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Oscillation of solutions to nonlinear first-order delay differential equations

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In this article, we present sufficient conditions for the oscillation of all solutions to the delay differential equation

$$x'(t)+\sum_{i=1}^{n} f_i\big(t,x(\tau_i(t))\big)=0,\quad t\ge t_0.$$

In particular, we extend known results from linear to nonlinear equations, and improve the bounds of previous criteria.

1. Introduction

In this article, we study the delay differential equation

$$x'(t)+\sum_{i=1}^{n} f_i\big(t,x(\tau_i(t))\big)=0,\quad t\ge t_0, \tag{1-1}$$

where $f_i:[t_0,\infty)\times\mathbb{R}\to\mathbb{R}$ and $\tau_i:[t_0,\infty)\to\mathbb{R}$ are continuous functions satisfying conditions stated below. We establish sufficient conditions for all solutions to oscillate.

When $f_i(t,x)=p_i(t)x$, equation (1-1) becomes linear and it is easy to show that all solutions oscillate or tend to zero, under the assumption

$$\int_{t_0}^{\infty}\sum_{i=1}^{n} p_i(s)\,ds=\infty. \tag{1-2}$$

This result has been extended to delay equations of several types: nonlinear, nonhomogeneous, higher order, neutral equations, etc.; see, for example, [Dix et al. 2008; Elbert and Stavroulakis 1995; Erbe et al. 1995; Gil' 2014; Győri and Ladas 1991; Hale 1977; Ladde et al. 1987; Zhou 2011]. Since we want to ensure oscillation, we impose conditions stronger than the one above.

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For $n = 1$ and $f_1(t,x) = p_1(t)x$, there are two well-known conditions for the oscillation of all solutions: [Ladde et al. 1987, Theorem 2.1.3],

$$\limsup_{t\to\infty} \int_{\tau_1(t)}^{t} p_1(s)\,ds > 1, \tag{1-3}$$

and [Ladde et al. 1987, Theorem 2.1.1],

$$\liminf_{t\to\infty} \int_{\tau_1(t)}^{t} p_1(s)\,ds > \frac{1}{e}. \tag{1-4}$$

Some authors try to narrow the gap between these two lower bounds, while others extended the above criteria for covering more general equations. In this article, we try both of these tasks.

Braverman and Karpuz [2011] showed that when applying (1-3), the conditions that $\tau_1(t) < t$ and τ_1 be nondecreasing are necessary. They also modified (1-3) by using Grönwall's inequality. Chatzarakis and Öcalan [2015] applied the modified condition to multiple delay equations. We extend these results to nonlinear equations.

For $f_i(t,x) = p_i(t)x$, Grammatikopoulos et al. [2003] assumed that τ_i is monotonic. We do not use the monotonicity assumption. Győri and Ladas [1991] stated conditions using a nondecreasing upper bound for the delayed arguments, similar to our σ defined below. Hunt and Yorke [1984] proved oscillation of solutions assuming that

$$\liminf_{t\to\infty} \sum_{i=1}^{n} p_i(s)(t - \tau_i(t)) > \frac{1}{e}$$

and that $t - \tau_i(t)$ is bounded. They did not assume monotonicity of τ_i, and used an inequality of differentials in their proof. We extend their result to nonlinear equations; see Theorem 4.6 below. Li [1996] used a logarithmic inequality to obtain a condition weaker than (1-4) for constant delays. We use the same logarithmic inequality for variable delays in nonlinear equations. Fukagai and Kusano [1984] considered retarded and advanced nonlinear equations with $f_i(t,x) = p_i(t)g_i(x)$, where g_i satisfies conditions similar to those in (H2) below. We assume that $f_i(t,x) \ge p_i(t)g_i(x)$, and then apply the Grönwall and logarithmic inequalities.

In this article, we use the hypotheses

(H1) $\tau_i(t) < t$ for $t \ge t_0$, and $\lim_{t\to\infty} \tau_i(t) = \infty$ for $i = 1, 2, \dots, n$.

(H2) $x f_i(t,x) \ge 0$, and there exist continuous functions $p_i(t) \ge 0$ and $g_i(x)$ such that

$$|f_i(t,x)| \ge p_i(t)|g_i(x)| \quad \forall x \in \mathbb{R},\ t \ge t_0,$$

where $x g_i(x) > 0$ for $x \ne 0$ and $\limsup_{x\to 0} x/g_i(x) < \infty$. Without loss of generality, we assume that

$$\limsup_{x\to 0} \frac{x}{g_i(x)} < 1. \tag{1-5}$$

If $\limsup_{x\to 0} x/g_i(x) = M_1 \geq 1$, we multiply p_i by a constant greater than M_1, and divide g_i by the same constant; so the assumption is satisfied without modifying f_i.

We define the functions

$$\tau_0(t) = \max_{1\leq i\leq n} \tau_i(t), \quad \sigma(t) = \max_{t_0\leq s\leq t} \tau_0(s).$$

Then σ is nondecreasing. Also by (H1), we have $\tau_i(t) \leq \tau_0(t) \leq \sigma(t) < t$, and

$$\lim_{t\to\infty} \tau_0(t) = \infty, \quad \lim_{t\to\infty} \sigma(t) = \infty.$$

Let $t_{-1} = \min_{1\leq i\leq n} \inf_{t_0\leq t} \tau_i(t)$. Then the initial condition for (1-1) is

$$x(t) = \phi(t) \quad \text{for } t \in [t_{-1}, t_0], \tag{1-6}$$

where $\phi : [t_{-1}, t_0] \to \mathbb{R}$ is a continuous function.

By a solution we mean a function that is continuous on $[t_{-1}, \infty)$, differentiable on $[t_0, \infty)$, and satisfies (1-1) and (1-6).

A unique solution x can be obtained by the method of steps: Using the information on $[t_{-1}, t_0]$, define x by integrating (1-1) for $t \in [t_0, t_1]$, where t_1 is the largest value such that $\tau_i(t) \leq t_0$ for all $t \leq t_1$, where $i = 1, 2, \dots, n$. Then we repeat the process for $[t_1, t_2)$ and so on.

A function is said to be oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. A function x is said to be eventually positive if there exists t^* such that $x(t) > 0$ for all $t \geq t^*$. We define eventually negative similarly.

Lemma 1.1. *Under assumptions (H1), (H2) and (1-2), if x is an eventually positive solution of (1-1), then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau_i(t)) > 0$, x is nonincreasing, and $|x(\tau_i(t))| \leq |g_i(x(\tau_i(t)))|$ for $t \geq t_1$ and $i = 1, 2, \dots, n$.*

Proof. Since x is eventually positive, there exists $t^* \geq t_0$ such that $x(t) > 0$ for $t \geq t^*$. Since $\lim_{t\to\infty} \tau_i(t) = \infty$ for $i = 1, 2, \dots, n$, there exists $t^{**} \geq t^*$ such that $\tau_i(t) \geq t^*$; thus $x(\tau_i(t)) > 0$ for $t \geq t^{**}$ and $i = 1, 2, \dots, n$.

From (H2), $f_i(t, x(\tau_i(t)) \geq 0$, and from (1-1), $x'(t) \leq 0$. Therefore, x is nonincreasing. Since x is nonnegative and nonincreasing, it must converge to a number $\alpha \geq 0$ as $t \to \infty$. We claim that $\alpha = 0$. To reach a contradiction, assume that $\lim_{t\to\infty} x(t) = \alpha > 0$. Then $0 < \alpha \leq x \leq x_{\max}$. Since g_i is continuous and positive on $[\alpha, x_{\max}]$, there exists $\gamma_i > 0$ such that $\gamma_i \leq g(x(\tau_i(t)))$ for all $t \geq t^{**}$. By (1-1) and (H2),

$$0 \geq x'(t) + \sum_{i=1}^{n} p_i(t) g_i(x(\tau_i(t))) \geq x'(t) + \sum_{i=1}^{n} p_i(t)\gamma_i.$$

Integrating from t^{**} to t,

$$\alpha - x(t^{**}) \leq x(t) - x(t^{**}) \leq -\int_{t^{**}}^{t} \sum_{i=1}^{n} p_i(s)\gamma_i \, ds.$$

Note that as $t \to \infty$, by (1-2), the right-hand side approaches $-\infty$, while the left-hand side is constant. This contradiction implies $\lim_{t\to\infty} x(t) = 0$. From (1-5) and $\lim_{t\to\infty} \tau_i(t) = \infty$, there exists $t_1 \geq t^{**}$ such that $x(\tau_i(t)) \leq g_i(x(\tau_i(t)))$ for all $t \geq t_1$. □

Under the assumptions of Lemma 1.1, from the definitions of τ_0 and σ, for all $t \geq t_1$, we have the inequalities

$$0 = x'(t) + \sum_{i=1}^{n} f_i\big(t, x(\tau_i(t))\big) \geq x'(t) + \sum_{i=1}^{n} p_i(t) x(\tau_i(t)) \tag{1-7}$$

$$\geq x'(t) + x(\tau_0(t)) \sum_{i=1}^{n} p_i(t) \tag{1-8}$$

$$\geq x'(t) + x(\sigma(t)) \sum_{i=1}^{n} p_i(t) \tag{1-9}$$

$$\geq x'(t) + x(t) \sum_{i=1}^{n} p_i(t). \tag{1-10}$$

For the rest of this article, we reserve the symbol t_1 for the value obtained in Lemma 1.1. Note that a similar value t_1 can be obtained for eventually negative solutions. In such case, inequalities (1-7)–(1-10) need to be reversed.

2. Conditions using the limit superior

A direct application of [Ladde et al. 1987, Theorem 2.1.3] to (1-9) states that

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \sum_{i=1}^{n} p_i(s)\, ds > 1 \tag{2-1}$$

implies the oscillation of all solutions to (1-1). This corresponds to [Ladde et al. 1987, Remark 2.7.3], where the assumption that τ_i is nondecreasing needs to be added.

Regarding the necessity of σ being monotonic and $\sigma(t) < t$, Braverman and Karpuz [2011] considered the single delay equation

$$x'(t) + p_1 x(\tau_1(t)) = 0, \tag{2-2}$$

with the assumption

$$\limsup_{t\to\infty} \int_{\tau_1(t)}^{t} p_1\, ds > A, \tag{2-3}$$

where A and p_1 are positive constants. They showed that for every A, there exists a p_1 and a nonmonotonic delay τ_1, with $\tau_1(t) = t$ on some intervals, such that (2-3) is satisfied, but (2-2) has a nonoscillatory solution. We shall show a similar result for (1-1), when τ_1 remains monotonic; see Theorem 2.5 below.

As in [Braverman and Karpuz 2011, Corollary 1] and [Chatzarakis and Öcalan 2015, Theorem 1], we use Grönwall's inequality to obtain a condition weaker than (2-1).

Lemma 2.1. *Assume that* (H1), (H2) *and* (1-2) *hold and that x is an eventually positive solution of* (1-1). *Then*

$$\int_{\sigma(t)}^{t} \sum_{i=1}^{n} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma(t)} \sum_{j=1}^{n} p_j(r)\, dr\right) ds < 1 \quad \forall t \geq t_1, \tag{2-4}$$

where t_1 is defined by Lemma 1.1

Proof. Grönwall's inequality applied to (1-10) with $x > 0$ and $\tau_i(s) \leq \sigma(t)$ yields

$$x(\tau_i(s)) \geq x(\sigma(s)) \exp\left(\int_{\tau_i(s)}^{\sigma(t)} \sum_{i=j}^{n} p_j(r)\, dr\right). \tag{2-5}$$

Integrating (1-1) from $\sigma(t)$ to t and using (H2) and (2-5) yields

$$\begin{aligned} 0 &\geq x(t) - x(\sigma(t)) + \int_{\sigma(t)}^{t} \sum_{i=1}^{n} p_i(s) x(\tau_i(s))\, ds \\ &\geq x(t) - x(\sigma(t)) + x(\sigma(t)) \int_{\sigma(t)}^{t} \sum_{i=1}^{n} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma(t)} \sum_{j=1}^{n} p_j(r)\, dr\right) ds. \end{aligned} \tag{2-6}$$

Denoting the outer integral by $\mathbb{P}(t)$,

$$0 < x(t) \leq x(\sigma(t))(1 - \mathbb{P}(t)) \quad \forall t \geq t_1. \tag{2-7}$$

Therefore, $\mathbb{P}(t) < 1$ for all $t \geq t_1$, which completes the proof. □

Theorem 2.2. *Assume* (H1), (H2) *and* (1-2). *If there exists a sequence $\{u_k\} \to \infty$ such that*

$$\int_{\sigma(u_k)}^{u_k} \sum_{i=1}^{n} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma(u_k)} \sum_{j=1}^{n} p_j(r)\, dr\right) ds \geq 1 \quad \forall k, \tag{2-8}$$

then all solutions of (1-1) *are oscillatory.*

Proof. To reach a contradiction, assume that there is a nonoscillatory solution x, and initially assume x is eventually positive. Let t_1 be defined by Lemma 1.1. Then by Lemma 2.1, inequality (2-4) is satisfied, which contradicts (2-8). Therefore x cannot be eventually positive.

When x is eventually negative, we prove a variation of Lemma 1.1 in which $x(t) < 0$, $x(\tau_i(t)) < 0$, x is nondecreasing, and $|x(\tau_i(t))| \leq |g_i(x(\tau_i(t)))|$ for $t \geq t_1$. Then we show that Lemma 2.1 still holds. In its proof, we need to reverse inequalities (2-5), (2-6) and (2-7). With these two lemmas, we obtain again a contradiction to (2-8), which implies that x cannot be eventually negative. □

Remark 2.3. Note that (2-8) is implied by

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \sum_{i=1}^{n} p_i(s) \exp\Big(\int_{\tau_i(s)}^{\sigma(t)} \sum_{j=1}^{n} p_j(r)\, dr \Big)\, ds > 1. \tag{2-9}$$

Since the exponent in (2-9) is not negative, it follows that (2-9) is implied by (2-1). In summary, (2-8) is less restrictive than (2-2).

Remark 2.4. When the equal sign in (1-1) is replaced by $\le$, the new equation cannot have eventually positive solutions under assumption (2-8). Similarly when the equal sign in (1-1) is replaced by $\ge$, the new equation cannot have eventually negative solutions under assumption (2-8).

Regarding the necessity of the hypothesis $\sigma(t) < t$ in Theorem 2.2, we consider the single delay equation

$$x'(t) + p_1 x(\tau_1(t)) = 0 \tag{2-10}$$

with the assumption

$$\limsup_{t\to\infty} \int_{\tau_1(t)}^{t} p_1 \exp\Big(\int_{\tau_1(s)}^{\sigma(t)} p_1\, dr \Big)\, ds > A, \tag{2-11}$$

where A and p_1 are positive constants.

Theorem 2.5. *For each p_1 and each $A < e$, there exists a monotonic delay with $\tau_1(t) = t$ on certain intervals such that (2-11) is satisfied, but (2-10) has a nonoscillatory solution.*

Proof. Since the continuous mapping $y \mapsto ye^y$ is strictly increasing and maps zero to zero and 1 to e, there exists $\beta < 1$ such that $\beta e^\beta = A$. Since for positive integers, $\frac{m-1}{m} < 1$ and $\lim_{m\to\infty} \frac{m-1}{m} = 1$, there exists m such that $\beta < \frac{m-1}{m} < 1$. Then

$$\frac{m\beta}{(m-1)p_1} < \frac{1}{p_1}.$$

By the completeness of the real numbers, there exists α such that

$$\frac{m\beta}{(m-1)p_1} < \alpha < \frac{1}{p_1}.$$

In summary, for some integer m, we have

$$\alpha p_1 < 1 \quad \text{and} \quad \beta < \frac{(m-1)\alpha p_1}{m}. \tag{2-12}$$

As a delayed argument, we define the piecewise linear function

$$\tau_1(t) = \begin{cases} t & \text{if } 0 \le t \le \alpha, \\ \alpha & \text{if } \alpha < t < \frac{2m-1}{m}\alpha, \\ 2\alpha + m(t - 2\alpha) & \text{if } \frac{2m-1}{m}\alpha \le t \le 2\alpha. \end{cases}$$

For $t \in (2\alpha, 4\alpha]$, we use the formula $\tau_1(t) = 2\alpha + \tau_1(t-2\alpha)$, and a similar formula for $t \in (4\alpha, 6\alpha]$, etc. Note that τ_1 is continuous, nondecreasing, $\lim_{t\to\infty} \tau_1(t) = \infty$, and $\tau_1(t) = \tau_0(t) = \sigma(t)$. To define a solution to (2-10), we use an initial condition $x(t) = x_0 > 0$ for $t \le 0$.

On the interval $[0, \alpha]$, equation (2-10) becomes an ordinary differential equation whose solution is $x(t) = x_0 e^{-p_1 t}$, which is positive and decreasing.

On the interval $\left[\alpha, \frac{2m-1}{m}\alpha\right]$, the delayed argument is $\tau_1(t) = \alpha$. Then (2-10) has the solution

$$x(t) = x(\alpha) - p_1 x(\alpha)(t-\alpha) = x(\alpha)\big(1 - (t-\alpha)p_1\big), \tag{2-13}$$

which is decreasing. From the inequality $t \le \frac{2m-1}{m}\alpha < 2\alpha$, we obtain the lower bound

$$x(t) > x(\alpha)(1 - \alpha p_1),$$

which is positive because of (2-12).

So far the solution is positive on $\left[0, \frac{2m-1}{m}\alpha\right]$. Next we show that the solution cannot have zeros in $\left(\frac{2m-1}{m}\alpha, 2\alpha\right]$. To reach a contradiction, let t_2 be the smallest zero in $\left(\frac{2m-1}{m}\alpha, 2\alpha\right]$. By the mean value theorem, there exists t^* in $\left(\frac{2m-1}{m}\alpha, t_2\right)$ such that

$$x'(t^*) = \frac{x\left(\frac{2m-1}{m}\alpha\right) - 0}{\frac{2m-1}{m}\alpha - t_2}.$$

From $t_2 < 2\alpha$, it follows that

$$x'(t^*) < \frac{x\left(\frac{2m-1}{m}\alpha\right)}{-\frac{\alpha}{m}}. \tag{2-14}$$

Note that for $t \le t_2$, we have $\tau_1(t) < t_2$. Since $x(t) \ge 0$ for all $t \le t_2$, by (2-10), $x'(t) \le 0$ so that x is nonincreasing for all $t \le t_2$. Because x is nonincreasing and $\alpha \le \tau_1(t^*)$, we have $x(\tau_1(t^*)) \le x(\alpha)$. This and (2-14) imply

$$0 = x'(t^*) + p_1 x(\tau_1(t^*)) < \frac{x\left(\frac{2m-1}{m}\alpha\right)}{-\frac{\alpha}{m}} + p_1 x(\alpha). \tag{2-15}$$

From (2-13),

$$x\left(\frac{2m-1}{m}\alpha\right) = x(\alpha)\left(1 - \left(\frac{2m-1}{m}\alpha - \alpha\right)p_1\right) = x(\alpha)\left(1 - \frac{m-1}{m}\alpha p_1\right).$$

Substituting this value in (2-15) yields

$$x(\alpha)\left(1 - \frac{m-1}{m}\alpha p_1\right) < \frac{\alpha}{m} p_1 x(\alpha),$$

which implies $1 - \frac{m-1}{m}\alpha p_1 < \frac{\alpha}{m} p_1$. This in turn implies $1 < \alpha p_1$, and contradicts (2-11). Therefore, $x(t) > 0$ on $[0, 2\alpha]$.

Next we set $x(2\alpha)$ as the initial value, and solve (2-10) on $[2\alpha, 4\alpha]$. Repeating this process, we have a positive solution on $[0, \infty)$.

It remains to show that (2-11) is satisfied. From the definition of τ_1, when $t = u_1 = \frac{2m-1}{m}\alpha$, we have $\sigma(u_1) = \frac{2m-1}{m}\alpha$. For $\alpha \le s \le \frac{2m-1}{m}\alpha$, we have that $\tau_1(s) = \alpha$. Then (2-11) becomes

$$\int_\alpha^{\frac{2m-1}{m}\alpha} p_1 \exp\Big(\int_\alpha^{\frac{2m-1}{m}\alpha} p_1\, dr\Big)\, ds = \frac{m-1}{m}\alpha p_1 \exp\Big(\frac{m-1}{m}\alpha\, p_1\Big).$$

Since the mapping $y \mapsto e^y$ is increasing, by (2-12),

$$\frac{m-1}{m}\alpha p_1 \exp\Big(\frac{m-1}{m}\alpha p_1\Big) > \beta e^\beta > A.$$

Repeating this process at $u_k = 2k\alpha + \frac{2m-1}{m}\alpha$, we obtain a sequence at which the above inequality holds. The presence of this sequence implies (2-8) and (2-11) are satisfied. □

3. Conditions using the limit inferior

A direct application of [Ladde et al. 1987, Theorem 2.1.1] to (1-8) states that

$$\liminf_{t\to\infty} \int_{\sigma(t)}^t \sum_{i=1}^n p_i(s)\, ds > \frac{1}{e} \tag{3-1}$$

implies the oscillation of all solutions of (1-1). Also note that (3-1) implies (1-2).

Grammatikopoulos et al. [2003] showed that for (1-1) with $f_i(t,x) = p_i(t)x$, all solutions are oscillatory when the τ_i are nondecreasing, and

$$\begin{gathered}\int_0^\infty |p_i(s) - p_j(s)|\, ds < \infty,\\ \liminf_{t\to\infty} \int_{\tau_i(t)}^t p_i(s)\, ds = \beta_i > 0, \quad \sum_{i=1}^n \liminf_{t\to\infty} \int_{\tau_i(t)}^t p_i(s)\, ds > \frac{1}{e}.\end{gathered} \tag{3-2}$$

As in the previous part, we use Grönwall's inequality for finding a condition less restrictive than (3-1).

Lemma 3.1. *Assume (H1), (H2). If x is an eventually positive solution of (1-1), and*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^t \sum_{i=1}^n p_i(s) \exp\Big(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^n p_j(r)\, dr\Big)\, ds > \frac{1}{e}, \tag{3-3}$$

then $\lim_{t\to\infty} x(\sigma(t))/x(t) = \infty$.

Proof. By a contrapositive argument, we can show that (3-3) implies (1-2), so we let t_1 be defined by Lemma 1.1. Applying Grönwall's inequality to (1-10) yields

(2-5), which is substituted in (1-7) to obtain

$$0 \geq x'(t) + \sum_{i=1}^{n} p_i(t)x(\sigma(t)) \exp\left(\int_{\tau_i(t)}^{\sigma(t)} \sum_{j=1}^{n} p_j(r)\,dr\right) \quad \forall t \geq t_1. \tag{3-4}$$

Dividing by $x(t)$ and integrating from $\sigma(t)$ to t, we obtain

$$\ln\left(\frac{x(t)}{x(\sigma(t))}\right) + \int_{\sigma(t)}^{t} \sum_{i=1}^{n} p_i(s)\frac{x(\sigma(t))}{x(s)} \exp\left(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{n} p_j(r)\,dr\right) ds \leq 0. \tag{3-5}$$

From (3-3), there exist constants $t_2 \geq t_1$ and α such that

$$\int_{\sigma(t)}^{t} \sum_{i=1}^{n} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma(s)} \sum_{j=1}^{n} p_j(r)\,dr\right) ds \geq \alpha > \frac{1}{e} \quad \forall t \geq t_2.$$

Since $\sigma(s) < s$ and x is nonincreasing, $x(\sigma(s))/x(s) \geq 1$. Then (3-5) and the above inequality yield

$$\ln\left(\frac{x(t)}{x(\sigma(t))}\right) + \alpha \leq 0.$$

Since $\alpha e \leq e^{\alpha}$ for all α,

$$\alpha e \leq e^{\alpha} \leq \frac{x(\sigma(t))}{x(t)} \quad \forall t \geq t_2. \tag{3-6}$$

Since $\lim_{t\to\infty} \sigma(t) = \infty$, there exists $t_3 \geq t_2$ such that $\sigma(t) \geq t_2$ for all $t \geq t_3$. Using (3-6) in (3-5), we obtain

$$(\alpha e)^2 \leq \frac{x(\sigma(s))}{x(s)} \quad \forall t \geq t_3.$$

Repeating this process, we obtain

$$(\alpha e)^k \leq \frac{x(\sigma(s))}{x(s)}$$

for all t sufficiently large. Since $\alpha e > 1$, the assertion of the lemma follows. □

Theorem 3.2. *Under assumptions* (H1), (H2) *and* (3-3), *all solutions to* (1-1) *are oscillatory.*

Proof. To reach a contradiction, assume that there is a nonoscillatory solution x, which initially is assumed to be eventually positive. By a contrapositive argument, we can show that (3-3) implies (1-2), so we let t_1 be defined by Lemma 1.1. To simplify notation, we define

$$\mathbb{P}(s) = \sum_{i=1}^{n} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma(s)} \sum_{i=1}^{n} p_i(r)\,dr\right).$$

Then from (3-3), there exist constants $t_2 \geq t_1$ and α such that

$$\int_{\sigma(t)}^{t} \mathbb{P}(s)\,ds \geq \alpha > \frac{1}{e} \quad \forall t \geq t_2.$$

Using the intermediate value theorem, we can show that there exists $t^* \in (\sigma(t), t)$ such that

$$\int_{\sigma(t)}^{t^*} \mathbb{P}(s)\,ds \geq \frac{\alpha}{2} \quad \text{and} \quad \int_{t^*}^{t} \mathbb{P}(s)\,ds \geq \frac{\alpha}{2}. \tag{3-7}$$

Integrating (1-7) from $\sigma(t)$ to t^* and using (2-5) yield

$$x(t^*) - x(\sigma(t)) + x(\sigma(t^*)) \int_{\sigma(t)}^{t^*} \mathbb{P}(s)\,ds \leq 0.$$

Using that $x(t^*) > 0$ and (3-7), we obtain

$$x(\sigma(t^*)) \leq \frac{2}{\alpha} x(\sigma(t)). \tag{3-8}$$

Integrating (1-7) from t^* to t and using (2-5) yield

$$x(\sigma(t)) - x(t^*) + x(\sigma(t)) \int_{t^*}^{t} \mathbb{P}(s)\,ds \leq 0.$$

Using that $x(\sigma(t)) > 0$ and (3-7), we obtain

$$x(\sigma(t)) \leq \frac{2}{\alpha} x(t^*).$$

Using this inequality in (3-8) yields

$$\frac{x(\sigma(t^*))}{x(t^*)} \leq \left(\frac{2}{\alpha}\right)^2.$$

Because $\sigma(t) \leq t^* \leq t$ and $\lim_{t\to\infty} \sigma(t) = \infty$, the above inequality contradicts Lemma 3.1; so the solution x cannot be eventually positive.

When x is eventually negative, as in Lemma 1.1, there exist $t_1 \geq t_0$ such that $x(t) < 0$, $x(\tau_i(t)) < 0$, $x(t)$ is nondecreasing and $|x(\tau_i(t))| \leq |g_i(x(\tau_i(t)))|$ for $t \geq t_1$. Then Lemma 3.1 holds, but in its proof we need to reverse inequality (3-4). Again we reach a contradiction indicating that x cannot be eventually negative. □

Remark 3.3. Note that the exponent in (3-3) is nonnegative; therefore, condition (3-1) is more restrictive than (3-3). Also the statements in Remark 2.4 apply to condition (3-3).

4. Estimates using a logarithmic inequality

Li [1996] used the inequality $e^{rx} \geq x + \frac{1}{r}(1 + \ln r)$ to show that all solutions to (1-1) are oscillatory when $f_i(t,x) = p_i(t)x$ and the delays have the form $\tau_i(t) = t - k_i$

with positive constants k_i. There, the key assumption is

$$\int_{t_0}^{\infty} \sum_{i=1}^{n} p_i(s)\left(1+\ln\left(\int_s^{s+k_i} \sum_{j=1}^{n} p_j(r)\,dr\right)\right) ds = \infty. \tag{4-1}$$

We want to extend the result in [Li 1996] to (1-1) that are nonlinear and have variable delays. The variable delays cause some difficulties when obtaining a condition similar to (4-1).

First we define a function that is the inverse of σ almost everywhere. Under assumption (H1), the function σ is continuous; thus for each s, the set $\sigma^{-1}(s)$ is closed. Since σ is monotonic and $\lim_{t\to\infty}\sigma(t)=\infty$, the set $\sigma^{-1}(s)$ is a closed and bounded interval. There are at most countably many of those closed intervals that do not consist of a single point. Let

$$\sigma_{\text{inv}}(s) = \max\{t : \sigma(t)=s\}.$$

Note that σ_{inv} is strictly increasing and has at most countably many discontinuities. Also $s < \sigma_{\text{inv}}(s)$, and σ_{inv} is bounded on bounded intervals. Under these conditions, σ_{inv} is Riemann integrable, and expressions of the form $\int_a^b p(s)\int_s^{\sigma_{\text{inv}}(s)} \lambda(r)\,dr\,ds$ are well-defined for all continuous functions λ, p. Also the value of this integral remains the same when $\sigma_{\text{inv}}(s)$ is replaced by any t as long as $\sigma(t)=s$. This happens because the integrand would change only at countably many points.

Lemma 4.1. *Under assumption (H1), for $a \le \sigma(b)$ and any continuous nonnegative functions λ and p, we have*

$$\int_a^b p(s)\int_{\sigma(s)}^{s} \lambda(r)\,dr\,ds \ge \int_a^{\sigma(b)} \lambda(s)\int_s^{\sigma_{\text{inv}}(s)} p(r)\,dr\,ds. \tag{4-2}$$

Proof. Interchanging the order of integration on the left-hand side of (4-2) gives

$$\begin{aligned}\int_a^b p(s)\int_{\sigma(s)}^{s} \lambda(r)\,dr\,ds &= \int_{\sigma(a)}^{a} \lambda(r)\int_a^{\sigma_{\text{inv}}(r)} p(s)\,ds\,dr\\ &\quad+\int_a^{\sigma(b)} \lambda(r)\int_r^{\sigma_{\text{inv}}(r)} p(s)\,ds\,dr+\int_{\sigma(b)}^{b} \lambda(r)\int_r^{b} p(s)\,ds\,dr.\end{aligned}$$

Since all these integrals are nonnegative, we use the second integral in the right-hand side as a lower bound. Renaming the variables r and s, we obtain the assertion of the lemma. ☐

Lemma 4.2. *Under assumptions (H1), (H2) and (1-2), if (1-1) has an eventually positive solution, then*

$$\int_t^{\sigma_{\text{inv}}(t)} \sum_{i=1}^{n} p_i(s)\,ds < 1 \quad \forall t \ge t_1,$$

where t_1 is defined by Lemma 1.1

Proof. Let x be an eventually positive solution of (1-1). Recall that x is nonincreasing, σ is nondecreasing, and $t < \sigma_{\text{inv}}(t)$. Integrating (1-9) from t to $\sigma_{\text{inv}}(t)$, we have

$$x(\sigma_{\text{inv}}(t)) - x(t) + x(t) \int_t^{\sigma_{\text{inv}}(t)} \sum_{i=1}^n p_i(s)\,ds \le 0. \tag{4-3}$$

Then

$$0 < x(\sigma_{\text{inv}}(t)) \le x(t)\Big(1 - \int_t^{\sigma_{\text{inv}}(t)} \sum_{i=1}^n p_i(s)\,ds\Big) \quad \forall t \ge t_1. \tag{4-4}$$

The assertion of the lemma follows. □

Lemma 4.3. *Under assumptions* (H1), (H2) *and* (1-2), *if x is an eventually positive solution of* (1-1) *and*

$$\limsup_{t\to\infty} \int_t^{\sigma_{\text{inv}}(t)} \sum_{i=1}^n p_i(s)\,ds > 0, \tag{4-5}$$

then $\liminf_{t\to\infty} x(\sigma(t))/x(t) < \infty$.

Proof. Let t_1 be defined by Lemma 1.1. From (4-5), there exist a constant α and a sequence $\{t_k\}_{k=2}^{\infty} \to \infty$ such that

$$\int_{t_k}^{\sigma_{\text{inv}}(t_k)} \sum_{i=1}^n p_i(s)\,ds \ge \alpha > 0 \quad \forall k \ge 2.$$

Using the intermediate value theorem, we can show that there exists t_k^* in the interval $(t_k, \sigma_{\text{inv}}(t_k))$ such that

$$\int_{t_k}^{t_k^*} \sum_{i=1}^n p_i(s)\,ds \ge \frac{\alpha}{2} \quad \text{and} \quad \int_{t_k^*}^{\sigma_{\text{inv}}(t_k)} \sum_{i=1}^n p_i(s)\,ds \ge \frac{\alpha}{2}. \tag{4-6}$$

Integrating (1-9) from t_k to t_k^*, and using that σ is nondecreasing while x is nonincreasing, yields

$$x(t_k^*) - x(t_k) + x(\sigma(t_k^*)) \int_{t_k}^{t_k^*} \sum_{i=1}^n p_i(s)\,ds \le 0.$$

Using that $x(t_k^*) > 0$ and (4-6), we have

$$x(\sigma(t_k^*)) \le \frac{2}{\alpha} x(t_k). \tag{4-7}$$

Integrating (1-9) from t_k^* to $\sigma_{\text{inv}}(t_k)$ yields

$$x(\sigma_{\text{inv}}(t_k)) - x(t_k^*) + x(t_k) \int_{t_k^*}^{\sigma_{\text{inv}}(t_k)} \sum_{i=1}^n p_i(s)\,ds \le 0.$$

Using that $x(\sigma_{\text{inv}}(t_k)) > 0$ and (4-6), we have

$$x(t_k) \le \frac{2}{\alpha} x(t_k^*). \tag{4-8}$$

Using (4-8) in (4-7), it follows that

$$\frac{x(\sigma(t_k^*))}{x(t_k^*)} \le \left(\frac{2}{\alpha}\right)^2 \quad \forall k \ge 2.$$

The assertion of the lemma follows by calculating the limit inferior as $k \to \infty$. □

Theorem 4.4. *Assume* (H1), (H2), *and*

$$\int_s^{\sigma_{\text{inv}}(s)} \sum_{j=1}^n p_j(r)\,dr > 0 \quad \forall s \ge t_0, \tag{4-9}$$

$$\int_{t_0}^{\infty} \sum_{i=1}^n p_i(s)\left(1 + \ln\left(\int_s^{\sigma_{\text{inv}}(s)} \sum_{j=1}^n p_j(r)\,dr\right)\right) ds = \infty. \tag{4-10}$$

Then every solution of (1-1) *is oscillatory.*

Proof. To reach a contradiction, assume that there is a nonoscillatory solution x, which initially is assumed to be eventually positive. By a contrapositive argument, we can show that (4-10) implies (1-2), so we let t_1 be defined by Lemma 1.1. Let

$$\lambda(t) = \frac{-x'(t)}{x(t)} \quad \text{for } t \ge t_1.$$

Then λ is a continuous and nonnegative function. Integrating λ from a value t^* to t, we have $x(t) = x(t^*)\exp\big(-\int_{t^*}^t \lambda(s)\,ds\big)$. Then

$$x'(t) = -\lambda(t)x(t^*)\exp\left(-\int_{t^*}^t \lambda(s)\,ds\right).$$

Substituting this expression in (1-1) yields

$$\lambda(t) = \frac{1}{x(t^*)} \sum_{i=1}^n f_i\big(t, x(\tau_i(t))\big)\exp\left(\int_{t^*}^t \lambda(s)\,ds\right).$$

For $t^* = \sigma(t) < t$, using (H2) and $x(\sigma(t)) \le x(\tau_i(t))$, we obtain

$$\lambda(t) \ge \sum_{i=1}^n p_i(t)\exp\left(\int_{\sigma(t)}^t \lambda(r)\,dr\right). \tag{4-11}$$

Note that the corresponding inequality on [Li 1996, page 3734] is incorrect, but it does not affect their proof of Theorem 1. Next as in [Li 1996], we use the inequality

$$e^{\gamma\beta} \ge \gamma + \frac{1}{\beta}(1 + \ln(\beta)) \quad \forall \beta > 0, \tag{4-12}$$

which can be shown by fixing β and minimizing $e^{\gamma\beta} - \gamma - \frac{1}{\beta}(1 + \ln(\beta))$ with respect to γ. Let

$$\beta(s) = \int_s^{\sigma_{\mathrm{inv}}(s)} \sum_{i=1}^{n} p_i(r)\,dr,$$

which is positive. Then by (4-11) and (4-12),

$$\begin{aligned}\lambda(s) &\ge \sum_{j=1}^{n} p_j(s) \exp\Bigl(\frac{1}{\beta(s)} \int_{\sigma(s)}^{s} \lambda(r)\,dr\,\beta(s)\Bigr)\\ &\ge \sum_{i=1}^{n} p_i(s) \frac{1}{\beta(s)} \Bigl(\int_{\sigma(s)}^{s} \lambda(r)\,dr + \bigl(1 + \ln(\beta(s))\bigr)\Bigr).\end{aligned}$$

Multiplying by $\beta(s)$ and integrating from t_1 to t,

$$\int_{t_1}^{t} \lambda(s)\beta(s)\,ds \ge \int_{t_1}^{t} \sum_{i=1}^{n} p_i(s) \int_{\sigma(s)}^{s} \lambda(r)\,dr\,ds + \int_{t_1}^{t} \sum_{i=1}^{n} p_i(s)\bigl(1+\ln(\beta(s))\bigr)\,ds.$$

By Lemma 4.1, with $a = t_1$ and $b = t$, we have

$$\begin{aligned}&\int_{t_1}^{t} \lambda(s)\beta(s)\,ds\\ &\qquad \ge \int_{t_1}^{\sigma(t)} \lambda(s) \int_s^{\sigma_{\mathrm{inv}}(s)} \sum_{i=1}^{n} p_i(r)\,dr\,ds + \int_{t_1}^{t} \sum_{i=1}^{n} p_i(s)\bigl(1 + \ln(\beta(s))\bigr)\,ds.\end{aligned}$$

Substituting $\beta(s)$ by its value on the left-hand side, and combining integrals, gives

$$\int_{t}^{\sigma(t)} \lambda(s) \int_s^{\sigma_{\mathrm{inv}}(s)} \sum_{i=1}^{n} p_i(r)\,dr\,ds \ge \int_{t_1}^{t} \sum_{i=1}^{n} p_i(s)\bigl(1 + \ln(\beta(s))\bigr)\,ds.$$

By Lemma 4.2, the coefficient of $\lambda(s)$ is at most 1. Then

$$\ln\Bigl(\frac{x(\sigma(t))}{x(t)}\Bigr) = \int_{t}^{\sigma(t)} \lambda(s)\,ds \ge \int_{t_1}^{t} \sum_{i=1}^{n} p_i(s)\bigl(1 + \ln(\beta(s))\bigr)\,ds.$$

In the limit as $t \to \infty$, the right-hand side approaches ∞ because of (4-10). Therefore, $\lim_{t\to\infty} x(\sigma(t))/x(t) = \infty$, which contradicts Lemma 4.3. This shows that the solution cannot be eventually positive.

When x is eventually negative, as in Lemma 1.1, we obtain a $t_1 \ge t_0$ such that $x(t) < 0$, $x(\tau_i(t)) < 0$, x is nondecreasing, and $|x(\tau_i(t))| \le |g_i(x(\tau_i(t)))|$ for $t \ge t_1$. Lemma 4.1 holds; it is independent of x. Lemma 4.2 holds, but in its proof we need to reverse the inequalities in (4-3) and (4-4). Lemma 4.3 holds, but in its proof we need to reverse the inequalities in (4-6), (4-7) and (4-8). In the first part of this proof, we need to reverse inequality (4-10). Again, we reach a contradiction indicating that the solution cannot be eventually negative. □

Remark 4.5. If $t - \tau_i(t) = k_i$, a positive constant, then (3-1) implies (4-10). In general, conditions (2-1), (3-1) and (4-10) are independent of each other. Here we present an example where (4-10) is satisfied, but (2-1) and (3-1) are not satisfied.

Consider (1-1) with only one delay, $f_1(t,x) = p_1(t)x$, $\tau_1(t) = t - \frac{1}{e}$, and

$$p_1(t) = \begin{cases} 4et & \text{if } 0 \le t \le \frac{1}{2e}, \\ 2 & \text{if } \frac{1}{2e} < t < 1 - \frac{1}{2e}, \\ -4e(t-1) & \text{if } 1 - \frac{1}{2e} \le t \le 1. \end{cases}$$

For $t \ge 1$, extend p_1 with period 1. Then

$$\frac{1}{e} \le \int_s^{s+\frac{1}{e}} p_1(r)\,dr \le \frac{2}{e}.$$

Note that the lower bound is attained when s is an integer minus $\frac{1}{2e}$; therefore

$$\liminf_{t\to\infty} \int_{t-\frac{1}{e}}^{t} p_1(r)\,dr = \frac{1}{e},$$

and (2-1) is not satisfied. The upper bound is attained when s equals an integer plus $\frac{1}{2e}$; thus

$$\limsup_{t\to\infty} \int_{t-\frac{1}{e}}^{t} p_1(r)\,dr = \frac{2}{e} < 1,$$

and (3-1) is not satisfied. Condition (4-10) is satisfied, because

$$\int_0^\infty p_1(s)\left(1 + \ln\left(\int_s^{s+\frac{1}{e}} p_1(r)\,dr\right)\right) ds \ge \sum_{k=0}^{\infty} \int_{k+\frac{1}{2e}}^{k+1-\frac{3}{2e}} 2\left(1 + \ln\frac{2}{e}\right) = \infty.$$

Now we extend the results in [Hunt and Yorke 1984] from the linear to the nonlinear case of equation (1-1). However, the Grönwall and the logarithmic inequalities cannot be applied in this case.

Theorem 4.6. *Assume* (H1), (H2) *and that there exists a constant* β *such that*

$$0 < t - \tau_i(t) \le \beta \quad \forall t \ge t_0,\ 1 \le i \le n, \tag{4-13}$$

$$\liminf_{t\to\infty} \sum_{i=1}^{n} p_i(t)(t - \tau_i(t)) > \frac{1}{e}. \tag{4-14}$$

Then all solutions of (1-1) *are oscillatory.*

Proof. To reach a contradiction, assume that there is a nonoscillatory solution x, which initially is assumed to be eventually positive. First we show that (4-14) implies (1-2), which allows us to use Lemma 1.1. From (4-14), there exist $t^* \ge t_0$ such that

$$\sum_{i=1}^{n} p_i(t)(t-\tau_i(t)) \ge \frac{1}{e}$$

for all $t \ge t^*$. Then by (4-13),

$$\beta \int_{t^*}^{\infty} \sum_{i=1}^{n} p_i(t)\,dt \ge \int_{t^*}^{\infty} \sum_{i=1}^{n} p_i(t)(t-\tau_i(t))\,dt \ge \int_{t^*}^{\infty} \frac{1}{e}\,dt = \infty.$$

Let t_1 be defined by Lemma 1.1.

From (4-14), there exist constants $t_2 \ge t_1$ and α such that

$$\sum_{i=1}^{n} p_i(t)(t-\tau_i(t)) \ge \alpha > \frac{1}{e} \quad \forall t \ge t_2.$$

Let $y(t) = -\ln(x(t))$. Then $x(t) = \exp(-y(t))$ and from (1-7), we have

$$y'(t) \ge \sum_{i=1}^{n} p_i(t) \exp\bigl(y(t) - y(\tau_i(t))\bigr) \quad \forall t \ge t_2. \tag{4-15}$$

As in [Hunt and Yorke 1984], we construct a solution u to a delay differential equation such that $u(t) \le y(t)$ and u blows up in finite time. Let u be the solution to the delay equation

$$u'(t) = \alpha \inf_{t-\beta \le r < t} \frac{1}{t-r} \exp\bigl(u(t) - u(r)\bigr) \quad \forall t \ge t_2 + \beta, \tag{4-16}$$

with the constant initial condition

$$u(t) = u(t_2+\beta) \le \min_{t_2 \le s \le t_2+\beta} y(s) \quad \text{for } t \le t_2 + \beta.$$

The rest of the proof is the same as that of [Hunt and Yorke 1984, Theorem 1]; so we just outline the steps. First justify the existence of the solution to (4-16), and denote by $r(t)$ the value at which the infimum is attained. Then show that u and u' are increasing, and that, $r(t)$, being a minimizer, satisfies either $t - r(t) = 1/u'(t)$ or $(t-\beta) \le 1/u'(\beta)$ when $r(t) = \beta$. Then construct a recurrence sequence $\{t_n\}$ increasing to a value t^*, while $u(t_n) \to \infty$. This implies $\lim_{t \to t^*} -\ln(x(t)) = \infty$ and $x(t^*) = 0$, which contradicts x being eventually positive.

When x is eventually negative, as in Lemma 1.1, we obtain $t_1 \ge t_0$ such that $x(t) < 0$, $x(\tau_i(t)) < 0$, x is nonincreasing, and $|x(\tau_i(t))| \le |g_i(x(\tau_i(t)))|$ for $t \ge t_1$. We redefine $y(t) = -\ln(-x(t))$; thus $-x(t) = \exp(-y(t))$. From (1-7) with the inequality reversed, we obtain (4-15). The rest of the proof is as for the eventually positive case. □

Remark 4.7. Note that the integral in (3-1) satisfies

$$\int_{\sigma(t)}^{t} \sum_{i=1}^{n} p_i(s)\,ds \le \sum_{i=1}^{n} \int_{\tau_i(t)}^{t} p_i(s)\,ds,$$

and that for $p_i(t)$ constant, the right-hand side of this inequality is $p_i(t)(t-\tau_i(t))$, which is used in (4-14). Therefore when $p_i(t)$ is constant, (3-1) implies (4-14). When $p_i(t)$ is constant and $\tau_i(t)=\beta$, conditions (3-1) and (4-14) are the same. In general, (4-14) is independent of both (3-1) and (3-3).

The above conditions are only sufficient for the oscillation of all solutions; finding necessary conditions may be a direction for future research.

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