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On weak lattice point visibility

Neil R. Nicholson and Rebecca Rachan

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We say that a point Q in a specific rectangular array of lattice points is weakly visible from a lattice point P not in the array if no point in the array other than Q lies on the line connecting the external point P to Q. A necessary and sufficient condition for determining if a point in the array is weakly viewable by the external point, as well as the number of points that are weakly visible from the external point, is determined.

1. Introduction

Imagine a photographer attempting to capture a picture in which every member of a band in a rectangular array formation is visible, with all persons, including the photographer, standing on lattice points. The photographer must stand at a fixed position, and each band member must have a straight-line view of the photographer, unobstructed by all other band members. Laison and Schick [2007] describe this situation and prove that there are positions for the photographer to stand but these may be quite far away from the marching band. If the photographer decides to stand closer, how can we determine which band members she can see?

These are examples of the questions arising in lattice point visibility that have been investigated for decades. For example, another question that has gotten much attention considers two sets A and B of lattice points. When is every point in A visible from every point in B? Are there relationships between the sizes of the sets when this is the case, and if so, as the set B grows does its size act predictably? Much work has been done looking at questions such as these [Adhikari and Granville 2009; Adhikari and Balasubramanian 1996; Adhikari and Chen 1999; 2002; Chen and Cheng 2003; Herzog and Stewart 1971].

Here, we fix this second set to be a single point P, playing the role of the photographer trying to see the members of the marching band, those points in set A. In [Nicholson and Sharp 2010], a lower bound was placed on the distance P must be from a rectangular array of lattice points to weakly view every point in the array.

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Our main result can be used to prove this result in an alternate manner and provides another tool to address one of the primary questions in weak visibility: is there a formula for this minimum distance only dependent upon the dimensions of the lattice points?

2. Definitions

In this paper, all points are assumed to be lattice points in the first quadrant. Let $m, n \in \mathbb{Z}^+$ with $n \leq m$. Define $\Delta_{m,n} = \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$. We say $Q \in \Delta_{m,n}$ is *weakly visible* from a point $P \notin \Delta_{m,n}$ if no other point in $\Delta_{m,n}$ lies on the segment PQ.

It was proven in [Nicholson and Sharp 2010] that the $m \times m$ square of lattice points with its lower left-hand corner on (m, n) (that is, the square of lattice points with corners (m, n), $(m, n+m-1)$, $(2m-1, n)$, and $(2m-1, n+m-1)$), called the *adjacency square to* $\Delta_{m,n}$ and denoted $\mathrm{Adj}_{m,n}$, contains no point that weakly views every point in $\Delta_{m,n}$. As a corollary to this, there is a lower bound that can be placed on how close a point viewing every point in $\Delta_{m,n}$ can be to $\Delta_{m,n}$:

Theorem 2.1 [Nicholson and Sharp 2010]. *If a point P weakly views every point in $\Delta_{m,n}$, then P is at least $\sqrt{m^2+1}$ units from (m, n).*

What follows is a complete classification of which points in $\Delta_{m,n}$ are weakly visible from a general point $P \notin \Delta_{m,n}$ as well as two corollaries that follow from that classification.

3. Determining weak visibility

We begin this section with our main result: necessary and sufficient conditions for a point $Q \in \Delta_{m,n}$ to be weakly visible from a point $P \notin \Delta_{m,n}$.

Theorem 3.1. *The point $Q = (x_0, y_0) \in \Delta_{m,n}$ is not weakly visible by the point $P = (a, b)$ if and only if all of the following conditions hold*:

(1) $\gcd(a - x_0, b - y_0) > 1$.

(2) $m - x_0 \geq (a - x_0)/\gcd(a - x_0, b - y_0)$.

(3) $n - y_0 \geq (b - y_0)/\gcd(a - x_0, b - y_0)$.

Proof. Suppose Q is not weakly viewable by P. Then, there exist $t \geq 1$ points on the interior of the segment PQ, and let $R = (x_1, y_1)$ be the first of these points to the right of Q. Thus,

$$\begin{aligned} b - y_0 &= (y_1 - y_0)(t+1), \\ a - x_0 &= (x_1 - x_0)(t+1), \end{aligned} \tag{1}$$

implying that

$$\gcd(a-x_0, b-y_0) \geq t+1 \\ > 1. \tag{2}$$

Moreover, in order to have $(x_1, y_1) \in \Delta_{m,n}$, we have

$$\begin{aligned} m - x_0 &\geq x_1 - x_0 \\ &= \frac{a-x_0}{t+1} \\ &\geq \frac{a-x_0}{\gcd(a-x_0, b-y_0)}, \end{aligned} \tag{3}$$

with the third property following similarly.

Now, assume the three properties hold and $d = \gcd(a-x_0, b-y_0) > 1$, with

$$\begin{aligned} a - x_0 &= dp, \\ b - y_0 &= dq. \end{aligned} \tag{4}$$

We claim that $(x_0 + p, y_0 + q) \in \Delta_{m,n}$ lies on the segment PQ. To see this, note that the slope of PQ is q/p, so that PQ has equation

$$y - y_0 = \frac{q}{p}(x - x_0). \tag{5}$$

The point $(x_0 + p, y_0 + q)$ satisfies this equation, and

$$\begin{aligned} x_0 + p &= x_0 + \frac{a-x_0}{d} \\ &\leq x_0 + m - x_0 \\ &= m. \end{aligned} \tag{6}$$

Similarly, $y_0 + q \leq n$, showing $(x_0 + p, y_0 + q) \in \Delta_{m,n}$ and consequently that Q is not weakly viewable by P. □

What points then can a particularly chosen external point P weakly view? It is only natural to insist P lies strictly above the line $y = n$ and to the right of the line $x = m$. Thus, the closest such point (with distance measured to the nearest point, (m, n), in $\Delta_{m,n}$) would be $P = (m+1, n+1)$. The aforementioned result from [Nicholson and Sharp 2010] guarantees P cannot weakly view every point in $\Delta_{m,n}$ (for sufficiently large values of m and n). Which points then can P weakly view? Corollary 3.2 follows immediately from Theorem 3.1.

Corollary 3.2. *The point* $(x, 1) \in \Delta_{m,n}$ *is weakly viewable by the point* $P = (m+1, n+1)$ *if and only if* $\gcd((m+1) - x, n) = 1$.

This corollary states that the number of points in the first row of $\Delta_{m,n}$ that are weakly visible by P is the number of positive integers less than or equal to m that are relatively prime to n. This is a variation of the *Euler totient function* (or *Euler phi function*, $\phi(m)$, defined on positive integers m as the number of positive integers less than or equal to m that are relatively prime to m). For $n \leq m$, call this $\phi(n, m)$, precisely the number of points in the first row of $\Delta_{m,n}$ weakly viewable by this particular point P. This allows us to count the total number of points in $\Delta_{m,n}$ that P weakly views:

Corollary 3.3. *The number of points of $\Delta_{m,n}$ weakly viewable by the point $P = (m+1, n+1)$ is $\sum_{i=1}^{n} \phi(i, m)$.*

Proof. The number of points in the j-th row of $\Delta_{m,n}$ that are weakly viewable by P is $\phi(n - j + 1, m)$. □

We conclude by noting that the main question, amongst numerous other interesting questions, related to the results here remains open. Is there a formula dependent only upon m and n for the point closest to $\Delta_{m,n}$ that weakly views every point of $\Delta_{m,n}$? Such a formula would lend itself not only to deeper development in other lattice point visibility questions and graph theory but would potentially have applications in a multitude of fields [Ghosh and Goswami 2013].

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