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A combinatorial proof of a decomposition property of reduced  
residue systems

Yotsanan Meemark and Thanakorn Prinyasart



# A combinatorial proof of a decomposition property of reduced residue systems

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In this paper, we look at three common theorems in number theory: the Chinese remainder theorem, the multiplicative property of the Euler totient function, and a decomposition property of reduced residue systems. We use a grid of squares to give simple transparent visual proofs.

## 1. Introduction

Let  $m$  and  $n$  be positive integers. Construct an  $m \times n$  grid of squares. We place the sequence of positive integers  $1, 2, 3, \dots$  into the grid beginning with the upper left-hand corner cell and moving from the cell numbered  $i$  to the cell numbered  $i + 1$  by going one box down and one to the right. If this is not possible (at the last row or the rightmost column of our  $m \times n$  table), we wrap around to the opposite edge and continue. It is easy to see that the  $i$ -th row has numbers that are congruent to  $i$  modulo  $m$  and the  $j$ -th column has numbers that are congruent to  $j$  modulo  $n$ .

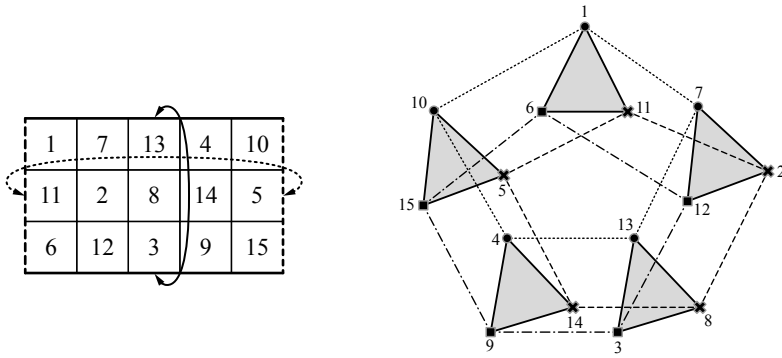
We observe that two positive integers  $x$  and  $y$  fill the same cell if and only if  $x \equiv y \pmod{m}$  and  $x \equiv y \pmod{n}$ , which is equivalent to  $x - y$  is divisible by  $[m, n]$ , the least common multiple of  $m$  and  $n$ . From this, it follows that there is a repetition after we get to  $[m, n]$  and, of course, that  $[m, n]$  is the first integer to arrive at the lower right-hand corner. Thus we have the positive integers from 1 to  $[m, n]$  in the table. Notice that we can number all  $mn$  boxes in this way if and only if  $m$  and  $n$  are relatively prime. This follows from  $(m, n)[m, n] = mn$ . Here  $(m, n)$  denotes the greatest common divisor of  $m$  and  $n$ . When  $m = 3$  and  $n = 5$ , the above explanation can be illustrated by a glued  $3 \times 5$  table and a discrete torus, which appear in [Terras 1999]; see Figure 1.

In what follows, we point out some applications of this elementary construction. It provides not only a visual verification of two common theorems in number theory, namely, the Chinese remainder theorem and the multiplicative property of the Euler

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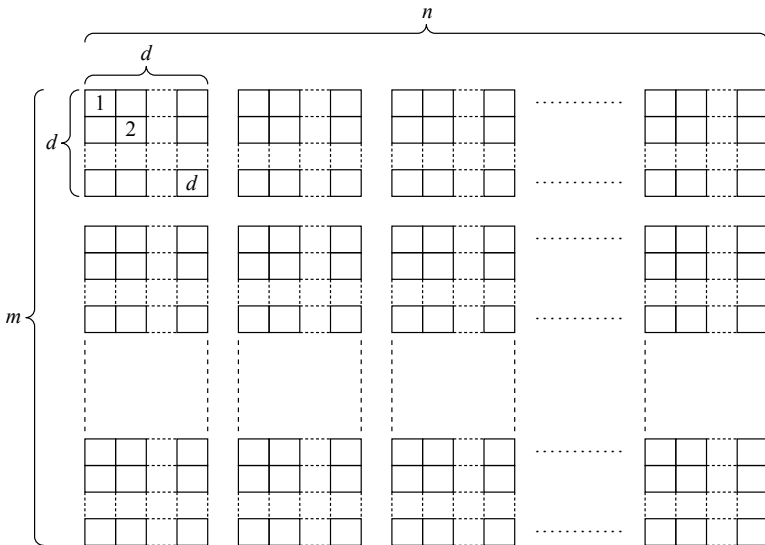
**Figure 1.** A glued  $3 \times 5$  table and its corresponding discrete torus.

totient  $\phi$ -function, but also gives a constructive proof for a decomposition property of reduced residue systems, to be defined below. The results are presented in Sections 2 and 3, respectively.

### 2. The Chinese remainder theorem

Let  $d = (m, n)$ . We can split the  $m \times n$  table into  $(m/d) \times (n/d)$  subtables so that each of them is a square  $d \times d$  table as shown in Figure 2.

By the above filling method, each subtable has numbers only in its diagonal. For example, the upper left-hand corner subtable will be filled with integers from 1 to  $d$ . We move from one subtable to another by going one subtable down and one



**Figure 2.** Our division of the  $m \times n$  table into  $d \times d$  subtables, where  $d = (m, n)$ .

to the right and wrap around as explained before. Hence a square  $d \times d$  subtable can be viewed as a block in an  $(m/d) \times (n/d)$  table. Since  $(m/d, n/d) = 1$ , all  $d \times d$  cells have the subsequence

$$(l - 1)d + 1, (l - 1)d + 2, \dots, ld \quad \text{for some } l \in \left\{1, 2, \dots, \frac{mn}{d^2}\right\}$$

in their diagonals. Thus, the  $m \times n$  table is transformed into an  $(m/d) \times (n/d)$  table with  $(m/d, n/d) = 1$  and we can now number all of the  $mn/d^2$  boxes with  $1, 2, \dots, mn/d^2$ . Now observe that the integers in the original table appear only in the positions  $(k + id, k + jd)$ , where  $k \leq d, i \leq m/d - 1$  and  $j \leq n/d - 1$ . In other words, the positions of the integers are  $(a, b)$  with  $a \equiv b \pmod d$ , that is,  $d \mid (a - b)$ . Furthermore, as mentioned earlier, there is a repetition of solutions modulo  $[m, n]$ . Therefore we have proved the Chinese remainder theorem:

**Theorem 1.** *Let  $m$  and  $n$  be positive integers. For integers  $a$  and  $b$ , the congruences*

$$x \equiv a \pmod m \quad \text{and} \quad x \equiv b \pmod n$$

*admit a simultaneous solution if and only if  $(m, n)$  divides  $a - b$ . Moreover, if a solution exists, then it is unique modulo  $[m, n]$ .*

The result when  $(m, n) = 1$  was also described by Ledet [2007]. We demonstrate Theorem 1 by the following example.

**Example 2.** Let  $m = 6$  and  $n = 8$ . Then  $(m, n) = 2$  and  $[m, n] = 24$ . Filling the  $6 \times 8$  table with the numbers from 1 to 24 as previously described, we obtain

1		19		13		7	
	2		20		14		8
9		3		21		15	
	10		4		22		16
17		11		5		23	
	18		12		6		24

According to this table, one easily sees that  $x \equiv 22 \pmod{24}$  is a simultaneous solution for  $x \equiv 4 \pmod 6$  and  $x \equiv 6 \pmod 8$ , and there is no  $x$  for which both  $x \equiv 5 \pmod 6$  and  $x \equiv 4 \pmod 8$ . □

If  $m$  is a positive integer, the *Euler totient function*  $\phi(m)$  is defined to be the number of positive integers not exceeding  $m$  which are relatively prime to  $m$ . By a *reduced residue system modulo  $m$* , we mean any set of  $\phi(m)$  integers, pairwise incongruent modulo  $m$ , each of which is relatively prime to  $m$ . Notice that if  $p$  is a prime, then  $\phi(p) = p - 1$  and  $\{1, 2, \dots, p - 1\}$  is a reduced residue system modulo  $p$ . It is also immediate that  $\phi(p^s) = p^s - p^{s-1}$  for all  $s \in \mathbb{N}$ .

Next, we investigate the decomposition property of the reduced residue systems by our combinatorial technique. Let  $a = mn$ , where  $m$  and  $n$  are positive integers.

We arrange the positive integers  $1, 2, \dots, [m, n]$  into the  $m \times n$  grid of squares by using the above filling method and delete the  $i$ -th rows and  $j$ -th columns of the table for all  $i$  and  $j$  with  $(m, i) > 1$  and  $(n, j) > 1$ . For a better understanding of this construction, one may erase all even (second, fourth, ...) rows and all even columns of the table in [Example 2](#). Recall that the  $i$ -th row has numbers that are congruent to  $i$  modulo  $m$  and the  $j$ -th column has numbers that are congruent to  $j$  modulo  $n$ .

Let  $l$  be a remaining positive integer in the table. Notice that  $l \equiv i \pmod{m}$  with  $(m, i) = 1$  and  $1 \leq i \leq m$ ; that is,  $l = i + km$  for some nonnegative integer  $k$ . Since  $(m, i) = 1$ , there exist integers  $x$  and  $y$  such that  $mx + iy = 1$ . Consequently, we choose  $x' = x - ky \in \mathbb{Z}$  and  $y' = y \in \mathbb{Z}$ . Then  $mx' + ly' = 1$ , so we have  $(l, m) = 1$ . Similarly, we can show that  $(l, n) = 1$ . Since  $a = mn$ , we also have  $(l, a) = 1$ . Hence all positive integers left in the table after deletion are relatively prime to  $a$  and less than  $[m, n]$ .

For  $(m, n) = 1$ , we can place the positive integers from 1 to  $[m, n] = mn = a$  in the  $m \times n$  grid by the means above. Erase the  $i$ -th rows that are not relatively prime to  $m$  and cross out the  $j$ -th columns that are not relatively prime to  $n$ . Then we obtain  $\phi(m)\phi(n)$  undeleted cells and eliminate all numbers that are not relatively prime to  $m$  and  $n$ . Since  $(m, n) = 1$ , the entries left in the table coincide with positive integers less than and relatively prime to  $a$ , so the number of these entries is equal to  $\phi(a)$ . Hence we can conclude the well-known multiplicative property of the Euler totient  $\phi$ -function, namely, if  $(m, n) = 1$ , then  $\phi(mn) = \phi(a) = \phi(m)\phi(n)$ . This combinatorial proof is the one given in the famous book on number theory [[Niven et al. 1991](#)]. Since  $\phi(p^s) = p^s - p^{s-1} = p^s(1 - p^{-1})$  when  $p$  is a prime and  $s \geq 1$ , the multiplicative property gives a formula for computing

$$\phi(M) = M \prod_{p|M} (1 - p^{-1})$$

for any positive integer  $M$ .

### 3. Decomposition property of reduced residue systems

Let  $m'$  be the product of primes in  $m$  not in  $n$  with the same exponents that they have in  $m$ . It is easy to see that  $m'$  and  $n$  are relatively prime. Place the positive integers from 1 to  $m'n$  in the  $m' \times n$  grid and erase the rows that are not relatively prime to  $m'$  and the columns that are not relatively prime to  $n$ . Let  $l$  be a positive integer left in the table after deletion. Then  $(l, m') = 1 = (l, n)$ . Assume that there exists a prime  $p$  dividing  $l$  and  $a = mn$ . Thus  $p | m$  or  $p | n$ . But  $(l, n) = 1$ , so  $p$  is not in  $n$  and thus  $p$  is in  $m$ . Therefore  $p | m'$ , which contradicts the fact

that  $(l, m') = 1$ . Hence the remaining  $\phi(m')\phi(n)$  positive integers in the table are relatively prime to  $a$ . Consider them as a  $\phi(m') \times \phi(n)$  matrix. The set of all members in each row of this matrix is a reduced residue system modulo  $n$  and  $x \equiv y \pmod n$  for all integers  $x$  and  $y$  that are in the same column.

Let  $A_0$  be the above  $\phi(m') \times \phi(n)$  matrix and

$$A_i = A_0 + i \begin{bmatrix} m'n & \dots & m'n \\ \vdots & \ddots & \vdots \\ m'n & \dots & m'n \end{bmatrix}_{\phi(m') \times \phi(n)} \quad \text{for } i = 0, 1, \dots, \frac{\phi(mn)}{\phi(m')\phi(n)} - 1.$$

The identity  $\phi(M) = M \prod_{p|M} (1 - p^{-1})$  shows that

$$\frac{\phi(mn)}{\phi(m')\phi(n)} = \frac{m}{m'},$$

so the index  $i$  ranges from 0 up to  $m/m' - 1$ , which implies that the entries of  $A_i$  do not exceed  $a$ . It is also obvious that each entry in  $A_i$  is relatively prime to  $a$ . We augment  $A_0$  by the matrices

$$A_1, \dots, A_{\frac{\phi(a)}{\phi(m')\phi(n)} - 1},$$

respectively, to form a new  $(\phi(a)/\phi(n)) \times \phi(n)$  matrix. Then the entries of this matrix are integers from 1 to  $a$ , relatively prime to  $a$ , with the condition that the set of the entries in each row is a reduced residue system modulo  $n$  and  $x \equiv y \pmod n$  for all integers  $x$  and  $y$  that are in the same column. Hence we have a constructive proof for a theorem on a decomposition property of reduced residue systems modulo  $a$  summarized as follows.

**Theorem 3.** *Let  $S$  be a residue system modulo  $a$ , and let  $n \geq 1$  be a divisor of  $a$ . Then we have the following decompositions of  $S$ :*

- (1)  $S$  is the union of  $\phi(a)/\phi(n)$  disjoint sets, each of which is a reduced residue system modulo  $n$ .
- (2)  $S$  is the union of  $\phi(n)$  disjoint sets, each of which consists of  $\phi(a)/\phi(n)$  numbers congruent to each other modulo  $n$ .

**Remark.** Another proof of this theorem and its application on character sums can be found in Apostol's book [1976].

**Example 4.** Consider  $a = 48$  with  $m = 6$  and  $n = 8$ . Since  $8 = 2^3$  and  $6 = 2 \cdot 3$ , let  $m' = 3$ . Filling a  $3 \times 8$  table with numbers by our technique, we obtain

1	10	19	4	13	22	7	16
17	2	11	20	5	14	23	8
9	18	3	12	21	6	15	24

Delete the rows that contain numbers not relatively prime to 3 and the columns that contain numbers not relatively prime to 8. We have then the  $2 \times 4$  matrix formed from the remaining numbers given by

$$A = \begin{bmatrix} 1 & 19 & 13 & 7 \\ 17 & 11 & 5 & 23 \end{bmatrix}.$$

Augment this matrix with  $\phi(3) = 2$  rows obtained by adding  $m'n$  to all entries of  $A$ , so we finally reach the decomposition

$$A' = \begin{bmatrix} 1 & 19 & 13 & 7 \\ 17 & 11 & 5 & 23 \\ 25 & 43 & 37 & 31 \\ 41 & 35 & 29 & 47 \end{bmatrix}$$

as desired. □

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[yotsanan.m@chula.ac.th](mailto:yotsanan.m@chula.ac.th)

*Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand*

[thanakorn\\_dpst@hotmail.com](mailto:thanakorn_dpst@hotmail.com)

*Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand*

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