involve
a journal of mathematics

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msp

2016 vol. 9, no. 3

INVOLVE 9:3 (2016)
dx.doi.org/10.2140/involve.2016.9.379

Generalized factorization in $\mathbb{Z}/m\mathbb{Z}$

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(Communicated by Vadim Ponomarenko)

Generalized factorization theory for integral domains was initiated by D. D. Anderson and A. Frazier in 2011 and has received considerable attention in recent years. There has been significant progress made in studying the relation τ_n for the integers in previous undergraduate and graduate research projects. In 2013, the second author extended the general theory of factorization to commutative rings with zero-divisors. In this paper, we consider the same relation τ_n over the modular integers, $\mathbb{Z}/m\mathbb{Z}$. We are particularly interested in which choices of $m, n \in \mathbb{N}$ yield a ring which satisfies the various τ_n-atomicity properties. In certain circumstances, we are able to say more about these τ_n-finite factorization properties of $\mathbb{Z}/m\mathbb{Z}$.

1. Introduction and background

D. D. Anderson and A. Frazier [2011] introduced a concept called τ-factorization. This provided a general theory which unified much of the existing literature on factorization theory in integral domains into one general notion of factorization theory. Recently, the second author has used several methods to extend this τ-factorization to commutative rings with zero-divisors; see [Mooney 2015a, 2015b; 2015c; 2016].

There has been a fair amount of research done on a particular τ-relation of interest especially in the integers, $\mathbb{Z}$. We discuss this in more depth in the following section. In particular, the dissertation of S. M. Hamon [2007] answered the following question, among others: for what $n \in \mathbb{N}$ is $\mathbb{Z}$ τ_n-atomic? A. Florescu [2013] investigated reduced τ_n-factorizations over $\mathbb{Z}$. These studies helped to give a concrete basis for τ-factorization over the integers.

In this paper, we carry out a similar investigation of $\mathbb{Z}/m\mathbb{Z}$. We again are interested in the τ_n-finite factorization properties, especially the question of τ_n-atomicity. We use the definitions and methods established by D. D. Anderson and S. Valdez-Leon [1996] and generalized by the second author [Mooney 2015a]. In Section 2, we present preliminary definitions and background information in a more rigorous

MSC2010: 13A05, 13E99, 13F15.
Keywords: modular integers, generalized factorization, zero-divisors, commutative rings.

and thorough manner. In Section 3, we present several important properties of $\mathbb{Z}/m\mathbb{Z}$ which play a role in the τ_n-finite factorization properties. In Section 4, we present the main results concerning τ_n-finite factorization properties of $\mathbb{Z}/m\mathbb{Z}$ for various choices of m and n. Finally, in Section 5, we present further thoughts on the remaining questions which were not answered in the present article.

2. Preliminaries

We assume R is a commutative ring with $1 \neq 0$. Let $R^* = R - \{0\}$, $U(R)$ be the set of units of R, and $R^\# = R^* - U(R)$ be the nonzero nonunits of R. As in [Anderson and Valdes-Leon 1996], we let

- $a \sim b$ if $(a) = (b)$,
- $a \approx b$ if there exists $\lambda \in U(R)$ such that $a = \lambda b$,
- $a \cong b$ if (1) $a \sim b$ and (2) $a = b = 0$ or if $a = rb$ for some $r \in R$ then $r \in U(R)$.

We say a and b are *associates* (resp. *strong associates, very strong associates*) if $a \sim b$ (resp. $a \approx b$, $a \cong b$). As in [Anderson et al. 2004], a ring R is said to be a *strongly associate* (resp. *very strongly associate*) ring if for any $a, b \in R$, $a \sim b$ implies $a \approx b$ (resp. $a \cong b$).

We leave the routine check that very strong associates are strong associates and strong associates are associates as an exercise for the reader. Both $\sim$ and $\approx$ are equivalence relations, while $\cong$ fails only to be reflexive. It is interesting to see why, in rings with zero-divisors, these associate relations are no longer equivalent. Any nontrivial idempotent $e \in R$ provides an example of an element such that $e \approx e$, but $e \ncong e$. We have $e = 1 \cdot e$, yet $e \ncong e$ because e is not a unit in $e = e \cdot e$. This also demonstrates why $\cong$ need not be reflexive. Examples of elements which are associate, but not strongly associate are more difficult to come by. We provide an example first given in [Fletcher 1969] and restated in [Anderson and Valdes-Leon 1996, Example 2.3], where the details are provided. Let $R = F[X, Y, Z]/(X - XYZ)$, where F is a field. Let x, y, and z be the images of X, Y, and Z respectively in R. Then $x = xyz$, so $x \sim xy$, but there is no unit $\lambda \in U(R)$ such that $x = \lambda xy$, so $x \not\approx xy$.

Let τ be a symmetric relation on $R^\#$; that is, $\tau \subseteq R^\# \times R^\#$ and if $(a, b) \in \tau$, then $(b, a) \in \tau$ and we will write $a \,\tau\, b$. For nonunits $a, a_i \in R$, and $\lambda \in U(R)$, $a = \lambda a_1 \cdots a_n$ is said to be a *τ-factorization* if $a_i \,\tau\, a_j$ for all $i \neq j$. If $n = 1$, then this is said to be a trivial *τ-factorization*. Given the above τ-factorization, we would say that *a_i is a τ-factor of a* or write $a_i \mid_\tau a$. We note that 0 cannot appear as a τ-factor, except in the trivial factorization $0 = \lambda 0$ for some $\lambda \in U(R)$.

We pause to provide some examples of τ-relations which have been of interest in the literature.

Example 2.1. Let R be a commutative ring with 1.

(1) $\tau_d = R^\# \times R^\#$. This yields the usual factorizations in R and $|_{\tau_d}$ is the same as the usual divides.

(2) $\tau = \varnothing$. For every $a \in R^\#$, there is only the trivial factorization and $a \,|_\tau\, b \iff a = \lambda b$ for $\lambda \in U(R) \iff a \approx b$.

(3) Let I be an ideal in R. Set $a \,\tau_I\, b$ if and only if $a - b \in I$.

(a) Let $R = \mathbb{Z}$ and $I = (n)$. Then this is τ_n, which was studied extensively in [Florescu 2013; Hamon 2007].

(b) In the present work, we are interested in the case when $R = \mathbb{Z}/m\mathbb{Z}$ and $I = (n)$. We note that $\tau_{(n)}$ is usually written as τ_n and this relation is indeed symmetric since $a - b \in I \iff b - a \in I$.

(4) We obtain the comaximal factorizations studied in [McAdam and Swan 2004] by $a \,\tau\, b$ if and only if $(a, b) = R$. Furthermore, for any $\star$-operation, we obtain $\star$-comaximal factorizations, studied in [Juett 2012], by $a \,\tau_\star\, b$ if and only if $(a, b)^\star = R$.

(5) Lastly, for any set S, such as the collection of irreducible or prime elements in a ring R, we can study τ_S-factorizations to obtain the atomic or prime factorizations respectively by saying $a \,\tau_S\, b$ if and only if $a \in S$ and $b \in S$.

We now summarize several definitions given in [Mooney 2015a; 2016]. Let $a \in R$ be a nonunit. Then a is said to be *τ-irreducible* or *τ-atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for some i. We say a is *τ-strongly irreducible* or *τ-strongly atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \approx a_i$ for some a_i. We say that a is *τ-m-irreducible* or *τ-m-atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for all i. Note: the "m" is for "maximal" since such an a is maximal among principal ideals generated by elements which occur as τ-factors of a. As in [Mooney 2016], $a \in R$ is said to be a *τ-unrefinable atom* if a admits only trivial τ-factorizations. We say that a is *τ-very strongly irreducible* or *τ-very strongly atomic* if $a \cong a$ and a has no nontrivial τ-factorizations. We refer the reader to [Mooney 2015a; 2016] for a further discussion and more equivalent definitions of these various forms of τ-irreducibility.

We have the following relationship between the various types of τ-irreducibles, which is proved in [Mooney 2015a, Theorem 3.9] as well as [Mooney 2016].

Theorem 2.2. *The following diagram illustrates the relationships between the various types of τ-irreducibility a might satisfy, where $\approx$ represents R being a strongly associate ring*:

$$\begin{tikzcd}
\tau\text{-v. s. irred.} \arrow[r, Rightarrow] & \tau\text{-unrefinably irred.} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \tau\text{-s. irred.} \arrow[r, Rightarrow] & \tau\text{-irred.} \\
& & \tau\text{-}m\text{-irred.} \arrow[u, Rightarrow, "\approx"'] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Let e be a nontrivial idempotent in R. Let $\tau_\varnothing = \varnothing$. Then there are no nontrivial $\tau_\varnothing$-factorizations. Thus every $a \in R^\#$ is $\tau_\varnothing$-unrefinably atomic. However, $e \cdot e = e$ shows that $e \not\cong e$ and thus e is not $\tau_\varnothing$-very strongly atomic. To see that none of the other reverse implications hold, we may set $\tau = R^\# \times R^\#$ to obtain the usual factorizations. Examples are provided in [Anderson and Valdes-Leon 1996] which show that the other implications are not reversible in rings with zero-divisors.

We are now able to summarize various τ_n-finite factorization properties that a ring may have.

Definition 2.3. Let $\alpha \in$ {atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic}. Let $\beta \in$ {associate, strongly associate, very strongly associate}.

(1) R is said to be τ-α if every nonunit has a τ-factorization into elements which are τ-α.

(2) R is said to satisfy *τ_n-ACCP* if for every nonunit $a_0 \in R$, any ascending chain of principal ideals

$$(a_0) \subseteq (a_1) \subseteq (a_2) \subseteq \cdots \subseteq (a_i) \subseteq (a_{i+1}) \subseteq \cdots$$

such that $a_{i+1} \mid_\tau a_i$ for each i becomes stationary.

(3) R is said to be a *τ_n-α-β-unique factorization ring* (*UFR*) if

- R is τ_n-α,
- every nonunit has a unique τ_n-α factorization up to rearrangement and β.

(4) R is said to be a *τ_n-α-half factorial ring* (*HFR*) if R is τ-α and for each nonunit, the length of every τ_n-α factorization is the same.

(5) R is said to be a *τ_n-bounded factorization ring* (*BFR*) if every nonunit has a finite bound on the length of any τ_n-factorization.

(6) R is said to be a *τ_n-β-finite factorization ring* (*FFR*) if every nonunit has only a finite number of τ_n-factorizations up to rearrangement and β.

(7) R is said to be a *τ_n-β-weak finite factorization ring* (*WFFR*) if every nonunit has only a finite number of τ_n-divisors up to β.

(8) R is said to be a *τ_n-α-β-divisor finite ring* (*df ring*) if every nonunit has only a finite number of τ_n-α-divisors up to β.

We include parts of the diagram from [Mooney 2016] to help the reader visualize the relationship between these τ-finite factorization properties. In the diagram below, ∇ represents τ being refinable and associate-preserving and we direct the reader to [Mooney 2016] for further details:

$$
\begin{tikzcd}
 & \tau\text{-}\alpha\text{-HFR} \arrow[dr, Rightarrow, "\nabla"] & & & \\
\tau\text{-}\alpha\text{-}\beta\text{-UFR} \arrow[ur, Rightarrow] \arrow[r, Rightarrow, "\nabla"] & \tau\text{-}\beta\text{-FFR} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \tau\text{-BFR} \arrow[r, Rightarrow, "\nabla"] & \tau\text{-ACCP} \arrow[r, Rightarrow, "\nabla"] & \tau\text{-atomic} \\
 & \tau\text{-}\beta\text{-WFFR} \arrow[dl, Rightarrow, "\nabla"'] \arrow[d, Rightarrow] & & & \\
\tau\text{-}\alpha\ \tau\text{-}\alpha\text{-}\beta\text{-df ring} \arrow[r, Rightarrow] & \tau\text{-}\alpha\text{-}\beta\text{-df ring} & & &
\end{tikzcd}
$$

3. $\mathbb{Z}/m\mathbb{Z}$ is strongly associate

We begin by studying the ring we are interested in, $\mathbb{Z}/m\mathbb{Z}$. As seen in the previous section, the main issue with factorization in rings with zero-divisors is the number of types of irreducibility and atomicity. We find that this ring has several nice properties, which makes our work slightly more manageable. We find that $\mathbb{Z}/m\mathbb{Z}$ is a strongly associate ring and if p is a prime and $e \in \mathbb{N}$, then $\mathbb{Z}/p^e\mathbb{Z}$ is présimplifiable. Equivalently, $\mathbb{Z}/p^e\mathbb{Z}$ is a very strongly associate ring. So if m is a prime power, then for any $a \in R^{\#}$, all the associate relations and hence types of τ-irreducibility coincide. In general, even if m has multiple prime divisors, we will know that associate and strongly associate coincide; hence τ_n-atomic and τ_n-strongly atomic also coincide.

It was proved, in [Kaplansky 1949], that any Artinian or principal ideal ring is strongly associate. This immediately gives us that our finite (hence Artinian) principal ideal ring, $\mathbb{Z}/m\mathbb{Z}$, is strongly associate. We outline an elementary proof for $\mathbb{Z}/m\mathbb{Z}$ being strongly associate as well as present other useful results about $\mathbb{Z}/m\mathbb{Z}$. We hope this is helpful for the reader, both to become familiar with the ring we are working in and to see the relationships between the various types of associate relations. Many of these results and similar techniques are used later when we analyze the question of τ_n-atomicity of $\mathbb{Z}/m\mathbb{Z}$.

We begin with a remark about the units of a direct product of commutative rings. This is a routine result, which can be found in any modern algebra text, and will be left as an exercise to the reader.

Remark. Let R_1 and R_2 be commutative rings with unity and let $R = R_1 \times R_2$. Then

$$U(R) = \{(\lambda_1, \lambda_2) \mid \lambda_1 \in U(R_1),\ \lambda_2 \in U(R_2)\} = U(R_1) \times U(R_2) := S.$$

That is, the units in a direct product of rings are the direct product of the collection of units in the individual rings.

Lemma 3.1. *$R = R_1 \times R_2$ is strongly associate if and only if R_1 and R_2 are both strongly associate.*

Proof. ($\Rightarrow$) Let $R = R_1 \times R_2$ be a strongly associate ring. Let (a), (b) be ideals in R_1 such that $a \sim b$, i.e., $(a) = (b)$. Consider the ideals $(a) \times R_2 = (a) \times (1)$ and $(b) \times R_2 = (b) \times (1)$. Since $(a) = (b)$, we have

$$((a, 1)) = (a) \times (1) = (b) \times (1) = ((b, 1)).$$

Now $R = R_1 \times R_2$ is strongly associate, so there is a unit $(\lambda_1, \lambda_2) \in U(R)$ such that $(a, 1) = (\lambda_1, \lambda_2)(b, 1)$. Thus $a = \lambda_1 b$. By the above remark, we have shown that $\lambda_1 \in U(R_1)$. Hence $a \approx b$. A symmetric argument demonstrates that R_2 is strongly associate.

($\Leftarrow$) Now suppose R_1 and R_2 are strongly associate rings. Let $a, b \in R$ with $a \sim b$. Suppose $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Now $a \sim b$ means $((a_1, a_2)) = ((b_1, b_2))$. We must prove that there exists a $(\lambda_1, \lambda_2) \in U(R)$ with $(a_1, a_2) = (\lambda_1, \lambda_2)(b_1, b_2)$. Now

$$(a_1) \times (a_2) = ((a_1, a_2)) = ((b_1, b_2)) = (b_1) \times (b_2).$$

Thus a_1 is associate with b_1 and a_2 is associate with b_2. Hence, R_1 and R_2 are strongly associate, so there exists $\lambda_1 \in U(R_1)$ and $\lambda_2 \in U(R_2)$ such that $a_1 = \lambda_1 b_1$ and $a_2 = \lambda_2 b_2$. Therefore $(\lambda_1, \lambda_2) \in U(R)$ with $(a_1, a_2) = (\lambda_1, \lambda_2)(b_1, b_2)$. This demonstrates R is strongly associate as desired. $\square$

A routine induction argument on n, the number of factors in the product, yields the following result since $R = (R_1 \times R_2 \times \cdots \times R_{n-1}) \times R_n = R_1 \times R_2 \times \cdots \times R_n$.

Lemma 3.2. *$R = R_1 \times R_2 \times \cdots \times R_n$ is strongly associate if and only if R_i is strongly associate for each $1 \leq i \leq n$.*

Lemma 3.3. *Let $a_1, \ldots, a_n \in R$. Then $(a_1 a_2 \cdots a_n) = (a_1)(a_2) \cdots (a_n)$.*

Proof. Let $x \in (a_1)(a_2) \cdots (a_n)$. Then

$$x = r_{11} a_1 r_{12} a_2 \cdots r_{1n} a_n + r_{21} a_1 r_{22} a_2 \cdots r_{2n} a_n + \cdots + r_{m1} a_1 r_{m2} a_2 \cdots r_{mn} a_n$$

for some $r_{ij} \in R$, with $1 \leq i, j \leq m$, is a typical element of $(a_1)(a_2) \cdots (a_n)$. Notice that we can factor out $a_1 a_2 \cdots a_n$ from each term yielding

$$x = (r_{11} r_{12} \cdots r_{1n} + r_{21} r_{22} \cdots r_{2n} + \cdots + r_{m1} r_{m2} \cdots r_{mn})(a_1 a_2 \cdots a_n). \quad (1)$$

The right-hand side of (1) demonstrates that $x \in (a_1 a_2 \cdots a_n)$. Thus $(a_1 a_2 \cdots a_n) \supseteq (a_1)(a_2) \cdots (a_n)$.

Let $x \in (a_1 a_2 \cdots a_n)$. Then $x = r a_1 a_2 \cdots a_n$ for some $r \in R$. Then we can write $x = r a_1 a_2 \cdots a_n = (r a_1)(1 a_2) \cdots (1 a_n)$, demonstrating $x \in (a_1)(a_2) \cdots (a_n)$. Thus $(a_1 a_2 \cdots a_n) \subseteq (a_1)(a_2) \cdots (a_n)$. $\square$

Lemma 3.4. *Let $p \in \mathbb{N}$ be a prime number and $e \in \mathbb{N}$. Then $R = \mathbb{Z}/p^e\mathbb{Z}$ is very strongly associate; equivalently, $\mathbb{Z}/p^e\mathbb{Z}$ is présimplifiable. Moreover, this means that $\mathbb{Z}/p^e\mathbb{Z}$ is a strongly associate ring.*

Proof. Suppose $a \sim b$. We will show $a \cong b$. Since $a \sim b$, we have $(a) = (b)$ by definition. Thus we must prove that either $a = b = 0$ or if $a = rb$ for some $r \in R$ then $r \in U(R)$.

If $a = 0$ or $b = 0$ we are done, so we may assume that neither a nor b is 0. If a or b are units, then $(a) = (b) = R$ and $r = ab^{-1}$, which is a unit. Thus we may assume a and b are nonzero nonunits. Thus $p \mid a$ and $p \mid b$. Let e_a be the largest integer such that p^{e_a} divides a, but no larger power still divides a. Define e_b similarly. Now $(a) = (b)$, so $a \mid b$ and $p^{e_a} \mid a$ and therefore $p^{e_a} \mid b$. This means $e_a \leq e_b$. Similarly, $b \mid a$ so $e_b \leq e_a$. This means $e_a = e_b$, but by comparing the number of factors of p in both sides of $a = rb$, we see that p cannot divide r. Thus $\gcd(r, p) = 1$ and $r \in U(\mathbb{Z}/p^e\mathbb{Z})$. Hence, R has been shown to be a very strongly associate ring, which is equivalent to présimplifiable in the language of Bouvier [1971; 1972a; 1972b; 1974]. Every présimplifiable ring is certainly a strongly associate ring. □

The following theorem now follows easily from the lemmas and the Chinese remainder theorem.

Theorem 3.5. *Let $m \in \mathbb{N}$ with $m \geq 2$ and $m = p_1^{e_1} \cdots p_n^{e_n}$. Then*

$$\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/p_1^{e_1}\mathbb{Z} \times \mathbb{Z}/p_2^{e_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_n^{e_n}\mathbb{Z}$$

is a strongly associate ring.

This means associate and strongly associate are always the same relation and hence τ_n-atomic and τ_n-strongly atomic coincide for our rings $\mathbb{Z}/m\mathbb{Z}$. We also needed R to be a strongly associate ring to conclude that τ_n-m-atomic implies τ_n-strongly atomic in Theorem 2.2. We find that this property of $\mathbb{Z}/m\mathbb{Z}$ greatly streamlines much of the research.

4. τ_n-factorization properties of $\mathbb{Z}/m\mathbb{Z}$

Here we begin our analysis of which choices of $m, n \in \mathbb{N}$ yield a τ_n-atomic (or -strongly atomic, -m-atomic, -unrefinably atomic, -very strongly atomic) ring. Moreover, when possible, we indicate if the ring satisfies other nice τ_n-finite factorization properties.

$\mathbb{Z}/p\mathbb{Z}$. We first consider the simplest case, $R = \mathbb{Z}/p\mathbb{Z}$ when p is prime.

Lemma 4.1. *Let $p \in \mathbb{N}$ be a prime number. Then $R = \mathbb{Z}/p\mathbb{Z}$ is a field.*

Proof. Let $a \in R^*$. Then $\gcd(a, p) = 1$, so by the Euclidean algorithm, there are integers $s, t \in \mathbb{Z}$ such that $as + pt = 1$. When reduced modulo p, we see that $as \equiv 1 \pmod{p}$. Thus $\mathbb{Z}/p\mathbb{Z}$ is a commutative ring with unity such that every nonzero element is a unit. Thus $\mathbb{Z}/p\mathbb{Z}$ is a field. □

Theorem 4.2. *Let $p \in \mathbb{N}$ be prime and set $R = \mathbb{Z}/p\mathbb{Z}$. Let $\alpha \in$ {atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic}. Let $\beta \in$ {associate, strongly associate, very strongly associate}. Then for any $n \in \mathbb{N}$, we have*:

(1) *R is τ_n-α.*

(2) *R satisfies τ_n-ACCP.*

(3) *R is a τ_n-BFR.*

(4) *R is a τ_n-α-β-UFR.*

(5) *R is a τ_n-α-HFR.*

(6) *R is a τ_n-β-FFR.*

(7) *R is a τ_n-β-WFFR.*

(8) *R is a τ_n-α-β-df ring.*

Proof. (1) Let $a \in R$ with a a nonunit. Then by Lemma 4.1, $a = 0$ since all nonzero elements are units in a field. The only τ_n-factorizations are $0 = \lambda 0$ since there are no other nonzero nonunits. Furthermore, R is a field, so (0) is a maximal ideal and therefore 0 is m-irreducible and thus τ_n-m-irreducible. Fields are integral domains, which are présimplifiable, so all of the other forms of τ_n-α coincide. Thus R is τ_n-α.

(2) The only proper ideal is (0) since R is a field, so it certainly satisfies ACCP and therefore τ_n-ACCP.

(3) There are no nonzero nonunits, so there can be no nontrivial τ_n-factorizations. Thus all τ_n-factorizations are trivial and have length 1, making R a τ_n-BFR.

(4)–(6) We know R is τ_n-α by (1). Moreover, 0 has only $0 = \lambda 0$ as a τ_n-factorization. Since R is a field, $0 \cong 0$, so we see this is the only factorization up to rearrangement and β. Hence R is a τ_n-α-β-UFR and a τ_n-α-HFR. Again, this is the only τ_n-factorization, not just the only τ_n-α factorization, so R is certainly a τ_n-β-FFR.

(7)–(8) R is a finite ring with p elements. Hence there are a finite number of τ_n- and τ_n-α-divisors in the whole ring. Thus R is a τ_n-β-WFFR and a τ_n-α-β-df ring. □

$\mathbb{Z}/p^e\mathbb{Z}$, where $e > 1$. For $\mathbb{Z}/m\mathbb{Z}$, with $m = p^e$ (where $e \in \mathbb{N}$ and p is prime), we found that $\mathbb{Z}/p^e\mathbb{Z}$ is présimplifiable, or equivalently very strongly associate. As in [Mooney 2016], we have the following, which we state without proof.

Lemma 4.3. *Let R be a présimplifiable ring. Let $a \in R^{\#}$ be a nonzero nonunit. Then the following are equivalent*:

(1) *a is τ_n-atomic.*

(2) *a is τ_n-strongly atomic.*

(3) *a is τ_n-m-atomic.*

(4) *a is τ_n-unrefinably atomic.*

(5) *a is τ_n-very strongly atomic.*

Lemma 4.4. *Let $R = \mathbb{Z}/p^e\mathbb{Z}$, where $p, e, n \in \mathbb{N}$ and p is prime. Then p is τ_n-m-atomic and therefore p is τ_n-atomic (-strongly atomic, -m-atomic, -unrefinably atomic, -very strongly atomic).*

Proof. Let $p \in R = \mathbb{Z}/p^e\mathbb{Z}$. We show that (p) is maximal. The following are equivalent:

- An element $a \in \mathbb{Z}/p^e\mathbb{Z}$ is a unit.
- $\gcd(a, p^e) = 1$.
- $\gcd(a, p) = 1$.
- p does not divide a.
- $a \notin (p)$.

Thus (p) is precisely the set of nonunits. If $J \supsetneq (p)$, then let $x \in J \setminus (p)$. Then p does not divide x, so $x \in J$ is a unit, and so $J = R$. This shows that (p) is a maximal ideal (not just among principal ideals). Thus p is m-atomic and therefore τ_n-m-atomic. Moreover, by Lemma 4.3 this means p is τ_n-atomic (-strongly atomic, -m-atomic, -unrefinably atomic, -very strongly atomic). □

Proposition 4.5. *Let $p, e, n \in \mathbb{N}$, where p is prime and $e > 1$. The only τ_n-atomic (-strongly atomic, -m-atomic, -unrefinably atomic, -very strongly atomic) elements of $R = \mathbb{Z}/p^e\mathbb{Z}$ are p and unit multiples of p.*

Proof. Let $a \in R$ be a τ_n-irreducible (equivalently, -strongly atomic, -m-atomic, -unrefinably atomic, -very strongly atomic) element. Since a must be a nonunit, we know $\gcd(a, p) = p > 1$. Therefore, $p \mid a$. Let j be the largest number of factors of p that we can factor out of a. That is, let j be the integer such that p^j divides a, but p^{j+1} does not divide a. Write $a = \lambda p^j$. Then $\gcd(\lambda, p) = 1$ or else $p^{j+1} \mid a$. This means $\lambda \in U(R)$. If $j > 1$, then $a = \lambda \cdot p^j = \lambda \cdot p \cdots p$ is a τ_n-factorization of a such that $(a) \neq (p)$. This means a is not τ_n-atomic and therefore a is also not τ_n-strongly atomic (-m-atomic, -unrefinably atomic, -very strongly atomic). Thus, $j = 1$ and $a = \lambda p$, showing any τ_n-atomic (or -strongly atomic, -m-atomic, -unrefinably atomic, -very strongly atomic) element of $R = \mathbb{Z}/p^e\mathbb{Z}$ must be a unit multiple of p. □

Theorem 4.6. *Let $R = \mathbb{Z}/p^e\mathbb{Z}$, where $p, e, n \in \mathbb{N}$ and p is prime. Then we have the following:*

(1) *R is τ_n-atomic.*

(2) *R is τ_n-strongly atomic.*

(3) *R is τ_n-m-atomic.*

(4) *R is τ_n-unrefinably atomic.*

(5) *R is τ_n-very strongly atomic.*

Proof. Let $a \in R$ be a nonunit. If a is not a unit, then $\gcd(a, p) > 1$; hence $p \mid a$. We let j represent the integer for which $p^j \mid a$, but p^{j+1} does not divide a. Thus $a = p^j \cdot \lambda$ for some $\lambda \in \mathbb{N}$. Moreover, p does not divide λ, so $\gcd(\lambda, p) = 1$ and λ is a unit. Then $a = \lambda p \cdots p$, where p occurs j times. Certainly $p \; \tau_n \; p$ for any $n \in N$ since $p - p = 0 \in (0) \subseteq I$ for any ideal I. Thus we have found a τ_n-atomic (-strongly atomic, -m-atomic, -unrefinably atomic, -very strongly atomic) factorization of a by Lemma 4.3. □

Proposition 4.7. *Let $R = \mathbb{Z}/p^e\mathbb{Z}$, where $p, e, n \in \mathbb{N}$ and p is prime. Let $\alpha \in \{$atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic$\}$ and let $\beta \in \{$associate, strongly associate, very strongly associate$\}$. Then we have the following:*

(1) *R is a τ_n-β-WFFR.*

(2) *R is a τ_n-α-β-idf ring.*

(3) *R satisfies τ_n-ACCP.*

Proof. This is immediate again since R is a finite ring. □

Remark. We note here that this ring nearly satisfies further τ_n-finite factorization properties; however, we have the following issue. For any $j \geq e$, we have $0 = p \cdots p = p^j$ is a τ_n-atomic (-strongly atomic, -m-atomic, -unrefinably atomic, -very strongly atomic) factorization of 0. This means that R fails to be a τ_n-BFR (or -α-HFR, -α-β-UFR, -β-FFR). We do, on the other hand, have some positive results for nonzero elements of $\mathbb{Z}/p^e\mathbb{Z}$.

Theorem 4.8. *Let $p, e, n \in \mathbb{N}$, where p is prime. Let $\alpha \in \{$atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic$\}$. Let $\beta \in \{$associate, strongly associate, very strongly associate$\}$. Let $a \in \mathbb{Z}/p^e\mathbb{Z}$, a nonzero nonunit. Then we have the following:*

(1) *Any two τ_n-α factorizations of a have the same length.*

(2) *The element a not only has a τ_n-α factorization, but it is unique up to rearrangement and β.*

(3) *The element a has a finite number of τ_n-factorizations up to rearrangement and β.*

(4) *There is a bound on the length of any τ_n-factorization of a.*

Proof. (1) Let $a \in R$ be a nonzero nonunit. We know by Theorem 4.6 that there is a τ_n-α factorization of a. As Proposition 4.5 demonstrated, p and unit multiples of p are the only τ_n-α elements in $\mathbb{Z}/p^e\mathbb{Z}$. Recall that from the construction of the τ_n-α factorization in Theorem 4.6, j is the unique integer such that $p^j \mid a$, but p^{j+1} does not divide a. It is clear then that any τ_n-α factorization of a must have precisely j factors, each being some unit multiple of p.

(2) By Proposition 4.5, the only τ_n-α elements are unit multiples of p. Now $\mathbb{Z}/p^e\mathbb{Z}$ is présimplifiable, so all choices of β are equivalent. Thus since all τ_n-α factorizations have the same length and all τ_n-α elements are β, it is clear that this τ_n-α factorization of a is unique.

(3) Since any τ_n-factorization of a is certainly a factorization of a, it suffices to show that there are only finitely many factorizations of a up to β. Again, let j be as in (1). We claim that j is the largest number of nonunit factors that any factorization can have. If each factor is a nonunit, then it must be divisible by p. By the definition of j, we have $p^j \mid a$, but p^{j+1} does not divide a. Thus there can be no more than j factors in any given factorization of a. In this way, all factorizations of a must come as some grouping of the j factors of p or some unit multiple of p. Hence the number of distinct factorizations up to β is certainly bounded by 2^j. A better bound would be $P(j)$, where $P(n)$ is the number of partitions of a set with n elements.

(4) Since there are only a finite number of τ_n-factorizations up to β, we can simply take the maximum length of these factorizations as the bound on the length of τ_n-factorizations of a. Alternatively, it is clear that j, as defined in the unique factorization in (1), is the longest possible τ_n-factorization since any other τ_n-factorization could be refined into this τ_n-α factorization and it would be at least as long. □

The above theorem shows that 0 is the only element preventing $\mathbb{Z}/p^e\mathbb{Z}$ from being a τ_n-α-β-UFR (or -α-HFR, -β-FFR, -BFR).

$\mathbb{Z}/m\mathbb{Z}$. When m has multiple distinct prime divisors, matters become more complicated. There are now nontrivial idempotent elements. For instance, consider $\mathbb{Z}/6\mathbb{Z}$ and the element 3. We can factor $3 = 3 \cdot 3 = 3 \cdot 3 \cdot 3 = \cdots$. Often the solution to dealing with issues that arise from idempotents is using U-factorization, as in [Mooney 2015b]. We are still able to say a few things about certain finite factorization properties in the affirmative, but further research will need to be conducted to completely answer this question.

We begin with a known result which sheds some light on the situation. If $\gcd(n, m) = 1$, then $(n) = R$ and we have the usual factorization since $\tau_n = \tau_d$, where $\tau_d = R^\# \times R^\#$ yields the usual factorizations. This situation was discussed in [Anderson and Valdes-Leon 1996] and we refer the reader here for the traditional case.

Proposition 4.9. *Let $R = \mathbb{Z}/m\mathbb{Z}$, where $m, n \in \mathbb{N}$. Let $\alpha \in \{$atomic, strongly atomic, m-atomic, unrefinably atomic, very strongly atomic$\}$. Let $\beta \in \{$associate, strongly associate, very strongly associate$\}$. Then we have the following:*

1. *R is a τ_n-β-WFFR.*
2. *R is a τ_n-α-β-idf ring.*
3. *R satisfies τ_n-ACCP.*

Proof. This is immediate again since R is a finite ring. □

Theorem 4.10. *Let* $\alpha \in$ {*atomic*, *strongly atomic*, *m-atomic*, *unrefinably atomic*, *very strongly atomic*} *and* $\beta \in$ {*associate*, *strongly associate*, *very strongly associate*}. *Let* $R = \mathbb{Z}/p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}\mathbb{Z}$, *where* $p_i, e_i, n, k \in \mathbb{N}$ *with* p_i *primes. Then we have the following*:

(1) *If* $k = 1$, *then* R *is as in the previous subsection.*

(2) *If* $e_i \neq 1$ *for at least one* i *and* $k > 1$, *then we have the following*:

 (a) *R fails to be a* τ_n*-BFR.*
 (b) *R fails to be a* τ_n*-*β*-FFR.*
 (c) *R fails to be a* τ_n*-*α*-HFR.*
 (d) *R fails to be a* τ_n*-*α*-*β*-UFR.*

(3) *If* $e_i = 1$ *for all* $1 \leq i \leq k$, *then* R *is a direct product of fields and we have the following*:

 (a) *R is not* τ_n*-unrefinably atomic* (*or -very strongly atomic*).
 (b) *R fails to be a* τ_n*-BFR.*
 (c) *R fails to be a* τ_n*-*β*-FFR.*
 (d) *R fails to be a* τ_n*-*α*-HFR.*
 (e) *R fails to be a* τ_n*-*α*-*β*-UFR.*

Proof. (1) is immediate.

(2) After reordering the primes if necessary, we may assume that $e_1 > 1$. Then consider the element $(0, 1, \ldots, 1)$ and the τ_n-factorizations

$$(0, 1, \ldots, 1) = (p, 1, \ldots, 1) \cdots (p, 1, \ldots, 1) = (p, 1, \ldots, 1)^j,$$

where $j \geq e_1$. We notice that this is indeed a τ_n-factorization for any choice of ideal (n) since $(p, 1, \ldots, 1) - (p, 1, \ldots, 1) = (0, 0, \ldots, 0) \in (n)$. Furthermore, $(p, 1, \ldots, 1)$ is both regular (not a zero-divisor) and generates a principal ideal which is maximal. This means $(p, 1, \ldots, 1)$ is τ_n-α and we have demonstrated arbitrarily long τ_n-α factorizations of a nonunit. This proves R is not a τ_n-BFR (or -β-FFR, -α-HFR, -α-β-UFR).

(3a) We observe that the element $e := (0, 1, \ldots, 1)$ is neither τ_n-unrefinably atomic nor τ_n-very strongly atomic. To see this, consider the τ_n-factorization

$$e = (0, 1, \ldots, 1) = (0, 1, \ldots, 1)(0, 1, \ldots, 1).$$

This demonstrates that e is an idempotent and hence $e \not\cong e$. Thus we have found a nontrivial τ_n-factorization of e. We now consider any factorization of e. We have

$$e = (0, 1, \ldots, 1) = (a_{11}, a_{12}, \ldots, a_{1k})(a_{21}, a_{22}, \ldots, a_{2k}) \cdots (a_{t1}, a_{t2}, \ldots, a_{tk}).$$

We have $0 = a_{11}a_{21} \cdots a_{t1}$ in $\mathbb{Z}/p_1^{e_1}\mathbb{Z}$, which is a field, so $a_{f1} = 0$ for some $1 \leq f \leq t$. In the other coordinates, we have factorizations of 1, and thus a_{ij} must be a unit for each i and $j \geq 2$. This tells us that any factorization of e must have a factor of the form $(0, \lambda_2, \ldots, \lambda_k)$, where $\lambda_2, \ldots, \lambda_k$ are units. But this means

$$e = (0, 1, \ldots, 1) = (1, \lambda_2^{-1}, \ldots, \lambda_k^{-1})(0, \lambda_2, \ldots, \lambda_k).$$

This factor is a strong associate of e which is neither τ_n-unrefinably atomic nor τ_n-very strongly atomic. Thus there is no possible τ_n-unrefinably atomic or τ_n-very strongly atomic factorization of e. On the other hand, $R/(e) \cong \mathbb{Z}/p_1\mathbb{Z}$, which is a field, and R is a strongly associate ring, so e is τ_n-atomic (-strongly atomic, -m-atomic).

(3b–3e) We again consider $e := (0, 1, \ldots, 1)$. We observe that $e = e^2 = e^3 = \cdots = e^j = \cdots$ yields τ_n-factorizations for any $j > 1$. This demonstrates that R is neither a τ_n-FFR nor a τ_n-BFR. Furthermore, this gives τ_n-atomic (-strongly atomic, -m-atomic) factorizations of e of different lengths, proving R is not a τ_n-atomic- (-strongly atomic-, -m-atomic-) HFR or a τ_n-atomic- (-strongly atomic-, -m-atomic-) β-UFR. Lastly, from (3a), we know R is not even τ_n-unrefinably atomic (or -very strongly atomic), so it is certainly not a τ_n-unrefinably atomic- (or -very strongly atomic-) HFR or a τ_n-unrefinably atomic- (or -very strongly atomic-) β-UFR. □

5. Further thoughts on $\mathbb{Z}/m\mathbb{Z}$ with multiple prime factors

We have answered many questions regarding τ_n-finite factorization properties in the negative; however, there are certainly some remaining open questions. When there are multiple prime divisors, the question of whether $R = \mathbb{Z}/m\mathbb{Z}$ is τ_n-atomic (or -strongly atomic, -m-atomic) appears much more complicated and sensitive to the choice of the ideal picked. Further research would need to be done. Indeed, this question appears difficult even in the integers; see [Florescu 2013; Hamon 2007]. For fixed $n \in \mathbb{Z}$, τ_n-atomicity and τ_n-finite factorization properties, even for small n, have been and continue to be studied in depth in $\mathbb{Z}$, especially by Reyes M. Ortiz Albino and many of his students at The University of Puerto Rico at Mayagüez. It seems fertile ground for future research.

The fact that $\mathbb{Z}/m\mathbb{Z}$ is strongly associate simplifies (or at least unifies) some of these questions to make it more tractable. The existence of idempotent elements when m has multiple prime divisors suggests that looking at τ-U-factorization, as in [Mooney 2015b], may be a better path to take. The τ-U-factorizations are particularly effective in dealing with direct products of rings. It was often idempotent elements that were preventing the ring from satisfying further τ_n-finite factorization properties. As initiated by C. R. Fletcher [1969; 1970] and studied extensively by M. Axtell, S. Forman, N. Roersma, and J. Stickles [Axtell 2002; Axtell et al. 2003], the method of U-factorizations is helpful for this. When using U-factorization, rings like $\mathbb{Z}/6\mathbb{Z}$ go from not being even bounded factorization rings ($3 = 3^i$ for all i) to being U-unique factorization rings.

Acknowledgments

The authors would like to thank Viterbo University, in particular, the Viterbo Summer Research Fellowship Program which provided the funding to carry out this research in the summer of 2014. Mooney would also like to acknowledge the work done at The University of Iowa with the VIGRE REU program under the supervision of Professor Daniel D. Anderson, which tackled this problem over $\mathbb{Z}$ and provided the inspiration for this particular study over $\mathbb{Z}/m\mathbb{Z}$ as a possible project suitable for undergraduate research. The authors would also like to thank the referee for diligent work and careful reading of the article. Their suggestions have improved the quality of the article.

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Received: 2014-09-27 Revised: 2015-04-07 Accepted: 2015-06-06

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2016 vol. 9 no. 3