Enumeration of $m$-endomorphisms

Louis Rubin and Brian Rushton
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An $m$-endomorphism on a free semigroup is an endomorphism that sends every generator to a word of length $\leq m$. Two $m$-endomorphisms are combinatorially equivalent if they are conjugate under an automorphism of the semigroup. In this paper, we specialize an argument of N. G. de Bruijn to produce a formula for the number of combinatorial equivalence classes of $m$-endomorphisms on a rank-$n$ semigroup. From this formula, we derive several little-known integer sequences.

1. Introduction

Let $D$ be a nonempty set of symbols, and let $D^+$ be the set of all finite strings of one or more elements of $D$. That is, $D^+ = \{d_1 \cdots d_k : k \in \mathbb{N}, d_i \in D\}$. Paired with the operation of string concatenation, $D^+$ forms the free semigroup on $D$. If $d_1, \ldots, d_k \in D$, then we refer to the natural number $k$ as the length of the string $d_1 \cdots d_k$. Denote the length of $W \in D^+$ by $|W|$.

By a semigroup endomorphism (or, simply, an endomorphism) on $D^+$, we mean a mapping $\phi : D^+ \to D^+$ satisfying $\phi(W_1 W_2) = \phi(W_1)\phi(W_2)$ for all $W_1, W_2 \in D^+$. Note that if $\phi$ is an endomorphism on $D^+$ and $d_1, \ldots, d_k \in D$, then $\phi(d_1 \cdots d_k) = \phi(d_1) \cdots \phi(d_k)$; this shows that an endomorphism on $D^+$ is determined by its action on the elements of $D$. On the other hand, any mapping $f : D \to D^+$ extends uniquely to the endomorphism $\phi_f : D^+ \to D^+$ defined by $\phi_f(d_1 \cdots d_k) = f(d_1) \cdots f(d_k)$, and it is straightforward to verify that $\phi_f$ is an automorphism (that is, a bijective endomorphism) precisely when $f$ is a bijection on $D$.

Example 1. Let $D = \{a, b\}$, and let $f : D \to D^+$ be defined by $f(a) = ab$ and $f(b) = a$. Then, for example,

$$\phi_f(ababa) = f(a)f(b)f(a)f(b)f(a) = abaabaab.$$
all $d \in D$. Note that the mapping $\phi_f$ from Example 1 is an $m$-endomorphism for all $m \geq 2$. Now let $\Gamma$ be the set of all $m$-endomorphisms on $D^+$. That is,

$$\Gamma = \{ \phi \in \text{End}(D^+) : \phi(D) \subseteq R \},$$

where $R = \{ W \in D^+ : |W| \leq m \}$. Consider the set $\Omega$ consisting of all mappings $f : D \to R$. Then we may write

$$\Gamma = \{ \phi_f : f \in \Omega \}.$$

We can put the set $\Gamma$ into one-to-one correspondence with $\Omega$ by sending each $m$-endomorphism to its restriction to $D$. Moreover, if $|D| = n \in \mathbb{N}$, then the size of these sets is easily evaluated in view of the fact that $|R| = \sum_{i=1}^{m} n^i$. In particular, if $n > 1$, then $|R| = (n^{m+1} - n)/(n - 1)$, and

$$|\Gamma| = |\Omega| = \left(\frac{n^{m+1} - n}{n-1}\right)^n.$$

However, in this paper we are interested in counting the number of classes of $m$-endomorphisms under a particular equivalence relation. To motivate our definition of equivalence on $\Gamma$, we define a relation $\sim$ on $\Omega$ as follows:

$$f_1 \sim f_2 \iff \text{there exists a bijection } g : D \to D \text{ such that } f_2 \circ g = \phi_g \circ f_1.$$

As an exercise, the reader may wish to verify that $\sim$ satisfies the reflexive, symmetric, and transitive properties required of any equivalence relation. In Section 1.1, however, it will be shown that $\sim$ is a specific instance of a well-known equivalence relation induced by a group acting on a nonempty set.

**Example 2.** Let $f$ be as in Example 1 (with $D = \{a, b\}$). Consider the bijection $g : D \to D$ defined by $g(a) = b$ and $g(b) = a$. Now let $f_1 : D \to D^+$ be given by $f_1(a) = b$ and $f_1(b) = ba$. Then

$$(f_1 \circ g)(a) = f_1(g(a)) = f_1(b) = ba = g(a)g(b) = \phi_g(ab) = \phi_g(f(a)) = (\phi_g \circ f)(a),$$

$$(f_1 \circ g)(b) = f_1(g(b)) = f_1(a) = b = g(a) = \phi_g(a) = \phi_g(f(b)) = (\phi_g \circ f)(b),$$

which shows that $f \sim f_1$.

**Remark 3.** Perhaps a more intuitive illustration of $\sim$ is as follows. If we let $f$ and $f_1$ be as in Example 2, then the respective graphs of $f$ and $f_1$ are $\{(a, ab), (b, a)\}$ and $\{(a, b), (b, ba)\}$. But the graph of $f_1$ can be obtained by applying the bijection $g$ to each element of $D$ that appears in the graph of $f$. In other words,

$$\{(g(a), g(a)g(b)), (g(b), g(a))\} = \{(a, b), (b, ba)\}.$$

Since the graphs of $f$ and $f_1$ are “the same” up to a permutation of $a$ and $b$, we wish to consider these mappings equivalent, and $\sim$ provides the desired equivalence relation.
Extending $\sim$ to an equivalence relation on $\Gamma$ leads to the following definition. If $f, h \in \Omega$, then $\phi_f$ is combinatorially equivalent to $\phi_h$ if and only if there exists a bijection $g : D \to D$ such that $\phi_h \circ \phi_g = \phi_g \circ \phi_f$. To state precisely the aim of this paper: given a set of symbols $D$ with $|D| = n$, we wish to produce a formula for the number of equivalence classes in $\Gamma$ under the relation of combinatorial equivalence. To this end, we shall specialize an argument of N. G. de Bruijn [1972] (namely, that used for his Theorem 1) to produce a formula for the number of classes in $\Omega$ under the relation $\sim$. But it is easy to check that for all $f, h \in \Omega$, we have $f \sim h$ if and only if $\phi_f$ is combinatorially equivalent to $\phi_h$. Hence, there is a well-defined correspondence given by $[f] \leftrightarrow [\phi_f]$ between the equivalence classes in $\Omega$ and those in $\Gamma$, and it follows that our formula will also provide the number of $m$-endomorphisms on $D^+$ up to combinatorial equivalence. Moreover, once this formula is obtained, we can fix one of the variables $n, m$ and let the other run through the natural numbers in order to derive integer sequences, many of which appear to be little-known.

1.1. Group actions. For the reader’s convenience, we review group actions. The following material (through Proposition 4) is paraphrased from [Malik et al. 1997]. Let $G$ be a group and $S$ a nonempty set. A left action of $G$ on $S$ is a function $\cdot : G \times S \to S$, $(g, s) \mapsto g \cdot s$, such that, for all $g_1, g_2 \in G$ and for all $s \in S$,

1. $(g_1g_2) \cdot s = g_1 \cdot (g_2 \cdot s)$, where $g_1g_2$ denotes the product of $g_1, g_2$ in $G$, and
2. $e \cdot s = s$, where $e$ is the identity element of $G$.

A left action induces the well-known equivalence relation $E$ on the set $S$ given by

$$(a, b) \in E \iff g \cdot a = b \text{ for some } g \in G$$

for all $a, b \in S$. We refer to the equivalence classes under this relation as the orbits of $G$ on $S$. The following result (known as Burnside’s lemma) gives an expression for the number of these, provided that $G$ and $S$ are finite.

**Proposition 4** [Malik et al. 1997]. Let $S$ be a finite, nonempty set, and suppose there is a left action of a finite group $G$ on $S$. Then the number of orbits of $G$ on $S$ is

$$\frac{1}{|G|} \sum_{g \in G} |\{s \in S : g \cdot s = s\}|.$$

Thus, the number of orbits of $G$ on $S$ equals the average number of elements of $S$ that are “fixed” by an element of $G$. We now show that the relation $\sim$ from Section 1 is a specific instance of the relation $E$ described above. To see this, let $D$
be a finite nonempty set, and let \( \text{Sym}(D) \) denote the symmetric group on \( D \) (i.e., the group of all bijections on \( D \)). Then \( \text{Sym}(D) \) acts on the set \( \Omega \) according to the rule 
\[
g \cdot f = \phi_g \circ f \circ g^{-1}
\]
for all \( g \in \text{Sym}(D), f \in \Omega \). (One can easily verify that \( \cdot \) defined in this way is indeed a left action.) Now, for any \( f_1, f_2 \in \Omega \), we have
\[
f_1 \sim f_2 \iff f_2 \circ g = \phi_g \circ f_1 \text{ for some } g \in \text{Sym}(D)
\]
\[
\iff f_2 = \phi_g \circ f_1 \circ g^{-1} \text{ for some } g \in \text{Sym}(D)
\]
\[
\iff g \cdot f_1 = f_2 \text{ for some } g \in \text{Sym}(D)
\]
\[
\iff (f_1, f_2) \in E.
\]

It follows that the equivalence classes in \( \Omega \) under the relation \( \sim \) are just the orbits of \( \text{Sym}(D) \) on \( \Omega \). Enumerating the elements of \( \text{Sym}(D) \) by \( g_1, \ldots, g_n! \), we find the number of orbits to be
\[
\frac{1}{n!} \sum_{r=1}^{n!} |\{ f \in \Omega : f \circ g_r = \phi_{g_r} \circ f \}|. \tag{1}
\]

For any permutation \( g \) of a finite set, and for each natural number \( j \), let \( c(g, j) \) denote the number of cycles of length\(^1 \) \( j \) occurring in the cycle decomposition of \( g \). (This notation comes from [de Bruijn 1972].) The quantities \( c(g, j) \) will play a role in the evaluation of \( |\{ f \in \Omega : f \circ g_r = \phi_{g_r} \circ f \}| \), which occurs in the next section. Our evaluation is a modification of de Bruijn’s counting argument [1964, § 5.12].

2. Main results

We now produce a formula for the number of equivalence classes in \( \Omega \) under the relation \( \sim \). Let \( D \) be a finite set, and suppose that \( g \in \text{Sym}(D) \) is the product of disjoint cycles of lengths \( k_1, k_2, \ldots, k_\ell \), where \( k_1 \leq k_2 \leq \cdots \leq k_\ell \). Then the sequence \( k_1, k_2, \ldots, k_\ell \) is called the cycle type of \( g \). For example, if \( D = \{a, b, c, d, e\} \), then the permutation \( g = (a)(b, c)(d, e) \) has cycle type 1, 2, 2. The following lemma will be useful.

**Lemma 5.** Let \( D \) be a finite set, and let \( g \in \text{Sym}(D) \) have cycle type \( k_1, k_2, \ldots, k_\ell \). For each \( 1 \leq i \leq \ell \), select a single \( d_i \in D \) from the cycle corresponding to \( k_i \). (Thus, \( k_i \) is the smallest natural number such that \( g^{k_i}(d_i) = d_i \).) Now suppose that \( f \in \Omega \). Then \( f \circ g = \phi_g \circ f \) if and only if for each \( 1 \leq i \leq \ell \),

\[
(1) \quad (f \circ g^j)(d_i) = (\phi_g^j \circ f)(d_i) \text{ for all } j \in \mathbb{N},
\]

\[
(2) \quad f(d_i) \text{ is of the form } d'_1 \cdots d'_{k \leq m}, \text{ where } d'_1, \ldots, d'_{k} \in D \text{ each belong to a cycle in } g \text{ whose length divides } k_i.
\]

\(^1\)There should be no confusion between the notions of “string length” and “cycle length”.
Proof. First assume that $f \circ g = \phi_g \circ f$. Then condition (1) follows from an inductive argument. But $f(d_i) = f(g^{k_i}(d_i)) = \phi_g^{k_i}(f(d_i))$. Write $f(d_i) = d'_1 \cdots d'_k$, where $d'_1, \ldots, d'_k \in D$ and $k \leq m$. Then

$$d'_1 \cdots d'_k = \phi_g^{k_i}(d'_1 \cdots d'_k) = g^{k_i}(d'_1) \cdots g^{k_i}(d'_k).$$

In particular, for each $1 \leq t \leq k$, we have $d'_t = g^{k_i}(d'_t)$. This implies that $$(d'_1, g(d'_1), g^2(d'_1), \ldots, g^{k_i-1}(d'_1))$$
is a cycle whose length divides $k_i$. The conclusion follows.

Conversely, suppose that condition (1) holds. (Condition (2) is superfluous here.) Let $d \in D$. Then there exist $i, j \in \mathbb{N}$ such that $d = g^j(d_i)$. Now,

$$f(g(d)) = f(g(g^j(d_i))) = f(g^{1+j}(d_i)) = \phi_g^{1+j}(f(d_i)) = \phi_g(\phi_g^j(f(d_i))) = \phi_g(f(g^j(d_i))) = \phi_g(f(d)).$$

Therefore, $f \circ g = \phi_g \circ f$, so the proof is complete. \qed

Once again, suppose that $|D| = n$, and label the elements of $\text{Sym}(D)$ by $g_1, \ldots, g_n!$. For each $1 \leq r \leq n!$, we can find the number of $f \in \Omega$ satisfying

$$f \circ g_r = \phi_{g_r} \circ f.$$  

(2)

Suppose that $g_r$ has cycle type $k_{r_1}, k_{r_2}, \ldots, k_{r_{\ell_r}}$. For each $1 \leq i \leq \ell_r$, select a single element $d_{ri} \in D$ from the cycle corresponding to $k_{ri}$. Then Lemma 5 implies that any $f \in \Omega$ satisfying (2) is determined by its values on each $d_{ri}$. Hence, to find the number of $f$ satisfying (2), we need only count the number of possible images of $d_{ri}$ under such an $f$, and then take the product over all $i$. But the $m$ or fewer elements of $D$ comprising the string $f(d_{ri})$ must each belong to a cycle in the decomposition of $g_r$ whose length divides $k_{ri}$. For each $1 \leq k \leq m$, there are

$$\left(\sum_{j \mid k_{ri}} j c(g_r, j)\right)^k$$
choices of $f(d_{ri})$ such that $|f(d_{ri})| = k$. Hence, there are

$$\sum_{k=1}^{m} \left(\sum_{j \mid k_{ri}} j c(g_r, j)\right)^k$$
total choices of $f(d_{ri})$. Taking the product over all $i$, it follows that the number of $f$ satisfying (2) is

$$\prod_{i=1}^{\ell_r} \left(\sum_{k=1}^{m} \left(\sum_{j \mid k_{ri}} j c(g_r, j)\right)^k\right).$$  

(3)
Thus, we’ve evaluated \(|\{ f \in \Omega : f \circ g_r = \phi_{g_r} \circ f \}|\), and putting together (1) and (3) gives an expression for the number of equivalence classes in \(\Omega\) under the relation \(\sim\). Recalling that these classes are in one-to-one correspondence with the classes in \(\Gamma\) under the relation of combinatorial equivalence, we obtain our main result:

**Theorem 6.** If \(|D| = n\), then the number of \(m\)-endomorphisms on \(D^+\), up to combinatorial equivalence, is the value of

\[
\frac{1}{n!} \sum_{r=1}^{n} \left( \prod_{i=1}^{\ell_r} \left( \sum_{k=1}^{m} j(c(g_r, j))^k \right) \right),
\]

where \(g_1, \ldots, g_n\) are the elements of \(\text{Sym}(D)\), and \(k_{r1}, \ldots, k_{r\ell_r}\) is the cycle type of \(g_r\).

**Example 7.** Let \(D = \{a, b\}\). We find the number of classes of 1-endomorphisms on \(D^+\). The elements of \(\text{Sym}(D)\) (in cycle notation) are \(g_1 = (a)(b)\) and \(g_2 = (a, b)\). Evidently, \(c(g_1, 1) = 2\), \(c(g_2, 1) = 0\), and \(c(g_2, 2) = 1\). Using Theorem 6, there are

\[
\frac{1}{2} (c(g_1, 1)^2 + 2c(g_2, 2)) = \frac{1}{2} (2^2 + 2) = 3
\]

classes of 1-endomorphisms on \(D^+\). These are given by

\[
\begin{align*}
\{ &a \to a \}, \quad \{ &a \to b \} \quad \text{and} \quad \{ &a \to a \equiv a \to b \}.
\end{align*}
\]

We can extend the result of Example 7 by fixing \(n = 2\) and letting \(m\) be arbitrary. From (4), we find that the number of classes of \(m\)-endomorphisms on \(D^+\), where \(|D| = 2\), is

\[
\frac{1}{2} ((2^{m+1} - 2)^2 + (2^{m+1} - 2)).
\]

Running \(m\) through the natural numbers, we obtain values 3, 21, 105, 465, 1953, \ldots. This is the sequence A134057 in the On-line Encyclopedia of Integers [OEIS 1996]. However, for \(n = 3\), the number of classes of \(m\)-endomorphisms becomes

\[
\frac{1}{6} \left( \left( \frac{3^{m+1} - 3}{2} \right)^3 + 3m \frac{3^{m+1} - 3}{2} + 2 \frac{3^{m+1} - 3}{2} \right).
\]

Letting \(m = 1, 2, 3, 4, \ldots\) gives values 7, 304, 9958, 288280, \ldots. This sequence appears to be little-known, and has been submitted by the authors to the OEIS.

**2.1. An alternative formulation of Theorem 6.** We now present a slight rewording of Theorem 6. In order to compute the number of equivalence classes of \(m\)-endomorphisms (where \(|D| = n\)), we need not, in practice, consider each element of \(\text{Sym}(D)\) individually. Rather, we need only consider the cycle types of these permutations. The following well-known result gives the number of permutations in \(\text{Sym}(D)\) of a given cycle type.
Proposition 8 [Dummit and Foote 2004]. Let $|D| = n$, and let $g \in \text{Sym}(D)$. Suppose that $m_1, m_2, \ldots, m_s$ are the distinct integers appearing in the cycle type of $g$. For each $j \in \{1, 2, \ldots, s\}$, abbreviate $c_j = c(g, m_j)$. Let $C_g$ be the set of all permutations in $\text{Sym}(D)$ whose cycle type is that of $g$. Then

$$|C_g| = \frac{n!}{\prod_{j=1}^{s} c_j! m_j^{c_j}}. \quad (5)$$

For convenience, we shall say that $g \in \text{Sym}(D)$ fixes the mapping $f \in \Omega$ if and only if $f \circ g = \phi_g \circ f$. Now, two bijections in $\text{Sym}(D)$ with the same cycle type must fix the same number of $f \in \Omega$. Therefore, in order to derive an expression for the number of classes of $m$-endomorphisms on $D^+$, we can select a single representative in $\text{Sym}(D)$ of each possible cycle type, then determine the number of $f \in \Omega$ fixed by each representative using expression (3), multiply this number by the corresponding value of (5), and then sum up over all of our representatives and divide by $n!$. But the cycle types in $\text{Sym}(D)$ are precisely the integer partitions of $n$, namely, the nondecreasing sequences of natural numbers whose sum is $n$. If $p(n)$ denotes the number of integer partitions of $n$, then we may restate Theorem 6 as follows.

Corollary 9. Let $|D| = n$, and suppose that $g_1, \ldots, g_{p(n)} \in \text{Sym}(D)$ have distinct cycle types. Then the number of $m$-endomorphisms on $D^+$, up to combinatorial equivalence, is the value of

$$\frac{1}{n!} \sum_{r=1}^{p(n)} \left( \frac{\ell_r}{c_{g_r}} \prod_{i=1}^{\ell_r} \left( \sum_{k=1}^{m} \left( \sum_{j \mid k_r i} j c(g_r, j) \right)^k \right) \right), \quad (6)$$

where $k_{r1}, \ldots, k_{r \ell_r}$ is the cycle type of $g_r$, and $C_{g_r}$ is as in Proposition 8.

Example 10. To illustrate Corollary 9, we compute the number of classes of $m$-endomorphisms when $|D| = 4$. Let $D = \{a, b, c, d\}$. As previously mentioned, the cycle types in $\text{Sym}(D)$ are the integer partitions of 4:

$$1 + 1 + 1 + 1, \quad 1 + 1 + 2, \quad 2 + 2, \quad 1 + 3, \quad 4.$$

Hence, the bijections

$$g_1 = (a)(b)(c)(d), \quad g_2 = (a)(b)(c, d), \quad g_3 = (a, b)(c, d), \quad g_4 = (a)(b, c, d), \quad g_5 = (a, b, c, d)$$

encompass all possible cycle types in $\text{Sym}(D)$. Direct calculation using (5) yields

$$|C_{g_1}| = 1, \quad |C_{g_2}| = 6, \quad |C_{g_3}| = 3, \quad |C_{g_4}| = 8, \quad |C_{g_5}| = 6.$$

Thus, by Corollary 9, the number of classes of $m$-endomorphisms when $n = 4$ is

$$\frac{1}{24} \left( \Lambda_4^4 + 6 \Lambda_2^2 \Lambda_4 + 3 \Lambda_4^2 + 8m \Lambda_4 + 6 \Lambda_4 \right),$$

where $\Lambda_k = (k^{m+1} - k)/(k - 1)$. 

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**ENUMERATION OF $m$-ENDOMORPHISMS**

429
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Table 1. Values of (6) for \(n, m \leq 6\).

Proceeding along the lines of Example 10, we find that there are
\[
\frac{1}{120} \left( \Lambda_5^5 + 10 \Lambda_3^3 \Lambda_5 + 15m \Lambda_5^2 + 20 \Lambda_2^2 \Lambda_5 + 20 \Lambda_2 \Lambda_3 + 30m \Lambda_5 + 24 \Lambda_5 \right)
\]
classes of \(m\)-endomorphisms when \(n = 5\), and
\[
\frac{1}{720} \left( \Lambda_6^6 + 15 \Lambda_4^4 \Lambda_6 + 45 \Lambda_2^2 \Lambda_6^2 + 15 \Lambda_6^3 + 40 \Lambda_3^3 \Lambda_6 \\
+ 120m \Lambda_3 \Lambda_4 + 40 \Lambda_6^2 + 90 \Lambda_2^2 \Lambda_6 + 90 \Lambda_2 \Lambda_6 + 144m \Lambda_6 + 120 \Lambda_6 \right)
\]
classes of \(m\)-endomorphisms when \(n = 6\). Letting \(m\) run through \(\mathbb{N}\) in these cases, we again obtain sequences that are not well-known. Table 1 displays the values of (6) for \(n, m \leq 6\).

Remark 11. The sequence 1, 3, 7, 19, 47, 130, \ldots is sequence A001372 in [OEIS 1996].

3. Two natural variations

In this section, we highlight two natural variations of Corollary 9. First, we restrict our attention to endomorphisms on \(D^+\) that send each element of \(D\) to a string of length exactly \(m\). We then consider \(m\)-endomorphisms of the so-called free monoid, which contains the empty string. Expressions analogous to those in Section 2 are derived in each case.

3.1. \(m\)-uniform endomorphisms. Fix \(n, m \in \mathbb{N}\), and suppose that \(|D| = n\). Then \(\phi \in \text{End}(D^+)\) is called an \(m\)-uniform endomorphism if and only if \(|\phi(d)| = m\) for
each $d \in D$. In this section, we produce a formula for the number of $m$-uniform endomorphisms on $D^+$ up to combinatorial equivalence. To begin, let $g_1, \ldots, g_{p(n)} \in \text{Sym}(D)$ have distinct cycle types. We now put $R = \{W \in D^+ : |W| = m\}$ and take $\Omega$ to be the set of all mappings of $D$ into $R$. For each $1 \leq r \leq p(n)$, we ask for the number of $f \in \Omega$ satisfying

$$f \circ g_r = \phi_{g_r} \circ f.$$ 

Once again, if $g_r$ has cycle type $k_{r1}, \ldots, k_{r\ell_r}$, then for each $1 \leq i \leq \ell_r$ we select an element $d_{ri}$ from the cycle corresponding to $k_{ri}$, and count the number of possible values of $f(d_{ri})$. In this case, we must have $|f(d_{ri})| = m$, where the elements of $D$ comprising the string $f(d_{ri})$ each belong to a cycle whose length divides $k_{ri}$. Hence, there are

$$\left( \sum_{j \mid k_{ri}} j c(g_r, j) \right)^m$$

choices of $f(d_{ri})$, and multiplying over all $i$ yields

$$\prod_{i=1}^{\ell_r} \left( \sum_{j \mid k_{ri}} j c(g_r, j) \right)^m$$

as the value of $|\{f \in \Omega : f \circ g_r = \phi_{g_r} \circ f\}|$. Noting that permutations in $\text{Sym}(D)$ of the same cycle type fix the same number of $f \in \Omega$, we multiply by $|C_{g_r}|$, sum with respect to $r$, and divide by $n!$ to obtain the following.

**Corollary 12.** If $|D| = n$ and $g_1, \ldots, g_{p(n)} \in \text{Sym}(D)$ have distinct cycle types, then the number of $m$-uniform endomorphisms on $D^+$, up to combinatorial equivalence, is the value of

$$\frac{1}{n!} \sum_{r=1}^{p(n)} \left( |C_{g_r}| \prod_{i=1}^{\ell_r} \left( \sum_{j \mid k_{ri}} j c(g_r, j) \right)^m \right),$$

(7)

where $k_{r1}, \ldots, k_{r\ell_r}$ is the cycle type of $g_r$, and $C_{g_r}$ is as in Proposition 8.

When $n = 2$, the number of $m$-uniform endomorphisms on $D^+$, up to combinatorial equivalence, is

$$\frac{1}{2} (2^{2m} + 2^m).$$

Letting $m = 1, 2, 3, 4, \ldots$ gives values 3, 10, 36, 136, \ldots. This is the sequence A007582 from [OEIS 1996]. Moreover, when $n = 3$ there are

$$\frac{1}{6} (3^{3m} + 3 \cdot 3^m + 2 \cdot 3^m)$$

classes of $m$-uniform endomorphisms, and letting $m$ run through $\mathbb{N}$ gives the sequence 7, 129, 3303, 88641, \ldots, which is not well known. Continuing, the
$n = 1$  $n = 2$  $n = 3$  $n = 4$

$m = 1$  1  3  7  19

$m = 2$  1  10  129  2,836

$m = 3$  1  36  3,303  700,624

$m = 4$  1  136  88,641  178,981,696

$m = 5$  1  528  7,973,053  45,813,378,304

$m = 6$  1  2,080  64,570,689  11,728,130,323,456

$m = 1$  47  130

$m = 2$  83,061  3,076,386

$m = 3$  254,521,561  141,131,630,530

$m = 4$  794,756,352,216  6,581,201,266,858,896

$m = 5$  2,483,530,604,092,546  307,047,288,863,992,988,160

$m = 6$  7,761,021,959,623,948,401  14,325,590,271,500,876,382,987,456

<table>
<thead>
<tr>
<th>$n = 5$</th>
<th>$n = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 1$</td>
<td>47</td>
</tr>
<tr>
<td>$m = 2$</td>
<td>83,061</td>
</tr>
<tr>
<td>$m = 3$</td>
<td>254,521,561</td>
</tr>
<tr>
<td>$m = 4$</td>
<td>794,756,352,216</td>
</tr>
<tr>
<td>$m = 5$</td>
<td>2,483,530,604,092,546</td>
</tr>
<tr>
<td>$m = 6$</td>
<td>7,761,021,959,623,948,401</td>
</tr>
</tbody>
</table>

Table 2. Values of (7) for $n$, $m \leq 6$.

expressions when $n = 4$, 5, 6 are

\[
\frac{1}{24} \left( 4^{4m} + 6 \cdot 2^{2m} \cdot 4^m + 3 \cdot 4^{2m} + 8 \cdot 4^m + 6 \cdot 4^m \right),
\]

\[
\frac{1}{120} \left( 5^{5m} + 10 \cdot 3^m \cdot 5^m + 15 \cdot 5^m \cdot 2^m \cdot 5^m + 20 \cdot 2^m \cdot 3^m + 30 \cdot 5^m + 24 \cdot 5^m \right),
\]

\[
\frac{1}{720} \left( 6^{6m} + 15 \cdot 4^{4m} \cdot 6^m + 45 \cdot 2^m \cdot 6^m + 15 \cdot 6^{3m} + 40 \cdot 3^m \cdot 6^m
\]

\[
+ 120 \cdot 3^m \cdot 4^m + 40 \cdot 6^{2m} + 90 \cdot 2^m \cdot 6^m + 90 \cdot 2^m \cdot 6^m + 144 \cdot 6^m + 120 \cdot 6^m \right),
\]

respectively. Table 2 displays the values of (7) for $n$, $m \leq 6$.

3.2. The free monoid. If we adjoin the unique string of length 0 (denoted by $\epsilon$) to the set $D^+$, then we form the set $D^*$. Paired with the operation of string concatenation, $D^*$ forms the free monoid on $D$. We refer to $\epsilon$ as the empty string, and it serves as the identity element in $D^*$. That is, for any $W \in D^*$,

\[
W \epsilon = W = \epsilon W.
\]

We define an endomorphism on $D^*$ to be a mapping $\phi : D^* \to D^*$ such that $\phi(W_1 W_2) = \phi(W_1) \phi(W_2)$ for all $W_1, W_2 \in D^*$.

Remark 13. Note that if $\phi$ is an endomorphism on $D^*$, then $\phi(\epsilon) = \epsilon$. This follows since for any $W \in D^*$, we have

\[
\phi(W) = \phi(\epsilon W) = \phi(\epsilon) \phi(W),
\]

which implies that $\phi(\epsilon)$ has length 0.
Now, an $m$-endomorphism on $D^*$ is an endomorphism such that $|\phi(d)| \leq m$ for all $d \in D$. Thus, an $m$-endomorphism on $D^*$ can map elements of $D$ to $\epsilon$. To determine the number of $m$-endomorphisms on $D^*$ up to combinatorial equivalence, we put $R = \{W \in D^* : |W| \leq m\}$, and for each $g \in \text{Sym}(D)$, we ask for the number of $f : D \rightarrow R$ that are fixed by $g$. Again, it suffices to count the number of possible images under such an $f$ of a single $d \in D$ from each cycle in the decomposition of $g$, and then multiply over all the cycles. But there is now one additional possible value of $f(d)$: the empty string. Hence, if $d$ belongs to a cycle of length $k_t$, then we have

$$1 + \sum_{k=1}^{m} \left( \sum_{j \mid k_t} j c(g_r, j) \right)^k = \sum_{k=0}^{m} \left( \sum_{j \mid k_t} j c(g_r, j) \right)^k$$

choices of $f(d)$. From this observation, we deduce the following.

**Corollary 14.** Let $|D| = n$, and suppose that $g_1, \ldots, g_{p(n)} \in \text{Sym}(D)$ have distinct cycle types. Then the number of $m$-endomorphisms on $D^*$, up to combinatorial equivalence, is the value of

$$\frac{1}{n!} \sum_{r=1}^{p(n)} \left( |C_{g_r}| \prod_{i=1}^{\ell_r} \left( \sum_{k=0}^{m} \left( \sum_{j \mid k_t} j c(g_r, j) \right)^k \right) \right),$$

where $k_{r1}, \ldots, k_{r\ell_r}$ is the cycle type of $g_r$, and $C_{g_r}$ is as in Proposition 8.

When $n = 2$, the number of $m$-endomorphisms on $D^*$, up to combinatorial equivalence, is

$$\frac{1}{2} \left( (2^{m+1} - 1)^2 + (2^{m+1} - 1) \right).$$

This is sequence A006516 from [OEIS 1996]. The corresponding expressions for $n = 3, 4, 5, 6$ are

$$\frac{1}{6} \left( \Delta_3^3 + 3(m + 1) \Delta_3 + 2 \Delta_3 \right),$$

$$\frac{1}{24} \left( \Delta_4^4 + 6 \Delta_2^2 \Delta_4 + 3 \Delta_4^2 + 8(m + 1) \Delta_4 + 6 \Delta_4 \right),$$

$$\frac{1}{120} \left( \Delta_5^5 + 10 \Delta_3^3 \Delta_5 + 15(m+1) \Delta_5^2 + 20 \Delta_2^2 \Delta_5 + 20 \Delta_2 \Delta_3 + 30(m+1) \Delta_5 + 24 \Delta_5 \right),$$

$$\frac{1}{720} \left( \Delta_6^6 + 15 \Delta_4^4 \Delta_6 + 45 \Delta_2^2 \Delta_6^2 + 15 \Delta_6^3 + 40 \Delta_3^3 \Delta_6 + 120(m+1) \Delta_3 \Delta_4 \\
+ 40 \Delta_6^2 + 90 \Delta_2^2 \Delta_6 + 90 \Delta_2 \Delta_6 + 144(m+1) \Delta_6 + 120 \Delta_6 \right),$$

where $\Delta_k = (k^{m+1} - 1)/(k - 1)$. Once again, the sequences given by these expressions appear to be little-known. Table 3 gives the values of (8) for $n, m \leq 6$.

### 4. $(\chi, \xi)$-patterns

In closing, we briefly place the relation $\sim$ from Section 1 into a more general context. Let $G$ be a finite group, and let $N$ and $M$ be finite nonempty sets. Suppose
Table 3. Values of (8) for $n, m \leq 6$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>6</td>
<td>16</td>
<td>45</td>
<td>121</td>
<td>338</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>28</td>
<td>390</td>
<td>8,442</td>
<td>244,910</td>
<td>8,967,034</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>120</td>
<td>10,760</td>
<td>2,180,845</td>
<td>770,763,470</td>
<td>419,527,164,799</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>496</td>
<td>295,603</td>
<td>563,483,404</td>
<td>2,421,556,983,901</td>
<td>19,636,295,549,860,505</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>2,016</td>
<td>8,039,304</td>
<td>144,651,898,755</td>
<td>2,370,422,688,990,078</td>
<td>916,720,535,022,517,503,173</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>8,128</td>
<td>217,629,416</td>
<td>37,057,640,711,850</td>
<td>23,683,244,198,577,149,289</td>
<td>42,775,066,732,111,188,868,070,978</td>
</tr>
</tbody>
</table>

that $\chi : G \to \text{Sym}(N)$ and $\zeta : G \to \text{Sym}(M)$ are group homomorphisms. Denote the set of all functions from $N$ into $M$ by $M^N$. This notation comes from de Bruijn [1972], who also introduced the equivalence relation $E_{\chi,\zeta}$ on $M^N$ defined by

$$(f_1, f_2) \in E_{\chi,\zeta} \iff f_2 \circ \chi(\gamma) = \zeta(\gamma) \circ f_1 \text{ for some } \gamma \in G.$$

**Example 15** [de Bruijn 1972]. Suppose that $N$ is a set of size $n \in \mathbb{N}$, and define an equivalence relation $S$ on the set of all mappings of $N$ into itself by

$$(f_1, f_2) \in S \iff f_2 \circ \gamma = \gamma \circ f_1 \text{ for some } \gamma \in \text{Sym}(N).$$

Letting $G = \text{Sym}(N)$, $M = N$, and $\chi = \zeta$ be the identity homomorphism on $\text{Sym}(N)$ shows that $S$ is a special case of the relation $E_{\chi,\zeta}$. Moreover, the sequence in Remark 11 gives the number of equivalence classes under $S$ for $n = 1, 2, 3 \ldots$ (See [de Bruijn 1972, § 3].)

The relation $E_{\chi,\zeta}$ stems from the left action of $G$ on $M^N$ given by

$$\gamma \cdot f = \zeta(\gamma) \circ f \circ \chi(\gamma^{-1})$$

for all $\gamma \in G, f \in M^N$. De Bruijn [1972] referred to the orbits of $G$ on $M^N$ as $(\chi, \zeta)$-patterns, and provided a formula for the number of these by applying Burnside’s lemma, and then evaluating $|\{f \in M^N : \gamma \cdot f = f\}|$ for each $\gamma \in G$. But the relation $\sim$ on the set $\Omega = \{\text{mappings of } D \text{ into } R\}$, where $0 < |D| < \infty$ and $R = \{W \in D^+ : |W| \leq m\}$, is a special instance of the relation $E_{\chi,\zeta}$. To see this,
take \( N = D, \ M = R, \) and \( G = \text{Sym}(D) \). Let \( \chi \) be the identity homomorphism on \( \text{Sym}(D) \), and define \( \zeta : G \to \text{Sym}(R) \) by

\[
\zeta(g) = \phi_g |_R
\]

for all \( g \in \text{Sym}(D) \). Then for any \( g, g' \in \text{Sym}(D) \),

\[
\zeta(g \circ g') = \phi_{g \circ g'} |_R = (\phi_g \circ \phi_{g'}) |_R = \phi_g |_R \circ \phi_{g'} |_R = \zeta(g) \circ \zeta(g'),
\]

so \( \zeta \) is a group homomorphism. Now, for any \( f_1, f_2 \in \Omega \), we have

\[
f_1 \sim f_2 \iff f_2 \circ g = \phi_g \circ f_1 = \phi_g |_R \circ f_1 \quad \text{for some } g \in \text{Sym}(D)
\]

\[
\iff f_2 \circ \chi(g) = \zeta(g) \circ f_1 \quad \text{for some } g \in \text{Sym}(D)
\]

\[
\iff (f_1, f_2) \in E_{\chi, \zeta}.
\]

It follows that the equivalence classes in \( \Omega \) under the relation \( \sim \) are \((\chi, \zeta)\)-patterns for \( \chi, \zeta \) chosen as above. In particular, our Theorem 6 is a special case of de Bruijn’s formula.

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### References


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