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On closed graphs, II

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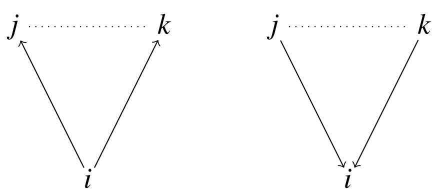
A graph is closed when its vertices have a labeling by $[n]$ with a certain property first discovered in the study of binomial edge ideals. In this article, we explore various aspects of closed graphs, including the number of closed labelings and clustering coefficients.

1. Introduction

Given a simple graph G with vertices $V(G)$ and edges $E(G)$, a *labeling* of G is a bijection $V(G) \simeq [n] = \{1, \ldots, n\}$. Given a labeling, we assume $V(G) = [n]$.

Definition 1.1. A labeling of G is *closed* when $\{j, i\}, \{i, k\} \in E(G)$ with $j > i < k$ or $j < i > k$ implies $\{j, k\} \in E(G)$. We say that G is *closed* if it has a closed labeling.

A labeling of G gives a direction to each edge $\{i, j\} \in E(G)$ where the arrow points from i to j when $i < j$; that is, the arrow points to the bigger label. In this context, closed means that when two edges point away from a vertex or towards a vertex, the remaining vertices are connected by an edge, as shown below:



(1-1)

Closed graphs were first encountered in the study of binomial edge ideals defined in [Herzog et al. 2010; Ohtani 2011]. Properties of these ideals are explored in [Ene et al. 2011; Saeedi Madani and Kiani 2012] and their relation to closed graphs features in [Crupi and Rinaldo 2011; Ene et al. 2014; 2015; Ene and Zarojanu 2015].

It is natural to ask for a characterization of those graphs that have a closed labeling. One solution was given in [Crupi and Rinaldo 2011], which characterizes closed graphs using the clique complex of G. Another approach, taken in our

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previous paper [Cox and Erskine 2015], shows that a connected graph is closed if and only if it is chordal, claw-free, and narrow (see [loc. cit., Definition 1.3] for the definition of narrow).

In this paper, we will use tools developed in [Cox and Erskine 2015] to study the combinatorial properties of closed graphs. Our main results include:

- Section 4: Theorem 4.3 counts the number of closed labelings of a closed graph.
- Section 5: Theorem 5.4 counts the number of closed graphs with fixed layer structure (see Section 2 for the definition of layer).
- Section 6: Theorem 6.3 gives a sharp lower bound for the clustering coefficient of a closed graph.

To prepare for these results, we will recall some relevant results and definitions in Section 2 and explore when a labeling remains closed after exchanging two labels in Section 3.

2. Notation and known results

We recall some notation and results from [Cox and Erskine 2015]. The *neighborhood* of $v \in V(G)$ is

$$N_G(v) = \{w \in V(G) \mid \{v, w\} \in E(G)\}.$$

When G is labeled and $i \in V(G) = [n]$, we have a disjoint union

$$N_G(i) = N_G^>(i) \cup N_G^<(i),$$

where

$$N_G^>(i) = \{j \in N_G(i) \mid j > i\} \quad \text{and} \quad N_G^<(i) = \{j \in N_G(i) \mid j < i\}.$$

Also, vertices $i, j \in [n]$ with $i \le j$ give the *interval* $[i, j] = \{k \in [n] \mid i \le k \le j\}$.

Here is a characterization of when a labeling of a connected graph is closed.

Proposition 2.1 [Cox and Erskine 2015, Proposition 2.4]. *A labeling on a connected graph G is closed if and only if for all $i \in [n]$, the set $N_G^>(i)$ is a complete subgraph and is an interval.*

When a connected graph G has a labeling with $V(G) = [n]$, we can decompose G into layers as follows. The *N-th layer of G* is the set L_N of all vertices that are distance N from vertex 1; i.e.,

$$L_N = \{i \in [n] \mid i \text{ is distance } N \text{ from } 1\}.$$

Since G is connected, we have a disjoint union

$$[n] = L_0 \cup L_1 \cup \cdots \cup L_h, \tag{2-1}$$

where $h = \max\{N \mid L_N \neq \varnothing\}$. Here is a simple property of layers.

Lemma 2.2 [Cox and Erskine 2015, Lemma 2.6]. *Let G be labeled and connected. If $i \in L_N$ and $\{i, j\} \in E(G)$, then $j \in L_{N-1}, L_N$, or L_{N+1}.*

When G is closed and connected, the layers are especially nice.

Proposition 2.3 [Cox and Erskine 2015, Proposition 2.7]. *If G is connected with a closed labeling, then*:

(1) *Each layer L_N is complete.*

(2) *If $d = \max\{L_N\}$, then $L_{N+1} = N_G^{>}(d)$.*

The diameter of G is denoted $\operatorname{diam}(G)$, and a *longest shortest path* of G is a shortest path of length $\operatorname{diam}(G)$. These concepts relate to layers as follows.

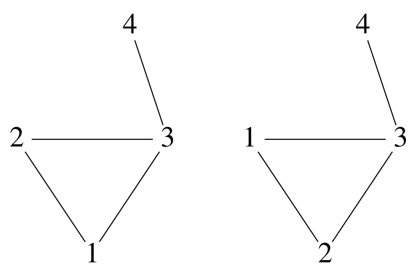
Proposition 2.4 [Cox and Erskine 2015, Proposition 2.8]. *If G is connected with a closed labeling, then*:

(1) $\operatorname{diam}(G)$ *is the integer h appearing in (2-1).*

(2) *If P is a longest shortest path of G, then one endpoint of P is in L_0 or L_1 and the other is in L_h, where $h = \operatorname{diam}(G)$.*

3. Exchangeable vertices

A closed graph with at least two vertices has at least two closed labelings, since the reversal of a closed labeling is clearly closed. But there may be other closed labelings, as shown by this simple example:



(3-1)

To explore what makes this example work, we need some definitions.

Definition 3.1. Let G be a graph.

(1) The *full neighborhood* of a vertex $v \in V(G)$ is $N_G^\star(v) = \{v\} \cup N_G(v)$.

(2) $v, w \in V(G)$ are *exchangeable*, written $v \sim w$, if $N_G^\star(v) = N_G^\star(w)$.

Vertices 1 and 2 are exchangeable in the left-hand graph of (3-1). Switching labels gives the right-hand graph, which is still closed. Here is the general result.

Proposition 3.2. *Let G have a closed labeling. If $i, j \in [n]$, where $i \neq j$, are exchangeable, then the labeling that switches i and j is also closed.*

Proof. Define $\phi : [n] \to [n]$ by $\phi(i) = j$, $\phi(j) = i$, and $\phi(k) = k$ for $k \in [n] \setminus \{i, j\}$. Pick $u, v, w \in V(G)$ with $\{u, v\}, \{v, w\} \in E(G)$, $u \neq w$, and $\phi(u) > \phi(v) < \phi(w)$ or $\phi(u) < \phi(v) > \phi(w)$. We need to prove that $\{u, w\} \in E(G)$.

If $\{i, j\} \cap \{u, v, w\} = \varnothing$, then $\{u, w\} \in E(G)$ since the original labeling is closed. Now suppose $\{i, j\} \cap \{u, v, w\} \neq \varnothing$ and $\phi(u) > \phi(v) < \phi(w)$. There are several cases to consider. First suppose that $i = v$. If $j \in \{u, w\}$, then without loss of generality we may assume $j = u$. Then

$$w \in N_G^\star(v) = N_G^\star(i) = N_G^\star(j) = N_G^\star(u)$$

implies $\{u, w\} \in E(G)$. If $j \notin \{u, w\}$, then $\phi(u) > \phi(i) < \phi(w)$ means that $u > j < w$. Then $\{u, w\} \in E(G)$ since the original labeling is closed and $j \sim i = v$.

The proof when $j = v$ is similar and is omitted. Then two cases remain:

- $i = u$ and $j \notin \{v, w\}$. Thus $\phi(u) > \phi(v) < \phi(w)$ means that $j > v < w$. Then $\{j, w\} \in E(G)$ since the original labeling is closed and $j \sim i = u$. Using $j \sim i = u$ again, we conclude that $\{u, w\} \in E(G)$.
- $i = u$ and $j = w$. Then $\phi(u) > \phi(v) < \phi(w)$ means $j > v < i$. Then $\{u, w\} = \{i, j\} \in E(G)$ since the original labeling is closed.

The proof when $\phi(u) < \phi(v) > \phi(w)$ is similar and is omitted. $\square$

Exchangeability, denoted $v \sim w$, is an equivalence relation on $V(G)$ with equivalence classes

$$e(v) = \{w \in V(G) \mid w \sim v\} = \{w \in V(G) \mid N_G^\star(w) = N_G^\star(v)\}.$$

Equivalence classes are complete, since $v \sim w$ implies $v \in N_G^\star(v) = N_G^\star(w)$, so that $\{v, w\} \in E(G)$ whenever $v \neq w$.

Since permutations are generated by transpositions, Proposition 3.2 implies that when G has a closed labeling, every permutation of an equivalence class yields a new closed labeling.

When G is connected and closed, equivalence classes have the following structure.

Proposition 3.3. *If G is connected with a closed labeling and $i \in [n]$, then the equivalence class $e(i)$ is an interval.*

Proof. It suffices to show that if i and j are exchangeable and $i < k < j$, then $N_G^\star(k) = N_G^\star(i)$. First note that $\{i, k\} \in E(G)$ since $j \in N_G^>(i)$ and $N_G^>(i)$ is an interval by Proposition 2.1. Then $\{j, k\} \in E(G)$ since $i \sim j$.

Now take $m \in N_G^\star(k)$. We need to show $m \in N_G^\star(i)$. If $m = k$, this follows from the previous paragraph. If $\{m, k\} \in E(G)$, there are two possibilities:

- If $m < k$, then $m < k > i$, so $\{m, i\} \in E(G)$ since the labeling is closed.
- If $m > k$, then $m > k < j$, so either $m = j$ or $\{m, j\} \in E(G)$ since the labeling is closed.

Since $N_G^\star(i) = N_G^\star(j)$, both possibilities imply $m \in N_G^\star(i)$.

Conversely, take $m \in N_G^\star(i)$. If $m = i$, then $m \in N_G^\star(k)$ since $\{i, k\} \in E(G)$ by the first paragraph of the proof. If $\{m, i\} \in E(G)$, then $\{m, j\} \in E(G)$ since $i \sim j$. Again, there are two possibilities:

- If $m < i$, then $m < i < k < j$, so $\{m, k\} \in E(G)$ since $N_G^>(m)$ is an interval.
- If $m > i$, then $m > i < k$, so either $m = k$ or $\{m, k\} \in E(G)$ since the labeling is closed.

Thus $m \in N_G^\star(k)$ and the proof is complete. □

4. Counting closed labelings

Some graphs have no nontrivial exchangeable vertices.

Definition 4.1. A graph G is *collapsed* if all exchangeable vertices are equal, i.e., $N_G^\star(v) = N_G^\star(w)$ implies $v = w$.

Proposition 4.2. *Let G be a closed graph with at least three vertices. Then the following are equivalent*:

(1) *G has exactly two closed labelings.*

(2) *G is connected and collapsed.*

Proof. The proof of (1) ⇒ (2) is easy. If G is not connected, then G is a disjoint union $G = G_1 \cup G_2$, where G_i is closed. We may assume G_1 has at least two vertices, so G_1 has at least two labelings. Then we get at least four closed labelings of G: two where 1 is in G_1, and two where 1 is in G_2. Also, if G is not collapsed, then some equivalence class $e(i)$ has at least two elements. If $|e(i)| \geq 3$, then switching labels within $e(i)$ gives at least six closed labelings, and if $|e(i)| = 2$, then G has at least one more vertex, which makes it easy to see that G has at least four closed labelings.

The proof of (2) ⇒ (1) will take more work. First note that $\operatorname{diam}(G) = h \geq 2$. This follows because $h = 1$ would imply that G is complete, which is impossible since G is collapsed with at least 3 vertices, and $h = 0$ is impossible since G is connected with at least 3 vertices.

Fix a closed labeling with $V(G) = [n]$. This gives layers $L_0 = \{1\}, L_1, \ldots, L_h$ associated with the labeling, and Proposition 2.4(2) implies that every longest shortest path has one endpoint in L_0 or L_1 and the other in L_h.

Let $\phi : [n] \to [n]$ be another closed labeling which we will call the ϕ-labeling. Pick $1' \in [n]$ such that $\phi(1') = 1$. Then some longest shortest path of G begins at $1'$. By the previous paragraph, $1' \in L_0 \cup L_1$ or $1' \in L_h$. Replacing ϕ with its reversal if necessary, we may assume that $1' \in L_0 \cup L_1$. We claim that ϕ is the identity function. This will prove the theorem.

We first show that $1' = 1$, i.e, $\phi(1) = 1$. Recall that $L_1 = N_G(1)$ and that L_1 is complete by Proposition 2.3(1). It follows that $N^{\star}_G(1) = L_0 \cup L_1$ is also complete. The same argument implies that $N^{\star}_G(1')$ is complete. Now suppose $1 \neq 1'$ and pick $m \in N^{\star}_G(1')$ different from 1. Then $\{1, m\} \in E(G)$ since $1 \in N^{\star}_G(1')$ and $N^{\star}_G(1')$ is complete. This implies $m \in L_1 = N_G(1)$, and then the inclusion $N^{\star}_G(1') \subseteq N^{\star}_G(1)$ follows easily. The opposite inclusion follows by interchanging the two labelings. Hence we have proved $N^{\star}_G(1') = N^{\star}_G(1)$. Since we are assuming $1 \neq 1'$, this contradicts the fact that G is collapsed. Hence we must have $1' = 1$, as claimed.

Now suppose that vertices $1, \dots, u-1 \in [n]$ have the same ϕ-label as in the original labeling, i.e., $\phi(j) = j$ for $1 \leq j \leq u-1$. Then pick $u' \in [n]$ such that $\phi(u') = u$. To prove that $u' = u$, i.e., $\phi(u) = u$, suppose that $u' \neq u$. Since ϕ is the identity on $1, \dots, u-1$ and $\phi(u') = u$, we have $u' > u$ and $\phi(u') < \phi(u)$.

We first show that $\{u, u'\} \in E(G)$. Since G is connected, Proposition 2.1 implies that every vertex is connected by an edge to its successor in any closed labeling. For the original labeling, this gives $\{u-1, u\} \in E(G)$, and for the ϕ-labeling, this gives $\{u-1, u'\} \in E(G)$ since $\phi(u-1) = u-1$ and $\phi(u') = u$. Proposition 2.1 implies that $N^{>}_G(u-1)$ (in the original labeling) is complete, and $\{u, u'\} \in E(G)$ follows.

We next prove that $N^{\star}_G(u) \subseteq N^{\star}_G(u')$. Pick $m \in N^{\star}_G(u)$. Then:

- If $m = u$, then $m \in N^{\star}_G(u')$ since $\{u, u'\} \in E(G)$.
- If $m > u$, then either $m = u'$, in which case $m \in N^{\star}_G(u')$ is obvious, or $m \neq u'$, in which case $m \in N^{\star}_G(u')$ since $m > u < u'$ implies $\{m, u'\} \in E(G)$ as the original labeling is closed.
- If $m < u$, then $m \in N^{\star}_G(u')$ since $\phi(m) = m < u < \phi(u) > \phi(u')$ implies $\{m, u'\} \in E(G)$ as the ϕ-labeling is closed.

This proves $N^{\star}_G(u) \subseteq N^{\star}_G(u')$. By symmetry, we get $N^{\star}_G(u') = N^{\star}_G(u)$, which contradicts $u' \neq u$ since G is collapsed. We conclude that $u' = u$, and then ϕ is the identity by induction on u. □

Now suppose that G is a connected graph with a closed labeling. Since each equivalence class is an interval by Proposition 3.3, we can order the equivalence classes

$$E_1 < E_2 < \cdots < E_r \tag{4-1}$$

so that if $i \in E_a$ and $j \in E_b$, then $i < j$ if and only if $a < b$. This induces an ordering on $V(G)/{\sim} = \{E_1, \dots, E_r\}$. Then define the graph $G/{\sim}$ with vertices

$$V(G/{\sim}) = V(G)/{\sim} = \{E_1, \dots, E_r\} \tag{4-2}$$

and edges

$$E(G/{\sim}) = \big\{\{E_a, E_b\} \mid \{i, j\} \in E(G) \text{ for some } i \in E_a,\ j \in E_b\big\}. \tag{4-3}$$

Since $i \sim i'$ and $j \sim j'$ imply that $\{i, j\} \in E(G)$ if and only if $\{i', j'\} \in E(G)$, we can replace "for some" with "for all" in (4-3).

Theorem 4.3. *Let G be connected with a closed labeling and exchangeable equivalence classes $E_1, \dots, E_r$. Then*:

(1) *The quotient graph $G/\!\sim$ defined in* (4-2) *and* (4-3) *is connected*, *collapsed*, *and closed with respect to the labeling* (4-1).

(2) *If $r > 1$, then G has precisely $2\prod_{a=1}^{r} |E_a|!$ closed labelings.*

Proof. For (1), we omit the straightforward proof that $G/\!\sim$ is connected and closed with respect to (4-1). To prove that $G/\!\sim$ is collapsed, we first observe that for vertices $u, v \in V(G)$,

$$u \in N_G^\star(v) \Longleftrightarrow e(u) \in N_G^\star(e(v)). \tag{4-4}$$

We leave the simple proof to the reader. Now suppose that equivalence classes $e(v), e(w)$ satisfy $e(v) \sim e(w)$. Then by (4-4), we have

$$u \in N_G^\star(v) \iff e(u) \in N_G^\star(e(v)) \iff e(u) \in N_G^\star(e(w)) \iff u \in N_G^\star(w).$$

This proves that $N_G^\star(v) = N_G^\star(w)$. Then $v \sim w$, which implies $e(v) = e(w)$. It follows that $G/\!\sim$ is collapsed.

For (2), first note that $r > 1$ implies $r \geq 3$, for if there were only two equivalence classes E_1 and E_2, then since G is connected there must be $\{v, w\} \in E(G)$ with $v \in E_1$ and $w \in E_2$. The observation following (4-3) implies that $\{s, t\} \in E(G)$ for all $s \in E_1$ and $t \in E_2$. It follows easily that G is complete, which implies $r = 1$, a contradiction. Hence $r \geq 3$.

According to Proposition 4.2, $G/\!\sim$ has exactly two closed labelings since it has $r \geq 3$ vertices by the previous paragraph and is connected, closed, and collapsed by (1). It follows from (4-1) that any closed labeling of G induces one of these two closed labelings of $G/\!\sim$. Hence all closed labelings of G arise from the two ways of ordering the equivalence classes, together with how we order elements within each equivalence class. Proposition 3.2 and the remarks following the proposition imply that we can use any of the $|E|!$ orderings of the elements of an equivalence class E. Since different equivalence classes can be ordered independently of each other, we get the desired formula for the total number of closed orderings of G. □

5. Counting closed graphs

In Theorem 4.3, we fixed a connected graph and counted the number of closed labelings. Here we change the point of view, where we fix a labeling and count the number of connected graphs for which the given labeling is closed.

Here is how a layer of a connected closed graph connects to the next layer.

Definition 5.1. Let G be a connected graph with a closed labeling. Let the layers of G be $L_0 = \{1\}, L_1, \dots, L_h$, $h = \operatorname{diam}(G)$.

(1) Let $a_N = |L_N|$ for $N = 0, \dots, h$. Note that $a_0 = 1$.

(2) If $N < h$, write the vertices of L_N in order. For $1 \le s \le a_N$, let b_s be the number of edges of G connecting the s-th vertex of L_N to a vertex of L_{N+1}.

(3) The *sequence* of L_N is the sequence $S_N = (b_1, b_2, \dots, b_{a_N})$.

Here is some further notation we will need. First, let $m_N = \min\{L_N\}$. Propositions 2.1 and 2.3 imply that L_N is complete and is an interval. Thus $L_N = [m_N, m_N + a_N - 1]$, and the s-th vertex of L_N is $u_s = m_N + s - 1$.

We can now show that the sequence $S_N = (b_1, b_2, \dots, b_{a_N})$ determines precisely how L_N is connected to L_{N+1}.

Proposition 5.2. *Let G be connected with a closed labeling. If $u_s = m_N + s - 1 \in L_N$ is the s-th vertex of L_N and $b_s > 0$, then*

$$\{v \in L_{N+1} \mid \{u_s, v\} \in E(G)\} = [m_{N+1}, m_{N+1} + b_s - 1].$$

Thus b_s determines how u_s links to L_{N+1}.

Proof. Let $A = \{v \in L_{N+1} \mid \{u_s, v\} \in E(G)\}$. Note that every $v \in A$ satisfies $v > u_s$ by Proposition 2.3(2). It follows easily that

$$A = N_G^{>}(u_s) \cap L_{N+1}.$$

We know that L_{N+1} is an interval, and the same is true for $N_G^{>}(u_s)$ by Proposition 2.1. Hence A is an interval. However, if $v \in A$ and $v \neq m_{N+1}$, then $m_{N+1} < v > u_s$ and the fact that the labeling is closed imply $\{u_s, m_{N+1}\} \in E(G)$ since $\{m_{N+1}, v\} \in E(G)$ by the completeness of L_{N+1}. Hence $m_{N+1} \in A$, and from here, the proposition follows without difficulty. □

Here is an important property of the sequence S_N.

Proposition 5.3. *Let G be connected with a closed labeling. If $N < \operatorname{diam}(G)$, then the sequence $S_N = (b_1, b_2, \dots, b_{a_N})$ of the layer L_N has the following properties*:

(1) *The last element of S_N is a_{N+1}; i.e., $b_{a_N} = a_{N+1}$.*

(2) *S_N is increasing; i.e., $b_s \le b_{s+1}$ for $s = 1, \dots, a_N - 1$.*

Proof. For (1), note that the last vertex of L_N connects to every vertex of L_{N+1} by Proposition 2.3(2). It follows that $b_{a_N} = |L_{N+1}| = a_{N+1}$.

For (2), let u_s be the s-th vertex of L_N, with $1 \le s \le a_N - 1$. If $b_s = 0$, then $b_s \le b_{s+1}$ clearly holds. If $b_s > 0$, then u_s connects to $m_{N+1} + b_s - 1$ by Proposition 5.2, and it connects to u_{s+1} since L_N is complete. Then $m_{N+1} + b_s - 1 > u_s < u_{s+1}$

implies that u_{s+1} connects to $m_{N+1}+b_s-1$ since the labeling is closed. Using Proposition 5.2 again, we obtain

$$m_{N+1}+b_s-1 \in [m_{N+1}, m_{N+1}+b_{s+1}-1],$$

and $b_s \leq b_{s+1}$ follows. □

We now come to the main result of this section.

Theorem 5.4. *Fix n and an integer partition $n = a_0 + a_1 + \cdots + a_h$, with $a_0 = 1$ and $a_N \geq 1$ for $N = 1, \ldots, h$. Also set $\mathcal{L}_0 = \{1\}$ and*

$$\mathcal{L}_N = [a_0 + \cdots + a_{N-1} + 1, a_0 + \cdots + a_N] \tag{5-1}$$

for $N = 1, \ldots, h$, so that $|\mathcal{L}_N| = a_N$. Then the number of graphs G satisfying the conditions

1. *$V(G) = [n]$,*
2. *G is connected and closed with respect to the labeling $V(G) = [n]$, and*
3. *the N-th layer of G is $\mathcal{L}_N$ for $N = 0, \ldots, h$*

is given by the product

$$\prod_{N=0}^{h-1} \binom{a_{N+1}+a_N-1}{a_N-1}.$$

Proof. Let G satisfy (1), (2) and (3). Each layer of G is complete, and every edge of G connects to the same layer or an adjacent layer by Lemma 2.2. Then Proposition 5.2 shows that the edges of G are uniquely determined by $S_0, \ldots, S_{h-1}$.

By Proposition 5.3, each $S_N = (b_1, b_2, \ldots, b_{a_N})$ is an increasing sequence of nonnegative integers of length a_N that ends at a_{N+1}. It is well known that the number of such sequences equals the binomial coefficient

$$\binom{a_{N+1}+a_N-1}{a_N-1}.$$

It follows that the product in the statement of the proposition is an upper bound for the number of graphs satisfying (1), (2) and (3).

To complete the proof, we need to show that every sequence counted by the product corresponds to a graph G satisfying (1), (2) and (3). First note that the minimal element of $\mathcal{L}_N$ is

$$m_N = a_0 + \cdots + a_{N-1} + 1$$

when $N > 0$. Now suppose we have sequences $S_0, \ldots, S_{h-1}$, where each $S_N = (b_1, b_2, \ldots, b_{a_N})$ is an increasing sequence of nonnegative integers of length a_N that ends at a_{N+1}. This determines a graph G with $V(G) = [n]$ and the following edges:

(A) All possible edges connecting elements in the same level $\mathcal{L}_N$.

(B) For each $N = 0, \dots, h-1$, all edges $\{u_s, v\}$, where u_s is the s-th vertex of $\mathcal{L}_N$ and v is any vertex in the interval $[m_{N+1}, m_{N+1}+b_s-1] \subseteq \mathcal{L}_{N+1}$ from Proposition 5.2.

Once we prove that G is closed and connected with $\mathcal{L}_N$ as its N-th layer, the theorem will be proved.

Since $b_{a_N} = a_{N+1}$, we see that for $N = 0, \dots, h-1$, the last element of $\mathcal{L}_N$ connects to all elements of $\mathcal{L}_{N+1}$. This enables us to construct a path from 1 to any $u \in \mathcal{L}_N$ for $N = 1, \dots, h$. It follows that G is connected and that all $u \in \mathcal{L}_N$ have distance at most N from vertex 1. Since every edge of G connects elements of $\mathcal{L}_M$ to $\mathcal{L}_M$, $\mathcal{L}_{M+1}$, or $\mathcal{L}_{M-1}$, any path connecting 1 to $u \in \mathcal{L}_N$ must have length at least N. It follows that $\mathcal{L}_N$ is indeed the N-th layer of G.

It remains to show that G is closed with respect to the natural labeling given by $V(G) = [n]$. A vertex of G is the s-th vertex u_s of $\mathcal{L}_N$ for some s and N. We will show that $N_G^{>}(u_s)$ satisfies Proposition 2.1. The formula (5-1) for $\mathcal{L}_N$ and the description of the edges of G given in (A) and (B) make it clear that

$$\begin{aligned} N_G^{>}(u_s) &= [u_{s+1}, a_0+\cdots+a_N] \cup [m_{N+1}, m_{N+1}+b_s-1] \\ &= [u_{s+1}, m_{N+1}+b_s-1], \end{aligned}$$

where the second equality follows from $m_{N+1} = a_0 + \cdots + a_N + 1$. To show that $N_G^{>}(u_s)$ is complete, take distinct vertices $v, w \in N_G^{>}(u_s)$. If both lie in $\mathcal{L}_N$ or $\mathcal{L}_{N+1}$, then $\{v, w\} \in V(G)$ by (A). Otherwise, we may assume without loss of generality that $v = u_t$, where $t \geq s$, and $w \in [m_{N+1}, m_{N+1}+b_s-1]$. Note that u_t links to every vertex in $[m_{N+1}, m_{N+1}+b_t-1]$ by (B). We also have $b_s \leq b_t$ since S_N is increasing. It follows that $\{v, w\} = \{u_t, w\} \in E(G)$. Hence $N_G^{>}(u_s)$ is complete, so that G is closed by Proposition 2.1. $\square$

6. Local clustering coefficients

In a social network, one can ask how often a friend of a friend is also a friend. Translated into graph theory, this asks how often a path of length two has an edge connecting the endpoints of the path. The illustration (1-1) from the Introduction indicates that this should be a frequent occurrence in a closed graph.

There are several ways to quantify the "friend of a friend" phenomenon. For our purposes, the most convenient is the *local clustering coefficient* of vertex v of a graph G, which is defined by

$$C_v = \begin{cases} \dfrac{\text{number of pairs of neighbors of } v \text{ connected by an edge}}{\text{number of pairs of neighbors of } v} & \text{if } \deg(v) \geq 2, \\ 0 & \text{if } \deg(v) \leq 1. \end{cases}$$

Local clustering coefficients are discussed in [Newman 2010, pp. 201–204].

Proposition 6.1. *Let v be a vertex of a closed graph G of degree $d = \deg(v) \geq 2$. Then the local clustering coefficient C_v satisfies the inequality*

$$C_v \geq \frac{1}{2} - \frac{1}{2(d-1)}.$$

Furthermore, $d \geq 3$ *implies that* $C_v \geq \frac{1}{3}$.

Proof. Pick a closed labeling of G and let $a = |N_G^{>}(v)|$ and $b = |N_G^{<}(v)|$. Then $a + b = |N_G(v)| = \deg(v) = d$. Since the labeling is closed, any pair of vertices in $N_G^{>}(v)$ or in $N_G^{<}(v)$ is connected by an edge. It follows that at least

$$\tfrac{1}{2}a(a-1) + \tfrac{1}{2}b(b-1)$$

pairs of neighbors of v are connected by an edge. Since the total number of such pairs is $\frac{1}{2}d(d-1)$ and $d = a + b$, we obtain

$$C_v \geq \frac{a(a-1)+b(b-1)}{d(d-1)} = \frac{a^2+b^2-d}{d(d-1)} \geq \frac{\frac{1}{2}d^2-d}{d(d-1)} = \frac{1}{2} - \frac{1}{2(d-1)}, \tag{6-1}$$

where we use $a^2 + b^2 - \frac{1}{2}d^2 = \frac{1}{2}(a-b)^2 \geq 0$. When $d \geq 4$, this inequality for C_v easily gives $C_v \geq \frac{1}{3}$. When $d = 3$, then $a + b = 3$, with $a, b \in \mathbb{Z}$, implies that $a^2 + b^2 \geq 5$, in which case the left half of (6-1) gives

$$C_v \geq \frac{5-3}{3(3-1)} = \frac{1}{3}. \qquad \square$$

A global version of the clustering coefficient defined by Watts and Strogatz is

$$C_{\mathrm{WS}} = \frac{1}{n} \sum_{v \in V(G)} C_v, \qquad n = |V(G)|.$$

(See reference [323] of [Newman 2010]. A different global clustering coefficient is discussed in [loc. cit., pp. 199–204].) To estimate C_{WS} for a closed graph, we need the following lemma.

Lemma 6.2. *Let G be a connected closed graph.*

(1) *Set $h = \mathrm{diam}(G)$ and let c be the number of vertices $v \in G$ with $\deg(v) = 2$ and $C_v = 0$. Then $c \leq h - 1$.*

(2) *G has at most two leaves.*

Proof. For (1), fix a closed labeling for G with $V(G) = [n]$ and pick $v \in V(G)$ with $\deg(v) = 2$ and $C_v = 0$. We claim that v is in a layer of its own. To see why, let $v \in L_N$ and suppose there is $s \in L_N$ with $s \neq v$. Then $\{v, s\} \in E(G)$ since layers are complete by Proposition 2.3(1). Furthermore, $|L_N| \geq 2$, so $N > 0$. Then $\{s, d\}, \{v, d\} \in E(G)$ for $d = \max\{L_{N-1}\}$ by Proposition 2.3(2). Since $\deg(v) = 2$, we must have $N_G(v) = \{s, d\}$, and then $\{s, d\} \in E(G)$ contradicts $C_v = 0$. Thus $\{v\}$ is a layer when $\deg(v) = 2$ and $C_v = 0$.

Note that if $\{v\} = L_0$, then the two vertices in $N_G(v) = L_1$ would be linked by an edge. The same holds if $\{v\} = L_h$, for here the two vertices would be in L_{h-1} since L_h is the highest layer by Proposition 2.4(1). It follows that each of the c vertices with $\deg(v) = 2$ and $C_v = 0$ lies in a separate layer distinct from L_0 or L_h. Since there are only $h-1$ intermediate layers, we must have $c \le h-1$.

For (2), assume G has leaves u, v, w and fix a closed labeling of G. We may assume $u < v < w$, and let u', v', w' be the unique vertices adjacent to u, v, w respectively. A shortest path from u to v is directed (see [Herzog et al. 2010] or Proposition 2.1 of [Cox and Erskine 2015]) and must pass through u' and v', hence $u < u' \le v' < v$ since $u < v$. The same argument applied to v and w would imply $v < v' \le w' < w$. Thus $v' < v$ and $v < v'$, so three leaves cannot exist. □

We can now estimate the clustering coefficient C_{WS} of a closed graph.

Theorem 6.3. *If G is connected and closed with $n > 1$ vertices and diameter h, then*

$$C_{\mathrm{WS}} \ge \frac{1}{3} - \frac{h+1}{3n}.$$

Proof. Since $n > 1$ and G is connected, all vertices of G have degree ≥ 1. Thus we can write $V(G)$ as the disjoint union

$$V(G) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D},$$

where $\mathcal{A}$ consists of vertices of degree ≥ 3, $\mathcal{B}$ consists of vertices of degree 2 with $C_v = 1$, $\mathcal{C}$ consists of vertices of degree 2 with $C_v = 0$, and $\mathcal{D}$ consists of the leaves (which have $C_v = 0$). Since $C_v \ge \frac{1}{3}$ for $v \in \mathcal{A}$ by Proposition 6.1, we have

$$C_{\mathrm{WS}} \ge \tfrac{1}{n}\left(\tfrac{1}{3} \cdot |\mathcal{A}| + 1 \cdot |\mathcal{B}| + 0 \cdot |\mathcal{C}| + 0 \cdot |\mathcal{D}|\right) \ge \frac{|\mathcal{A}| + |\mathcal{B}|}{3n} = \frac{n - (|\mathcal{C}| + |\mathcal{D}|)}{3n}.$$

Then we are done since $|\mathcal{C}| \le h-1$ and $|\mathcal{D}| \le 2$ by Lemma 6.2. □

By Theorem 6.3, the clustering coefficient C_{WS} is large when the diameter is small compared to the number of vertices. At the other extreme, both sides of the inequality in Theorem 6.3 are zero when G is a path graph.

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