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We study the vertex-connectivity and edge-connectivity of the zero-divisor graph Γ_R associated to a finite commutative ring R . We show that the edge-connectivity of Γ_R always coincides with the minimum degree, and that vertex-connectivity also equals the minimum degree when R is nonlocal. When R is local, we provide conditions for the equality of all three parameters to hold, give examples showing that the vertex-connectivity can be much smaller than minimum degree, and prove a general lower bound on the vertex-connectivity.

1. Introduction

Let R be a commutative ring with unit element $1 \neq 0$. The set of zero-divisors in R does not in general possess a convenient algebraic structure; hence, nonalgebraic methods are often needed to study this set. One attempt in this direction involves the so-called *zero-divisor graph* Γ_R , whose definition was first given by Beck [1988] and later adjusted slightly by Anderson and Livingston [1999]. The vertices of Γ_R are precisely the nonzero zero-divisors of R , with two vertices adjacent if and only if the product of the ring elements they represent is zero. The idea is that by studying combinatorial properties of Γ_R , one might hope to draw conclusions about the structure of the set of zero-divisors in R . Since the paper [Anderson and Livingston 1999], considerable work has been done on this topic; for details, see the recent survey articles [Anderson et al. 2011; Coykendall et al. 2012].

One of the first results proved was that for any R , the graph Γ_R is connected, and in fact has diameter at most 3 [Anderson and Livingston 1999, Theorem 2.3]. A more refined combinatorial notion than connectedness is that of *connectivity*. For a graph G , the *vertex-connectivity*, denoted $\kappa(G)$, is the size of the smallest subset of vertices whose removal renders the graph disconnected or leaves a single vertex, while the *edge-connectivity*, denoted $\lambda(G)$, is the size of the smallest subset of edges whose removal renders the graph disconnected. In general, connectivity

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of either type is rather difficult to compute; however, when graphs have a lot of symmetry — as is the case with zero-divisor graphs — it is sometimes possible to perform calculations, or at least give meaningful bounds.

The vertex connectivity of $\Gamma(\mathbb{Z}_n)$, with $n \geq 2$, was studied by Aaron Lauve [1999], who later discovered a mistake in his proof of the key formula in Section 4. The present article started as a project to correct this mistake, but later developed into a more comprehensive study of both the vertex- and edge-connectivity of $\Gamma(R)$ for arbitrary finite rings. An obvious starting point is the set of bounds $\kappa(G) \leq \lambda(G) \leq \delta(G)$ (see Proposition 2.2), valid for any graph G ; here $\delta(G)$ is the minimum degree of a vertex in G . In this article, we show that for all finite rings R , we have $\lambda(\Gamma_R) = \delta(\Gamma_R)$, and for nonlocal R , we also have $\kappa(\Gamma_R) = \delta(\Gamma_R)$. When R is local, however, $\kappa(\Gamma_R)$ is not nearly as well-behaved. For example, if R is a principal ideal domain, we always have $\kappa(\Gamma_R) = \delta(\Gamma_R)$; however, one can construct infinite families of rings for which $\kappa(\Gamma_R)$ is of order $\delta(\Gamma_R)^{3/4}$. We give more precise conditions under which $\kappa(\Gamma_R) = \delta(\Gamma_R)$ holds, and show that for any R , the vertex-connectivity $\kappa(\Gamma_R)$ must at least be of order $\delta(\Gamma_R)^{1/3}$.

Problems related to the focus of the present article have been studied in the recent literature. The structure of minimal vertex cuts in Γ_R was studied in [Coté et al. 2011]; however, that article does not investigate the *size* of such cuts, as is the focus of the present article. Our results are of a distinctly different flavor and thus complement rather than duplicate those of [Coté et al. 2011]. The papers [Axtell et al. 2011; Redmond 2012] are more focused in scope, and study graphs whose vertex-connectivity is 1.

2. Preliminaries

Throughout this paper, all rings are finite and commutative with $1 \neq 0$, and all graphs are finite, with no loops or multiple edges.

If R is a ring, we denote by $Z(R)$ the set of zero-divisors in R .

Definition 2.1. Let R be a ring. The *zero-divisor graph* of R , denoted Γ_R , is the graph whose vertex set is the set $Z(R) - \{0\}$, and in which $\{x, y\}$ is an edge if x and y are distinct zero-divisors of R such that $xy = 0$.

By abuse of notation, we blur the distinction between elements of $Z(R) - \{0\}$ and elements of $V(\Gamma_R)$. For $x \in Z(R) - \{0\}$, we denote by $\text{ann } x$ the annihilator of x . Hence, the degree of x (viewed as a vertex of Γ_R) is $|\text{ann } x - \{0, x\}|$.

We also recall various conventions and definitions from graph theory; see [West 1996] or any reference on graph theory for further details. For a graph G , we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. For a vertex v , we denote by $N_G(v)$ (or simply $N(v)$ if the context is clear) the set of neighbors of v in G . We denote by $\delta(G)$ the minimum vertex degree in G .

If $S \subseteq V(G)$, we write $G - S$ to denote the graph with vertex set $\bar{S} = V(G) - S$ and edge set $E(G) - \{[x, y] : \{x, y\} \cap S \neq \emptyset\}$. If $T \subseteq E(G)$ is any subset, we denote by $G - T$ the graph with vertex set $V(G)$ and edge set $E(G) - T$. A *vertex cut* is a subset $S \subseteq V(G)$ such that $G - S$ is disconnected, and a *disconnecting set of edges* of G is a subset $T \subseteq E(G)$ such that the graph $G - T$ is disconnected; an *edge cut* is a disconnecting set of edges which is minimal (with respect to inclusion). Writing $[A, B]$ for the set of edges in G with one endpoint in each of the subsets A, B of $V(G)$, it is easily shown (see [West 1996, Remark 4.1.8]) that any edge cut in G must be of the form $[S, \bar{S}]$ for some subset $S \subseteq V(G)$. The *vertex-connectivity* of G , denoted $\kappa(G)$, is the size of the smallest set $S \subseteq V(G)$ such that S is a vertex cut or $G - S$ has only one vertex. Similarly, the *edge-connectivity* of G , denoted $\lambda(G)$, is the size of the smallest edge cut in G . For convenience, we write κ_R (respectively, λ_R, δ_R) instead of $\kappa(\Gamma_R)$ (respectively, $\lambda(\Gamma_R), \delta(\Gamma_R)$). A well-known result relating these parameters is the following statement, due to Whitney.

Proposition 2.2 [West 1996, Theorem 4.1.9]. *For any graph G , we have*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

3. Results

Theorem 3.1. *Let R be a finite nonlocal ring. Then $\kappa_R = \lambda_R = \delta_R$.*

Proof. By the structure theorem for Artin rings, $R \cong R_1 \times \dots \times R_k$, where $k \geq 2$ and each R_i is a finite local ring. In light of Proposition 2.2, it suffices to show $\kappa_R \geq \delta_R$. To this end, let $S \subseteq V(\Gamma_R)$ be a subset with $|S| < \delta_R$; we will show that $H = \Gamma_R - S$ is connected. For i , with $1 \leq i \leq k$, define

$$C_i = \{(0, \dots, 0, a_i, 0, \dots, 0) \in R_1 \times \dots \times R_k : a_i \in Z(R_i) - \{0\}\}.$$

We claim that every vertex in H is adjacent to a vertex in $C_i \cap V(H)$ for some $1 \leq i \leq k$. Since vertices of C_i are clearly adjacent to vertices of C_j when $i \neq j$, it will then follow that H is connected. Toward this goal, suppose $b = (b_1, \dots, b_k) \in V(H)$, and fix i , with $1 \leq i \leq k$, such that $b_i \neq 0$. If we define $b' = (1, \dots, 1, b_i, 1, \dots, 1)$, then clearly $N_{\Gamma_R}(b') \subseteq C_i$. In particular, this implies $|C_i| \geq \delta > |S|$, so H must contain some vertex $v \in N_{\Gamma_R}(b')$. Since $N_{\Gamma_R}(b) \supseteq N_{\Gamma_R}(b')$, we see that $v \in N_{\Gamma_R}(b) \cap C_i$, as desired. □

From this point on, R will denote a finite local ring with maximal ideal \mathfrak{m} . Since R is Artinian, it follows from Nakayama's lemma (see [Atiyah and Macdonald 1969, Proposition 8.6]) that $\mathfrak{m}^n = 0$ for some positive integer n . We will reserve the symbol r for the smallest $n > 0$ satisfying this property. If $r = 1$, then R is a field and Γ_R is the empty graph. If $r = 2$, then Γ_R is a complete graph; so clearly $\kappa_R = \lambda_R = \delta_R = |\mathfrak{m}| - 2$. For the balance of the article, we assume $r \geq 3$, so in

particular, $\mathfrak{m}^2 \neq 0$. Since $\mathfrak{m}^{r-1} \subseteq \text{ann } \mathfrak{m}$, it follows immediately that $A_R = \text{ann } \mathfrak{m} - \{0\}$ is nonempty, and also that Γ_R is not complete. Viewed as a subset of $V(\Gamma_R)$, we have that A_R is a dominating set in Γ_R . Clearly any vertex cut in Γ_R must contain A_R ; thus, writing $\alpha_R = |A_R|$ and using Proposition 2.2, we have the elementary bounds

$$\alpha_R \leq \kappa_R \leq \lambda_R \leq \delta_R. \tag{1}$$

The following condition is important in that its presence forces all the inequalities in (1) to be equalities, but its absence typically has the opposite effect:

$$\text{There exists } x \in \mathfrak{m} \text{ such that } \text{ann } x = \text{ann } \mathfrak{m}. \tag{2}$$

Proposition 3.2. *Suppose condition (2) holds. Then $\alpha_R = \kappa_R = \lambda_R = \delta_R$.*

Proof. If $x^2 = 0$, then $x \in \text{ann } x = \text{ann } \mathfrak{m}$. Thus, $\mathfrak{m} = \text{ann } x = \text{ann } \mathfrak{m}$, and so $\mathfrak{m}^2 = 0$. Hence, we may assume $x^2 \neq 0$. In this case,

$$\delta_R \leq \deg(x) = |\text{ann } x - \{x, 0\}| = |\text{ann } x - \{0\}| = |\text{ann } \mathfrak{m} - \{0\}| = \alpha_R. \quad \square$$

If R is a principal ideal ring, condition (2) is certainly satisfied; thus, we have this:

Corollary 3.3. *Let p be a prime number and $n \geq 3$. Then*

$$\kappa(\mathbb{Z}/p^n\mathbb{Z}) = \lambda(\mathbb{Z}/p^n\mathbb{Z}) = p - 1.$$

It turns out that for local rings, the edge-connectivity is much better behaved than the vertex-connectivity. Recalling that vertices of A_R are dominant in Γ_R , the determination of λ_R is strictly graph-theoretic and follows immediately from the following easily verified fact:

Proposition 3.4. *Let G be a graph with a dominant vertex. Then $\lambda(G) = \delta(G)$.*

Proof. Choose $S \subseteq V(\Gamma_R)$ such that $T = [S, \bar{S}] \subseteq E(\Gamma_R)$ is an edge cut. We may assume without loss of generality that \bar{S} contains a dominant vertex v . Since v is adjacent to all vertices of S , we must have $|T| \geq |S|$. On the other hand, every vertex in S has at least $\delta - |S| + 1$ neighbors in \bar{S} ; so $\delta \geq |T| \geq |S|(\delta - |S| + 1)$. Rearranging the inequality $|S|(\delta - |S| + 1) \leq \delta$ gives $\delta(|S| - 1) \leq |S|(|S| - 1)$. If $|S| > 1$, then cancellation gives $\delta \leq |S|$ and so $|S| = |T| = \delta$. If $|S| = 1$, then all edges incident at the sole vertex in S must be in T , so $|T| = \delta$ in this case also. \square

Corollary 3.5. *Let R be a local ring with $\mathfrak{m}^2 \neq 0$. Then $\lambda_R = \delta_R$.*

We now turn our attention to the vertex-connectivity of Γ_R . It is natural to ask how tight the bounds $\alpha_R \leq \kappa_R \leq \delta_R$ are. In the absence of condition (2), the lower bound is usually not met.

Proposition 3.6. *Let R be a local ring with $r \geq 4$ such that condition (2) fails. Then $\kappa_R > \alpha_R$.*

Proof. First suppose $r \geq 5$. Any vertex cut must contain A_R , so it suffices to show that $H = \Gamma_R - A_R$ is connected. Because $\mathfrak{m}^{r-1} = \mathfrak{m}^{r-2}\mathfrak{m} \neq 0$, there exists some $x \in \mathfrak{m}^{r-2}$ such that $x \notin A_R$. Moreover, x is a finite sum of products of the form uv , where $u \in \mathfrak{m}^{r-3}$ and $v \in \mathfrak{m}$. Since $x \neq 0$ and $A_R \cup \{0\}$ is an ideal (hence closed under addition), at least one of these products must not be in A_R . Thus, we may assume without loss of generality that $x = uv$, where $u \in \mathfrak{m}^{r-3}$ and $v \in \mathfrak{m}$. Clearly u and v are also vertices of H , and because $r \geq 5$, we have $ux \in \mathfrak{m}^{2r-5} \subseteq \mathfrak{m}^r = 0$, so u is adjacent to x in H .

We claim that there is a path in H from every $y \in V(H)$ to x . If $y = u$ or $y = x$, this is clear, so assume otherwise. Since condition (2) fails, y has a neighbor z in H , so $yz = 0$. Now consider the product zu . If $zu = 0$, then y, z, u, x is a path. If $zu \neq 0$ but $zu \in A_R$, then $zx = (zu)v = 0$ and y, z, x is a path. Finally, if $zu \neq 0$ and $zu \notin A_R$, then zu is a vertex of H ; moreover, $y(zu) = 0$ and $x(zu) = (xu)z = 0$, so y, zu, x is a path.

Now suppose $r = 4$. Then $\mathfrak{m}^4 = 0$ but $\mathfrak{m}^3 \neq 0$, so there exists $x \in \mathfrak{m}^2$ such that x is a vertex of $H = \Gamma_R - A_R$. It suffices to show that there is a path from any vertex of H to x . To this end, let y be a vertex of H distinct from x . Since condition (2) fails, y has a neighbor z in H , i.e., $yz = 0$. If $zm \subseteq A_R$, then $zm^2 = 0$ and z is adjacent to x . If $zm \not\subseteq A_R$, then there exists $w \in \mathfrak{m}$ such that zw is a vertex of H . Now zw is a neighbor of y ; however, $zw \in \mathfrak{m}^2$, so it is also a neighbor of x . \square

Remark. The hypothesis $r \geq 4$ in Proposition 3.6 is necessary: when $r = 3$, there exist rings R not satisfying condition (2) for which $\kappa_R = \alpha_R$ and others for which $\kappa_R > \alpha_R$.

As an example of the former, let \mathbb{F}_2 be the field with two elements and consider

$$R = \frac{\mathbb{F}_2[x, y]}{(x^2, y^2)}.$$

By abuse of notation, we will use elements of $\mathbb{F}_2[x, y]$ to describe the cosets they represent in R . Then $\mathfrak{m} = (x, y)$ has eight elements and $\mathfrak{m}^2 = \text{ann } \mathfrak{m} = \{0, xy\}$. Thus, Γ_R has seven vertices, with xy a dominant vertex; moreover, $\Gamma_R - \{xy\}$ is a graph on six vertices with three connected components $\{x, x + xy\}$, $\{y, y + xy\}$ and $\{x + y, x + y + xy\}$, so $\kappa_R = \alpha_R = 1$. Note also that for any $t \in R$, $\text{ann } t$ contains (t) . Since (t) has at least four elements for any $t \neq 0$, there is no way for the equality $\text{ann } t = \text{ann } \mathfrak{m}$ to hold for any $t \in V(\Gamma_R)$. Hence, condition (2) necessarily fails.

As an example of the latter, consider

$$R = \frac{\mathbb{F}_2[x, y, z, w]}{(x^2, y^2, z^2, w^2, xy, yz, zw, wx)}.$$

It is easily seen that R is a local ring satisfying $t^2 = 0$ for all $t \in R$, whose maximal ideal $\mathfrak{m} = (x, y, z, w)$ satisfies $\mathfrak{m}^3 = 0$, $\mathfrak{m}^2 \neq 0$. Moreover, $\text{ann } \mathfrak{m} = (xz, yw)$, so $\alpha_R = 3$. As in the previous example, $t \in \text{ann } t$ for all $t \in R$, so it is easily seen

that condition (2) is not satisfied. Now let $H = \Gamma_R - A_R$; we will show that H is connected, and hence that $\kappa_R > 3$. Observe first that every vertex of H is of the form $c_1x + c_2y + c_3z + c_4w + c_5xz + c_6yw$, where the c_i are elements of \mathbb{F}_2 , and c_1, \dots, c_4 are not all 0. Evidently each such vertex is adjacent to $c_1x + c_2y + c_3z + c_4w$. Since x, y, z, w, x is a cycle in H , it will suffice (to show that H is connected) to construct a path from any vertex of the form $c_1x + c_2y + c_3z + c_4w$ (with not all c_i equal to 0) to one of the vertices of the abovementioned cycle. If v_1, v_2 are distinct elements of $\{x, y, z, w\}$ which are adjacent in H , then $v_1 + v_2$ is adjacent to v_1 . If v_1, v_2 are not adjacent, then choose v_3 from this set, distinct from v_1 and v_2 ; then v_3 will be adjacent to $v_1 + v_2$. If v_1, v_2, v_3 are distinct elements of $\{x, y, z, w\}$, then we may assume without loss of generality that v_2 is adjacent to both v_1 and v_3 . It follows that $v_1 + v_2 + v_3$ is adjacent to v_2 . Finally, $x + y + z + w$ is adjacent to $x + z$. Thus H is connected, and so $\kappa_R > 3 = \alpha_R$.

The next family of examples shows that both bounds $\alpha_R \leq \kappa_R \leq \delta_R$ can be quite loose.

Proposition 3.7. *Let F be a field of order $f = 2^s$ and*

$$R = \frac{F[x, y, z]}{(x^2, y^2, z^2)}.$$

Then $\alpha_R = f - 1, \kappa_R = f^3 - 1$, and $\delta_R = f^4 - 2$.

Proof. Observe that R is a local ring with maximal ideal $\mathfrak{m} = (x, y, z)$ such that $t^2 = 0$ for all $t \in R$. Moreover, $\mathfrak{m}^2 = (xy, xz, yz), \mathfrak{m}^3 = (xyz),$ and $\mathfrak{m}^4 = 0$.

Clearly R is generated (as an F -vector space) by $\{1, x, y, z, xy, xz, yz, xyz\}$; from this description, it is easily seen that $|R| = f^8, |\mathfrak{m}| = f^7, |\mathfrak{m}^2| = f^4,$ and $|\mathfrak{m}^3| = f$. Also, $\text{ann } \mathfrak{m} = \mathfrak{m}^3$, so $\alpha_R = f - 1$. Now since $t^2 = 0$ for all $t \in R$, it follows that $\text{ann } t \supseteq (t)$; because $|\text{ann } t| \cdot |(t)| = |R|$, we have $|\text{ann } t| \geq |R|^{1/2} = f^4$ for all $t \in R$. Direct computation shows that $\text{ann } x = (x)$, so x is a vertex in Γ_R of minimum degree $\delta_R = f^4 - 2$.

Let $S = (\text{ann } x \cap \mathfrak{m}^2) - \{0\}$. Also, any element in $(x) - S - \{0\}$ is associate to x and hence has the same neighborhood in Γ_R ; in fact, $(x) - S - \{0\}$ is a clique and a connected component of $\Gamma_R - S$. Thus there is no path in $\Gamma_R - S$ from x to y , and so $\kappa_R \leq |S| = f^3 - 1$.

Now suppose $T \subseteq V(\Gamma_R)$ is a set of vertices such that $|T| < f^3 - 1$. Given $t \in \mathfrak{m}$, consider the multiplication-by- t map $\mathfrak{m}^2 \rightarrow t\mathfrak{m}^2$. This is an R -module homomorphism whose kernel is $\text{ann } t \cap \mathfrak{m}^2$; hence

$$|\mathfrak{m}^3| \geq |t\mathfrak{m}^2| = \frac{|\mathfrak{m}^2|}{|\text{ann } t \cap \mathfrak{m}^2|},$$

and so $|\text{ann } t \cap \mathfrak{m}^2| \geq |\mathfrak{m}^2|/|\mathfrak{m}^3| = f^3$. Taking into account that 0 and possibly t itself are elements of $\text{ann } t$, this implies that every vertex of $H = \Gamma_R - T$ has a

neighbor (in H) lying in \mathfrak{m}^2 . To show that H is connected, let a and b be vertices of H . Then a has a neighbor $c \in \mathfrak{m}^2$ in H and b has a neighbor $d \in \mathfrak{m}^2$ in H . Now $cd \in \mathfrak{m}^4 = 0$, so c and d are adjacent in H , proving that there exists a path from a to b .

This shows that $\kappa_R = f^3 - 1$. □

In the example of Proposition 3.7, κ_R is roughly $(1/|F|)\delta_R$, so by taking F to be arbitrarily large, we see that there is no hope for a general upper bound on κ_R which is linear in δ_R ; in fact, in this family, κ_R is roughly $\delta_R^{3/4}$. It is natural, then, to ask for the maximum value of a , with $0 < a \leq 3/4$, such that κ_R can be bounded below (for all finite rings R) by a function of order δ_R^a . As a first step in this direction, we offer this:

Proposition 3.8. *Let R be a finite ring. Then $\kappa_R \geq (\frac{1}{2}\delta_R)^{1/3} - (\sqrt{3})^{-1}$.*

The proof relies crucially on the following observation:

Lemma 3.9. *Let R be a ring and S a vertex cut of Γ_R such that $V(G)$ is the disjoint union of two nonempty sets A and B with no edges between A and B . Suppose $|S| < \delta_R$. If $a \in A$ and $b \in B$, then $ab \in S$, $|\text{ann } a| \geq |B|/|S|$ and $|\text{ann } b| \geq |A|/|S|$.*

Proof. The hypothesis $|S| < \delta_R$ implies that a has some neighbor $x \in A$ and that b has some neighbor $y \in B$. Then $ab \neq 0$, but ab is a neighbor of both $x \in A$ and $y \in B$; thus, $ab \in S$. Now let $B = \{b_1, \dots, b_n\}$. Since each of the products ab_1, \dots, ab_n is an element of S , some element $s \in S$ appears at least $|B|/|S|$ times in this list; without loss of generality, we may assume that $ab_1 = \dots = ab_k = s$, where $k \geq |B|/|S|$. Thus, $0, b_2 - b_1, \dots, b_k - b_1$ are distinct elements of $\text{ann } a$ and hence $|\text{ann } a| \geq k \geq |B|/|S|$. The proof of the remaining assertion is similar. □

Proof of Proposition 3.8. If $\kappa_R = \delta_R$, there is nothing to prove, so assume $\kappa_R < \delta_R$ and let $S \subseteq V(\Gamma_R) = \mathfrak{m} - \{0\}$ be a minimal vertex cut. Partition the vertices of $H = \Gamma_R - S$ into two disjoint nonempty sets A and B such that there are no edges between A and B ; we may assume without loss of generality that B is the larger of these two sets, i.e.,

$$|A| \leq \frac{|\mathfrak{m}| - 1 - |S|}{2} \leq |B|.$$

Now if $x \in A$ and $y \in B$, Lemma 3.9 implies that H contains no vertices from $\text{ann } x \cap \text{ann } y$. Since the zero element is not a vertex of Γ_R , we have, again using Lemma 3.9, that

$$|S| \geq |\text{ann } x \cap \text{ann } y| - 1 = \frac{|\text{ann } x| |\text{ann } y|}{|\text{ann } x + \text{ann } y|} - 1 \geq \frac{|B|/|S| \cdot |A|/|S|}{|\mathfrak{m}|} - 1.$$

Thus,

$$\begin{aligned} |S|^3 &\geq \frac{|A||B|}{|\mathfrak{m}|} - |S|^2 \geq |A| \frac{|\mathfrak{m}| - 1 - |S|}{2|\mathfrak{m}|} - |S|^2 \\ &= \frac{|A|}{2} - \frac{|S|}{2} \frac{|S| + 1}{|S|} \frac{|A|}{|\mathfrak{m}|} - |S|^2 \geq \frac{|A|}{2} - \frac{|S|}{2} - |S|^2. \end{aligned}$$

However, the neighbors of $x \in A$ in Γ_R are all members of $A \cup S$. Thus, $|A| + |S| \geq \delta_R + 1$ and so, continuing the calculation from above, we have

$$|S|^3 + |S|^2 + \frac{|S|}{2} \geq \frac{|A|}{2} \geq \frac{\delta_R - |S| + 1}{2},$$

which, upon rearrangement, gives

$$2(|S|^3 + |S|^2 + |S| + \frac{1}{2}) \geq \delta_R.$$

Hence, $2(|S| + 1/\sqrt{3})^3 \geq \delta_R$. Rearranging the inequality gives the desired result. \square

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References

- [Anderson and Livingston 1999] D. F. Anderson and P. S. Livingston, “The zero-divisor graph of a commutative ring”, *J. Algebra* **217**:2 (1999), 434–447. MR 2000e:13007 Zbl 0941.05062
- [Anderson et al. 2011] D. F. Anderson, M. C. Axtell, and J. A. Stickles, Jr., “Zero-divisor graphs in commutative rings”, pp. 23–45 in *Commutative algebra—Noetherian and non-Noetherian perspectives*, edited by M. Fontana et al., Springer, New York, 2011. MR 2012b:13023 Zbl 1225.13002
- [Atiyah and Macdonald 1969] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA, 1969. MR 39 #4129 Zbl 0175.03601
- [Axtell et al. 2011] M. Axtell, N. Baeth, and J. Stickles, “Cut vertices in zero-divisor graphs of finite commutative rings”, *Comm. Algebra* **39**:6 (2011), 2179–2188. MR 2012i:13043 Zbl 1226.13007
- [Beck 1988] I. Beck, “Coloring of commutative rings”, *J. Algebra* **116**:1 (1988), 208–226. MR 89i:13006 Zbl 0654.13001
- [Coté et al. 2011] B. Coté, C. Ewing, M. Huhn, C. M. Plaut, and D. Weber, “Cut-sets in zero-divisor graphs of finite commutative rings”, *Comm. Algebra* **39**:8 (2011), 2849–2861. MR 2012i:13014 Zbl 1228.13011
- [Coykendall et al. 2012] J. Coykendall, S. Sather-Wagstaff, L. Sheppardson, and S. Spiroff, “On zero-divisor graphs”, pp. 241–299 in *Progress in commutative algebra 2: Closures, finiteness and factorization*, edited by C. Francisco et al., Walter de Gruyter, Berlin, 2012. MR 2932598 Zbl 1243.13017
- [Lauve 1999] A. Lauve, *Zero-divisor graphs of finite commutative rings*, Senior thesis, University of Oklahoma, 1999.
- [Redmond 2012] S. P. Redmond, “Cut vertices and degree-one vertices of zero-divisor graphs”, *Comm. Algebra* **40**:8 (2012), 2749–2756. MR 2968909 Zbl 1263.13004
- [West 1996] D. B. West, *Introduction to graph theory*, Prentice Hall, Upper Saddle River, NJ, 1996. MR 96i:05001 Zbl 0845.05001

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akhtarr@miamioh.edu

*Department of Mathematics, Miami University,
Oxford, OH 45056, United States*

lucas@boldlee.com

*Department of Mathematics, Miami University,
Oxford, OH 45056, United States*

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