-1} d(x_i, x_{i+1}) \leq \left(\sum_{i=1}^{n-1} \tfrac{1}{4}(L(x_i) + L(x_{i+1}) + 3) \right) - \tfrac{2}{4} \leq \tfrac{1}{8}n^2 + \tfrac{3}{4}n - \tfrac{17}{8}.$$

Thus, by direct calculation we have,

$$\mathrm{rn}(P_n^4) \geq \begin{cases} \left\lceil \left(2k^2+\frac{1}{2}\right)+\frac{1}{4} \right\rceil = 2k^2+1 & \text{if } n \equiv 3 \pmod 8 \ \text{(i.e., } n = 4k-1 \text{ and } k \text{ is odd)}, \\ \left\lceil (2k^2+1)+\frac{1}{4} \right\rceil = 2k^2+2 & \text{if } n \equiv 5 \pmod 8 \ \text{(i.e., } n = 4k+1 \text{ and } k \text{ is odd)}, \\ \left\lceil \left(2k^2+\frac{1}{2}\right)+\frac{1}{4} \right\rceil = 2k^2+1 & \text{if } n \equiv 7 \pmod 8 \ \text{(i.e., } n = 4k-1 \text{ and } k \text{ is even)}. \end{cases}$$

Now assume $n \equiv 1 \pmod 8$ and $n \geq 17$; that is, $n = 4k + 1$, k is even and $k \geq 4$. Assume to the contrary that $f(x_n) = 2k^2 + 1$. Then only Case 1 is possible and all of the following must hold:

(1) $\{x_1, x_n\} = \{v_{2k+1}, v_{2k+2}\}$ or $\{v_{2k+1}, v_{2k}\}$. That is, $\{x_1, x_n\}$ is of the form $\{x_1, x_n\} = \{$center, a vertex right next to center$\}$.

(2) $f(x_{i+1}) = f(x_i) + k + 1 - d(x_i, x_{i+1})$ for all i.

(3) For all $i \geq 1$, the two vertices x_i and x_{i+1} are on opposites sides unless one of them is the center.

(4) There exist three t-values, $1 \leq t \leq n-1$, such that $L(x_t) + L(x_{t+1}) \equiv 2 \pmod 4$ while $L(x_t) + L(x_{t+1}) \equiv 1 \pmod 4$ for all other $i \neq t$.

By (1) and by symmetry, we can assume that $x_1 = v_{2k+1}$; i.e., x_1 is the center. Excluding the center, there are $\frac{1}{2}k$ vertices whose level is congruent to $0 \pmod 4$, $1 \pmod 4$, $2 \pmod 4$, and $3 \pmod 4$ on each side, respectively. Since x_n is of level one, by (2), (3), and (4) we have:

(5) The labeling pattern must be the arrangement of boxed blocks

$$\boxed{\mathrm{C}-(1,0)}-\boxed{(2,3)}-\boxed{(3,2)}-\boxed{(0,1)}.$$

Claim. $\{v_1, v_n\} = \{x_{k+1}, x_{3k+2}\}$ (*i.e.*, $\{v_1, v_n\}$ *consists of the last vertex whose level is congruent to* $0 \pmod 4$ *in the* $(1, 0)$*-block and the first vertex whose level is congruent to* $0 \pmod 4$ *in the* $(0, 1)$*-block*).

Proof of claim. Suppose $v_1 \notin \{x_{k+1}, x_{3k+2}\}$. Then v_1 is inside one of the (0, 1)- or (1, 0)-blocks, since $L(v_1) = 2k \equiv 0 \pmod 4$. Let $v_1 = x_c$ for some c, where x_{c-1} and x_{c+1} are both vertices on the right side. Thus, $L(x_{c-1}) \equiv L(x_{c+1}) \equiv 1 \pmod 4$. Let $L(x_{c-1}) = y$ and $L(x_{c+1}) = z$. By (2),

$$f(x_c) - f(x_{c-1}) = \tfrac{1}{2}k + 1 - \lceil \tfrac{1}{4}y \rceil,$$
$$f(x_{c+1}) - f(x_c) = \tfrac{1}{2}k + 1 - \lceil \tfrac{1}{4}z \rceil.$$

Therefore,

$$f(x_{c+1}) - f(x_{c-1}) = k + 2 - \lceil \tfrac{1}{4}y \rceil - \lceil \tfrac{1}{4}z \rceil,$$

contradicting that

$$f(x_{c+1}) - f(x_{c-1}) \geq k + 1 - \lceil \tfrac{1}{4}|z - y| \rceil \quad (\text{as } y \equiv z \equiv 1 \pmod 4, \text{ so } y, z \neq 0).$$

Therefore $v_1 \in \{x_{k+1}, x_{3k+2}\}$. Similarly, we can show that $v_n \in \{x_{k+1}, x_{3k+2}\}$. □

By the claim, we may assume that $v_n = x_{k+1}$ and $v_1 = x_{3k+2}$ (the proof for the other case is symmetric). By (5), $L(x_k) = a \equiv 1 \pmod 4$ and $L(x_{k+2}) = b \equiv 2 \pmod 4$. By (2), (3), the fact that k is even, and our assumption that $L(x_{k+1}) = L(v_n) = L(v_{4k+1}) = 2k$, we have

$$f(x_{k+1}) - f(x_k) = \tfrac{1}{2}k + 1 - \lceil \tfrac{1}{4}a \rceil,$$
$$f(x_{k+2}) - f(x_{k+1}) = \tfrac{1}{2}k + 1 - \lceil \tfrac{1}{4}b \rceil,$$

and so,

$$f(x_{k+2}) - f(x_k) = k + 2 - \lceil \tfrac{1}{4}a \rceil - \lceil \tfrac{1}{4}b \rceil.$$

By definition and by Lemma 3,

$$f(x_{k+2}) - f(x_k) \geq k + 1 - \lceil \tfrac{1}{4}|a - b| \rceil.$$

Therefore, a must equal 1. Thus $L(x_k) = 1$, which means x_k is the level-one vertex on the left side, since $x_{k+1} = v_n$ is a right-vertex. Thus $x_k = v_{2k}$. Similarly, we can show that x_{3k+3} is of level one and on the right side. Thus, $x_{3k+3} = v_{2k+2}$.

Now, x_n is a right-vertex since $x_{3k+2} = v_1$ is a left-vertex, and so $x_n = v_{2k+2}$. This implies that $x_n = v_{2k+2} = x_{3k+3}$ and therefore $k = 2$, contradicting the assumption $k \geq 4$. Therefore $\mathrm{rn}(P_n^4) \geq 2k^2 + 2$ if $n \equiv 1 \pmod 8$ and $n \geq 17$.

Similar techniques can be applied for the case $n \equiv 5 \pmod 8$. Assume that $n \equiv 5 \pmod 8$ and $n \geq 21$; that is, $n = 4k + 1$, k is odd, and $k \geq 5$. Assume to the contrary that $f(x_n) = 2k^2 + 1$. Then only Case 1 is possible and the same requirements (1), (2), (3), and (4) for the case $n = 1 \pmod 8$ and $n \geq 17$ must hold.

By (1) and by symmetry, we can assume that $x_1 = v_{2k+1}$; i.e., x_1 is the center. Excluding the center, there are $\frac{1}{2}(k-1)$ vertices whose level is congruent to $0 \pmod 4$, $\frac{1}{2}(k+1)$ vertices whose level is congruent to $1 \pmod 4$, $\frac{1}{2}(k+1)$ vertices whose level is congruent to $2 \pmod 4$, and $\frac{1}{4}(k-1)$ vertices whose level is congruent to $3 \pmod 4$, on each side. By (1), (2), (3), and the second part of (4), the labeling pattern must be the arrangement of boxed blocks

$$\boxed{\mathrm{C}-(1-0-1)}-\boxed{(2-3-2)}-\boxed{(2-3-2)}-\boxed{(1-0-1)}.$$

However, in this arrangement the three t-values for which $L(x_t) + L(x_{t+1})$ is not congruent to $1 \pmod 4$ are not all congruent to $2 \pmod 4$, which contradicts the first part of (4). Therefore, $\mathrm{rn}(P_n^4) \geq 2k^2 + 2$. □

5. Upper bound and optimal radio labelings

To establish Theorem 1, it suffices to give radio labelings achieving the desired spans. To this end, we will use the next lemma, which provides us with an easy way to verify that a given labeling of P_n^r is indeed a radio labeling of P_n^r.

Lemma 7. *Let P_n^r be an r-th power path graph on n vertices, where $k = \mathrm{diam}(P_n^r) = \lceil \frac{1}{r}(n-1) \rceil$. Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be a permutation of $V(P_n^r)$ such that for any $1 \leq i \leq n-2$,*

$$\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq \tfrac{1}{2}rk + 1$$

and $\max\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \not\equiv 1 \pmod r$ if k is even and the equality in the above holds. Let f be a function, $f : V(P_n^r) \longrightarrow \{0, 1, 2, \ldots\}$ with $f(x_1) = 0$ and $f(x_{i+1}) - f(x_i) = k + 1 - d(x_i, x_{i+1})$ for all $1 \leq i \leq n-1$. Then f is a radio labeling for P_n^r.

Before we present the proof of Lemma 7, note that Proposition 4 will be used frequently throughout the proof of Lemma 7 below. The construction of this proof is adapted from [Liu and Xie 2009].

Proof. Let f be a function satisfying the assumption. It suffices to prove that $f(x_j) - f(x_i) \geq k + 1 - d(x_i, x_j)$ for any $j \geq i + 2$. For $i = 1, 2, \ldots, n-1$, set

$$f_i = f(x_{i+1}) - f(x_i).$$

For any $j \geq i + 2$, it follows that $f(x_j) - f(x_i) = f_i + f_{i+1} + f_{i+2} + \cdots + f_{j-1}$. We divide the proof into three cases:

Case 1: $j = i+2$. Assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$ (the proof for $d(x_i, x_{i+1}) \leq d(x_{i+1}, x_{i+2})$ is similar). Then,

$$d(x_{i+1}, x_{i+2}) \leq \left\lceil \frac{\frac{1}{2}rk+1}{r} \right\rceil \leq \begin{cases} \frac{1}{2}(k+2) & \text{if } k \text{ is even,} \\ \frac{1}{2}(k+1) & \text{if } k \text{ is odd.} \end{cases}$$

Therefore, $d(x_{i+1}, x_{i+2}) \leq \frac{1}{2}(k+2)$. It suffices to consider the following subcases:

Case 1.1: x_i is between x_{i+1} and x_{i+2}. Then $d(x_i, x_{i+1}) \leq d(x_{i+1}, x_{i+2})$. Since we assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$, we have $d(x_i, x_{i+1}) = d(x_{i+1}, x_{i+2}) \leq \frac{1}{2}(k+2)$ and $d_{P_n}(x_i, x_{i+2}) \leq (r-1)$, from which we have $d(x_i, x_{i+2}) = 1$. Hence,

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= k+1-d(x_i, x_{i+1}) + k+1-d(x_{i+1}, x_{i+2}) \\ &\geq k+1-d(x_i, x_{i+2}). \end{aligned}$$

Case 1.2: x_{i+1} is between x_i and x_{i+2}. This implies

$$d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) - 1.$$

Similar to the calculations above, we have $f(x_{i+2}) - f(x_i) \geq k+1-d(x_i, x_{i+2})$.

Case 1.3: x_{i+2} is between x_i and x_{i+1}. Assume k is odd or

$$\min\bigl\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\bigr\} \leq \bigl(\tfrac{1}{2}rk+1\bigr) - 1,$$

then we have $d(x_{i+1}, x_{i+2}) \leq \frac{1}{2}(k+1)$ and $d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})$. Hence, $f(x_{i+2}) - f(x_i) \geq k+1-d(x_i, x_{i+2})$. If k is even and

$$\min\bigl\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\bigr\} = \tfrac{1}{2}rk+1,$$

then by our assumption, it must be that $d_{P_n}(x_{i+1}, x_{i+2}) = \frac{1}{2}rk+1 \equiv 1 \pmod{r}$ and $d_{P_n}(x_i, x_{i+1}) \not\equiv 1 \pmod{r}$. Thus we have,

$$d(x_i, x_{i+2}) = d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) + 1,$$

which implies

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= 2k+2-\bigl(d(x_i, x_{i+2}) + d(x_{i+1}, x_{i+2}) - 1\bigr) - d(x_{i+1}, x_{i+2}) \\ &\geq k+1-d(x_i, x_{i+2}). \end{aligned}$$

Case 2: $j = i+3$.

Case 2.1: The sum of some pair of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, and $d(x_{i+2}, x_{i+3})$ is at most $k+2$. Then,

$$\begin{aligned} f(x_{i+3}) - f(x_i) &\geq 3k+3-(k+2)-k \\ &> k+1-d(x_i, x_{i+3}). \end{aligned}$$

Case 2.2: The sum of any pair of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, and $d(x_{i+2}, x_{i+3})$ is greater than $k+2$. If we then assume that $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$ (the proof for $d(x_i, x_{i+1}) \leq d(x_{i+1}, x_{i+2})$ is similar), from the calculation in Case 1, we have $d(x_{i+1}, x_{i+2}) \leq \frac{1}{2}(k+2)$. By our hypothesis, it follows that $d(x_i, x_{i+1})$ and $d(x_{i+2}, x_{i+3})$ must both be greater than $\frac{1}{2}(k+2)$. This result, together with $\operatorname{diam}(P_n^r) = k$ and our assumption under this case, implies that x_i must appear before x_{i+2}, then x_{i+1}, then x_{i+3}, from left to right on the r-th power path (or x_{i+3} must appear before x_{i+1}, then x_{i+2}, then x_i). Therefore,

$$d(x_i, x_{i+3}) \geq d(x_i, x_{i+1}) + d(x_{i+2}, x_{i+3}) - d(x_{i+1}, x_{i+2}) - 1.$$

Therefore, we have

$$\begin{aligned} f(x_{i+3}) - f(x_i) &\geq 3k + 3 - d(x_i, x_{i+3}) - 2d(x_{i+1}, x_{i+2}) - 1 \\ &\geq k + 1 - d(x_i, x_{i+3}). \end{aligned}$$

Case 3: $j \geq i + 4$. Since

$$\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq \tfrac{1}{2}(k+2)$$

and $f_i \geq k + 1 - d(x_i, x_{i+1})$ for any i, we have $\max\{f_i, f_{i+1}\} \geq \frac{1}{2}k$ for any $1 \leq i \leq n-2$. Therefore,

$$\begin{aligned} f(x_j) - f(x_i) &\geq (f_i + f_{i+1}) + (f_{i+2} + f_{i+3}) \\ &\geq \left(\tfrac{1}{2}k + 1\right) + \left(\tfrac{1}{2}k + 1\right) > k + 1 - d(x_i, x_j). \end{aligned}$$ □

When $\operatorname{diam}(P_n^r)$ is odd, we have the following "looser" condition for checking that a given labeling is indeed a radio labeling:

Lemma 8. *Let P_n^r be an r-th power path graph on n vertices, where $k = \operatorname{diam}(P_n^r) = \left\lceil \frac{1}{r}(n-1) \right\rceil$ is odd. Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be a permutation of $V(P_n^r)$ such that for any $1 \leq i \leq n-2$,*

$$\min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} \leq \tfrac{1}{2}r(k+1).$$

Let f be a function, $f : V(P_n^r) \longrightarrow \{0, 1, 2, \ldots\}$ with $f(x_1) = 0$ and $f(x_{i+1}) - f(x_i) = k + 1 - d(x_i, x_{i+1})$ for all $1 \leq i \leq n-1$. Then f is a radio labeling for P_n^r.

Proof. Assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$ (the proof for $d(x_i, x_{i+1}) \leq d(x_{i+1}, x_{i+2})$ is similar). Then

$$d(x_{i+1}, x_{i+2}) \leq \left\lceil \frac{\frac{1}{2}r(k+1)}{r} \right\rceil = \tfrac{1}{2}k + 1 \leq \tfrac{1}{2}k + 2.$$

Note that this is the same conclusion we obtained in the beginning of the proof of Lemma 7. Therefore we can use exactly the same proof as above for the case when k is odd to prove this lemma. □

For each radio labeling f of P_n^4 given in the following, we shall first define a permutation (line-up) of the vertices $V(P_n^4) = \{x_1, x_2, x_3, \ldots, x_n\}$, then define f by $f(x_1) = 0$, and for all $1 \leq i \leq n-1$, $f(x_{i+1}) - f(x_i) = k + 1 - d(x_i, x_{i+1})$.

Case 1: $\mathrm{rn}(P^4_{8q+5}) \leq 2k^2+2$. Let $n = 8q+5$ for some $q \in \mathbb{N}$. Then $k = \mathrm{diam}(P^4_{8q+5}) = 2q+1$. We give a radio labeling with span $2k^2+2$. The line-up of $V(P^4_n) = \{x_1, x_2, \ldots, x_n\}$ is given by the arrows in the display below. That is, x_1 is the center, x_2 is the left-vertex of P^4_n whose level is equal to $4q+1, \ldots, x_n$ is the right-vertex of P^4_n whose level is equal to 2. The values above and below each arrow indicate the distances in P^4_n and P_n, respectively, between consecutively labeled vertices.

$$\mathrm{C} \xrightarrow[4q+1]{q+1} \mathrm{L}(4q+1) \xrightarrow[4q+5]{q+2} \mathrm{R}4 \xrightarrow[4q+1]{q+1} \mathrm{L}(4q-3) \xrightarrow[4q+5]{q+2} \cdots \xrightarrow[4q+1]{q+1} \mathrm{L}5 \xrightarrow[4q+5]{q+2} \mathrm{R}(4q) \xrightarrow[4q+1]{q+1} \mathrm{L}1$$
$$\xrightarrow[4q+2]{q+1} \mathrm{R}(4q+1) \xrightarrow[4q+5]{q+2} \mathrm{L}4 \xrightarrow[4q+1]{q+1} \mathrm{R}(4q-3) \xrightarrow[4q+5]{q+2} \mathrm{L}8 \xrightarrow[4q+1]{q+1} \cdots \xrightarrow[4q+1]{q+1} \mathrm{R}5 \xrightarrow[4q+5]{q+2} \mathrm{L}(4q) \xrightarrow[4q+1]{q+1} \mathrm{R}1$$
$$\xrightarrow[4q+3]{q+1} \mathrm{L}(4q+2) \xrightarrow[4q+5]{q+2} \mathrm{R}3 \xrightarrow[4q+1]{q+1} \mathrm{L}(4q-2) \xrightarrow[4q+5]{q+2} \mathrm{R}7 \xrightarrow[4q+1]{q+1} \cdots \xrightarrow[4q+1]{q+1} \mathrm{L}6 \xrightarrow[4q+5]{q+2} \mathrm{R}(4q-1) \xrightarrow[4q+1]{q+1} \mathrm{L}2$$
$$\xrightarrow[4q+4]{q+1} \mathrm{R}(4q+2) \xrightarrow[4q+5]{q+2} \mathrm{L}3 \xrightarrow[4q+1]{q+1} \mathrm{R}(4q-2) \xrightarrow[4q+5]{q+2} \mathrm{L}7 \xrightarrow[4q+1]{q+1} \cdots \xrightarrow[4q+1]{q+1} \mathrm{R}6 \xrightarrow[4q+5]{q+2} \mathrm{L}(4q-1) \xrightarrow[4q+1]{q+1} \mathrm{R}2.$$

By Lemma 8, f is a radio labeling for P^4_{8q+5}. Observe from the above display, there are two possible distances in P^4_{8q+5} between consecutively labeled vertices, namely, $q+1$ and $q+2$, with the number of occurrences $4q+4$ and $4q$, respectively. It follows by direct calculation that

$$f(x_{8q+5}) = (8q+4)(k+1) - \sum_{i=1}^{8q+4} d(x_i, x_{i+1}) = 2k^2+2.$$

Case 2: $\mathrm{rn}(P^4_{8q+4}) \leq 2k^2+2$. Let $n = 8q+4$ for some $q \in \mathbb{N}$. Then $k = \mathrm{diam}(P^4_{8q+4}) = 2q+1$. Let $G = P^4_{8q+5}$ and H be the subgraph of G induced by the vertices $\{v_1, v_2, \ldots, v_{8q+4}\}$. Then $H \cong P^4_{8q+4}$, $\mathrm{diam}(H) = \mathrm{diam}(G) = 2q+1$, and $d_G(u, v) = d_H(u, v)$ for every $u, v \in V(H)$. Let f be a radio labeling for G, then $f|_H$ is also a radio labeling for H. By Case 1, $\mathrm{rn}(P^4_{8q+4}) \leq \mathrm{rn}(P^4_{8q+5}) \leq 2k^2+2$.

Case 3: $\mathrm{rn}(P^4_{8q+3}) \leq 2k^2+1$. Let $n = 8q+3$ for some $q \in \mathbb{N}$. Then $k = \mathrm{diam}(P^4_{8q+3}) = 2q+1$. Similar to Case 1, we line up the vertices according to the display below.

$$\mathrm{C} \xrightarrow[4q+1]{q+1} \mathrm{L}(4q+1) \xrightarrow[4q+5]{q+2} \mathrm{R}4 \xrightarrow[4q+1]{q+1} \mathrm{L}(4q-3) \xrightarrow[4q+5]{q+2} \cdots \xrightarrow[4q+1]{q+1} \mathrm{L}5 \xrightarrow[4q+5]{q+2} \mathrm{R}(4q) \xrightarrow[4q+1]{q+1} \mathrm{L}1$$
$$\xrightarrow[4q+2]{q+1} \mathrm{R}(4q+1) \xrightarrow[4q+5]{q+2} \mathrm{L}4 \xrightarrow[4q+1]{q+1} \mathrm{R}(4q-3) \xrightarrow[4q+5]{q+2} \mathrm{L}8 \xrightarrow[4q+1]{q+1} \cdots \xrightarrow[4q+1]{q+1} \mathrm{R}5 \xrightarrow[4q+5]{q+2} \mathrm{L}(4q) \xrightarrow[4q+1]{q+1} \mathrm{R}1$$
$$\xrightarrow[4q-1]{q} \mathrm{L}(4q-2) \xrightarrow[4q+1]{q+1} \mathrm{R}3 \xrightarrow[4q-3]{q} \mathrm{L}(4q-6) \xrightarrow[4q+1]{q+1} \mathrm{R}7 \xrightarrow[4q-3]{q} \cdots \xrightarrow[4q-3]{q} \mathrm{L}2 \xrightarrow[4q+1]{q+1} \mathrm{R}(4q-1)$$
$$\xrightarrow[4q+2]{q+1} \mathrm{L}3 \xrightarrow[4q+1]{q+1} \mathrm{R}(4q-2) \xrightarrow[4q+5]{q+2} \mathrm{L}7 \xrightarrow[4q+1]{q+1} \mathrm{R}(4q-6) \xrightarrow[4q+5]{q+2} \cdots \xrightarrow[4q+5]{q+2} \mathrm{L}(4q-1) \xrightarrow[4q+1]{q+1} \mathrm{R}2.$$

By Lemma 7, f is a radio labeling for P^4_{8q+3}. If follows by direct calculation that

$$f(x_{8q+3}) = (8q+2)(k+1) - \sum_{i=1}^{8q+2} d(x_i, x_{i+1}) = 2k^2+1.$$

Case 4: $\mathrm{rn}(P^4_{8q+2}) \le 2k^2$. Let $n = 8q+2$ for some $q \in \mathbb{N}$. Then $k = \mathrm{diam}(P^4_{8q+2}) = 2q+1$. Similarly, we line up the vertices according to the display below.

$$\begin{aligned}
&\mathrm{R0}\xrightarrow[4q+1]{q+1}\mathrm{L}(4q)\xrightarrow[4q+5]{q+2}\mathrm{R4}\xrightarrow[4q+1]{q+1}\mathrm{L}(4q-4)\xrightarrow[4q+5]{q+2}\cdots\xrightarrow[4q+1]{q+1}\mathrm{L4}\xrightarrow[4q+5]{q+2}\mathrm{R}(4q)\\
&\xrightarrow[4q+2]{q+1}\mathrm{L1}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-1)\xrightarrow[4q+5]{q+2}\mathrm{L5}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-5)\xrightarrow[4q+5]{q+2}\cdots\xrightarrow[4q+5]{q+2}\mathrm{L}(4q-3)\xrightarrow[4q+1]{q+1}\mathrm{R3}\\
&\xrightarrow[4q+2]{q+1}\mathrm{L}(4q-2)\xrightarrow[4q+1]{q+1}\mathrm{R2}\xrightarrow[4q-3]{q}\mathrm{L}(4q-6)\xrightarrow[4q+1]{q+1}\mathrm{R6}\xrightarrow[4q-3]{q}\cdots\xrightarrow[4q-3]{q}\mathrm{L2}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-2)\\
&\xrightarrow[4q+2]{q+1}\mathrm{L3}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-3)\xrightarrow[4q+5]{q+2}\mathrm{L7}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-7)\xrightarrow[4q+5]{q+2}\cdots\xrightarrow[4q+5]{q+2}\mathrm{L}(4q-1)\xrightarrow[4q+1]{q+1}\mathrm{R1}\xrightarrow[2]{1}\mathrm{L0}.
\end{aligned}$$

By Lemma 7, f is a radio labeling for P^4_{8q+2}. If follows by direct calculation that

$$f(x_{8q+2}) = (8q+1)(k+1) - \sum_{i=1}^{8q+1} d(x_i, x_{i+1}) = 2k^2.$$

Case 5: $\mathrm{rn}(P^4_{8q+1}) \le 2k^2 + q$. Let $n = 8q+1$ for some $q \in \mathbb{N}$. Then $k = \mathrm{diam}(P^4_{8q+1}) = 2q$. Similarly, we line up the vertices according to the display below.

$$\begin{aligned}
&\mathrm{C}\xrightarrow[4q-3]{q}\mathrm{L}(4q-3)\xrightarrow[4q+1]{q+1}\mathrm{R4}\xrightarrow[4q-3]{q}\mathrm{L}(4q-7)\xrightarrow[4q+1]{q+1}\cdots\xrightarrow[4q-3]{q}\mathrm{L1}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q)\\
&\xrightarrow[8q-2]{2q}\mathrm{L}(4q-2)\xrightarrow[4q+1]{q+1}\mathrm{R3}\xrightarrow[4q-3]{q}\mathrm{L}(4q-6)\xrightarrow[4q+1]{q+1}\mathrm{R7}\xrightarrow[4q-3]{q}\cdots\xrightarrow[4q-3]{q}\mathrm{L2}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-1)\\
&\xrightarrow[8q-2]{2q}\mathrm{L}(4q-1)\xrightarrow[4q+1]{q+1}\mathrm{R2}\xrightarrow[4q-3]{q}\mathrm{L}(4q-5)\xrightarrow[4q+1]{q+1}\mathrm{R6}\xrightarrow[4q-3]{q}\cdots\xrightarrow[4q-3]{q}\mathrm{L3}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-2)\\
&\xrightarrow[8q-2]{2q}\mathrm{L}(4q)\xrightarrow[4q+1]{q+1}\mathrm{R1}\xrightarrow[4q-3]{q}\mathrm{L}(4q-4)\xrightarrow[4q+1]{q+1}\mathrm{R5}\xrightarrow[4q-3]{q}\cdots\xrightarrow[4q-3]{q}\mathrm{L4}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-3).
\end{aligned}$$

By Lemma 7, f is a radio labeling for P^4_{8q+1}. It follows by direct calculation that

$$f(x_{8q+1}) = (8q)(k+1) - \sum_{i=1}^{8q} d(x_i, x_{i+1}) = 2k^2 + q.$$

Case 6: $\mathrm{rn}(P^4_{8q}) \le 2k^2 + 1$. Let $n = 8q$ for some $q \in \mathbb{N}$. Then $k = \mathrm{diam}(P^4_{8q}) = 2q$. Similarly, we line up the vertices according to the display below.

$$\begin{aligned}
&\mathrm{R0}\xrightarrow[4q-3]{q}\mathrm{L}(4q-4)\xrightarrow[4q+1]{q+1}\mathrm{R4}\xrightarrow[4q-3]{q}\mathrm{L}(4q-8)\xrightarrow[4q+1]{q+1}\cdots\xrightarrow[4q-3]{q}\mathrm{L4}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-4)\\
&\xrightarrow[8q-6]{2q-1}\mathrm{L}(4q-3)\xrightarrow[4q+1]{q+1}\mathrm{R3}\xrightarrow[4q-3]{q}\mathrm{L}(4q-7)\xrightarrow[4q+1]{q+1}\mathrm{R7}\xrightarrow[4q-3]{q}\cdots\xrightarrow[4q-3]{q}\mathrm{L1}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-1)\\
&\xrightarrow[8q-2]{2q}\mathrm{L}(4q-2)\xrightarrow[4q+1]{q+1}\mathrm{R2}\xrightarrow[4q-3]{q}\mathrm{L}(4q-6)\xrightarrow[4q+1]{q+1}\mathrm{R6}\xrightarrow[4q-3]{q}\cdots\xrightarrow[4q-3]{q}\mathrm{L2}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-2)\\
&\xrightarrow[8q-2]{2q}\mathrm{L}(4q-1)\xrightarrow[4q+1]{q+1}\mathrm{R1}\xrightarrow[4q-3]{q}\mathrm{L}(4q-5)\xrightarrow[4q+1]{q+1}\mathrm{R5}\xrightarrow[4q-3]{q}\cdots\xrightarrow[4q-3]{q}\mathrm{L3}\xrightarrow[4q+1]{q+1}\mathrm{R}(4q-3)\xrightarrow[4q-2]{q}\mathrm{L0}.
\end{aligned}$$

By Lemma 7, f is a radio labeling for P^4_{8q}. It follows by direct calculation that

$$f(x_{8q}) = (8q-1)(k+1) - \sum_{i=1}^{8q-1} d(x_i, x_{i+1}) = 2k^2 + 1.$$

Case 7: $\text{rn}(P^4_{8q-2}) \le \text{rn}(P^4_{8q-1}) \le 2k^2+1$. Since $k = \text{diam}(P^4_{8q-2}) = \text{diam}(P^4_{8q-1}) = \text{diam}(P^4_{8q}) = 2q$, using the same subgraph argument as in Case 2, we have that $\text{rn}(P^4_{8q-2}) \le \text{rn}(P^4_{8q-1}) \le \text{rn}(P^4_{8q}) \le 2k^2+1$.

Cases 1–7, together with Lemmas 5 and 6, complete the proof of Theorem 1.

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