involve

a journal of mathematics

Affine hyperbolic toral automorphisms

Colin Thomson and Donna K. Molinek

msp

2016 vol. 9, no. 4

INVOLVE 9:4 (2016)
dx.doi.org/10.2140/involve.2016.9.541

Affine hyperbolic toral automorphisms

Colin Thomson and Donna K. Molinek

(Communicated by Michael E. Zieve)

A hyperbolic transformation of the torus is an example of a function that is Devaney chaotic; that is, it is topologically transitive and has dense periodic points. An irrational rotation of the torus, on the other hand, is not chaotic because it has no periodic points. We show that a hyperbolic transformation of the torus followed by a translation (an affine hyperbolic toral automorphism) has dense periodic points and maintains transitivity. As a consequence, affine toral automorphisms are chaotic, even when the translation is an irrational rotation.

1. Introduction

Değirmenci and Koçak [2010] showed that the cross-product of the double-angle map and an irrational rotation, which is a function on the torus, is transitive and has sensitive dependence to initial conditions, but no periodic points, and therefore is not chaotic. Linear hyperbolic toral automorphisms are known to be chaotic, so a natural question in light of [Değirmenci and Koçak 2010] (and the generalizations in [Li and Zhou 2013]) is whether a linear hyperbolic toral automorphism plus a translation is still chaotic. We will refer to such functions as affine hyperbolic toral automorphisms to indicate the translation. Our main goal will be to determine whether such an affine map has periodic points, even in the event that the rotation is irrational.

We find that affine hyperbolic toral automorphisms are chaotic; in fact, we can find the precise locations of periodic points in relation to the periodic points of the corresponding linear map. In this respect, we generalize statements about the transitivity and periodic points of linear hyperbolic toral automorphisms to affine hyperbolic toral automorphisms.

2. Definitions

Throughout this paper, $f : X \to X$ will be a continuous function on a complete metric space (X, d). We will examine the iterates of f using the notation f^n to represent the n-th iterate of f; that is, $f^1 = f$ and $f^{n+1} \equiv f \circ f^n$. The composition

MSC2010: primary 54H20; secondary 37B40.
Keywords: topological dynamics, chaos, toral automorphism.

of f is still a continuous function from X to X. For a specific point $x \in X$, we may refer to the n-th iterate of x under f by x^n, which means $x^0 = x$ is the initial point. In this paper, all points in the space will be specified as vectors, and as such the superscript notation will unambiguously denote an iterate, not raising to a power. In addition, subscripts on points in the space will refer to the corresponding coordinate value, with the basis specified in the case that it is unclear.

A function is *transitive* if for every pair of nonempty open sets $U, V \subseteq X$, there exists a positive integer n such that $f^n(U) \cap V \neq \varnothing$. An example of a transitive function is the irrational rotation on the circle. An irrational rotation is actually *totally transitive*, by which we mean that f^m is transitive for every positive integer m. A property of the irrational rotation that makes it useful for counterexamples is that it is transitive, but has no periodic points.

A *periodic point* $p \in X$ is one for which $f^n(p) = p$ for some n, a positive integer. The least such n is called the *period* of p, and if $n = 1$, we say that p is a *fixed point*. We can locate points with a given period m by finding fixed points of f^n, provided that there is no $k < n$ such that f^k also fixes that point.

A function is *Devaney chaotic* (henceforth, *chaotic*) if it is transitive, has *dense* periodic points, and has *sensitive dependence to initial conditions*. "Dense" refers to the presence of at least one periodic point in every nonempty open set. Sensitivity to initial conditions means that there exists an $\epsilon > 0$ so that for all $\delta > 0$ and $x \in X$, there exists a $y \in X$ with $d(x, y) < \delta$ and an $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \epsilon$. Banks et al. [1992] proved that the first two hypotheses are sufficient for the third, making transitivity and dense periodic points all that is necessary for chaos. As Crannell [1995] pointed out and by Banks et al. [1992], the elimination of the sensitivity hypothesis makes chaos an entirely topological concept: sensitive dependence on initial conditions is the only hypothesis of the three that relied on the metric.

In general, no other combination of two hypotheses implies the third, but on the unit interval, transitivity guarantees dense periodic points, and is therefore sufficient for chaos [Vellekoop and Berglund 1994]. Contrast this with the irrational rotation on the circle, which is transitive but has no periodic points and is not sensitive to initial conditions.

A torus of d dimensions $\mathbb{T}^d$ is the cartesian product of d copies of the circle, $S^1 \times S^1 \times \cdots \times S^1$. Since $S^1 = \mathbb{R}/\mathbb{Z}$, coordinates in $\mathbb{T}^d$ are real numbers from 0, inclusive, to 1, exclusive. A linear automorphism of $\mathbb{T}^d$ is matrix multiplication of the coordinates in $[0, 1) \times [0, 1) \times \cdots \times [0, 1)$, taken modulo 1. Since the corners of the unit d-cube are all identified on $\mathbb{T}^d$, their images under matrix multiplication must all have integer entries to ensure they are each mapped to the origin, modulo 1. Thus the matrix representing the linear transformation must have integer entries. In addition, this matrix must have determinant ± 1 so that the map is a bijection.

This paper is concerned with *hyperbolic* toral automorphisms. If A is the matrix representing the toral automorphism, the product of the d (not necessarily distinct) eigenvalues of A is the determinant, which we require to be ± 1. A toral automorphism is hyperbolic when none of the eigenvalues are equal in magnitude to 1.

3. Preliminary results

Lemma 3.1 [Katok and Hasselblatt 1995]. *Any hyperbolic toral automorphism with a largest eigenvalue whose eigenvector has rationally independent entries is transitive.*

Proof. Let $U, V \subset \mathbb{T}^d$ be nonempty open sets. The set U must contain a line segment parallel to the eigenvector associated with the largest eigenvalue. Since this eigenvalue is greater than 1, under iteration the line segment grows without bound while remaining parallel to the eigenvector. Since the line "wraps around" the torus whenever the value of a coordinate exceeds 1, the distances between points where the line intersects the i-axis take on values that are multiples of the i-th entry in the eigenvector. As with the irrational rotation of the circle, as the number of iterates tends towards infinity, these intersection points are dense on the i-axis. Since the line stays parallel to the eigenvector, and the entries are rationally independent, the orbit of the line is dense in $\mathbb{T}^d$. This guarantees that the line intersects V after a finite number of iterations, and therefore U and V have nontrivial intersection for some number of iterations of f. □

Lemma 3.2 [Katok and Hasselblatt 1995]. *The rational points on the torus are periodic for any hyperbolic toral automorphism.*

Proof. Let

$$p = \left(\frac{p_1}{q}, \dots, \frac{p_d}{q} \right),$$

with $p_1, \dots, p_d, q \in \mathbb{N}$, be a point in $\mathbb{T}^d$ with rational coordinates (not necessarily in lowest terms). Since the entries of the matrix corresponding to the hyperbolic toral automorphism are all integers, the image of p is also a rational point with common denominator q. Since there are precisely q^d rational points in the unit square with denominator q (again, not necessarily in lowest terms), every such point can take on only finitely many values under iterates of the automorphism. Thus, each rational point is either periodic, or preperiodic (in the sense that p is mapped into a periodic orbit, but that orbit does not contain p). Since the automorphism is invertible, no points are preperiodic and therefore must be periodic, with maximum period q^d. □

In fact, only the rational coordinates are periodic. To see this, consider that periodic points of period n are in the kernel of $A^n - I_d$, where I_d is the identity matrix of dimension d. Since $A^n - I_d$ has integer entries, its kernel is composed only of vectors with rational entries.

Example 3.3 [Elaydi 2008; Katok and Hasselblatt 1995]. The canonical example of a hyperbolic toral automorphism is the Arnold "cat" map

$$L_A : \mathbb{T}^2 \to \mathbb{T}^2, \quad x \mapsto \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x \bmod 1. \tag{1}$$

The eigenvalues of the matrix are $\frac{1}{2}(3+\sqrt{5})$ and $\frac{1}{2}(3-\sqrt{5})$ with respective eigenvectors $\left[\frac{1}{2}(1+\sqrt{5}),\, 1\right]^\top$ and $\left[\frac{1}{2}(1-\sqrt{5}),\, 1\right]^\top$. You can see that one of the eigenvalues is larger than 1 and the other less, while both eigenvectors have irrational slope.

4. Main results

With the previous two lemmas, we have enough machinery to prove the main theorem pertaining to affine hyperbolic toral automorphisms. As in the introduction, an affine hyperbolic toral automorphism is a hyperbolic toral automorphism followed by a translation. We give two proofs of the result. The first gives the precise location of periodic points. The second relies on the fact that chaos is entirely topological and uses topological conjugacy.

Theorem 4.1. *Any affine hyperbolic toral automorphism is chaotic.*

Proof. Let $v_1, v_2, \dots, v_d$ be the eigenvectors of A associated with $\lambda_1, \lambda_2, \dots, \lambda_d$, respectively. The eigenvectors form a basis for $\mathbb{R}^d$, so for any translation $b \in \mathbb{R}^d$, b can be written as $b = b_1 v_1 + b_2 v_2 + \cdots + b_d v_d$ and any point on $x \in \mathbb{T}^d$ as $\sum_{i=1}^d x_i v_i$. So instead of $x^{n+1} = Ax^n + b$, we may write

$$x^{n+1} = A \sum_{i=1}^d x_i^n v_i + \sum_{i=1}^d b_i v_i.$$

We wish to find a closed form of x^m. For any point $x^0 \in \mathbb{T}^d$,

$$x^1 = Ax^0 + b = A \sum_{i=1}^d x_i^0 + \sum_{i=1}^d b_i v_i = \sum_{i=1}^d \lambda_i x_i^0 v_i + \sum_{i=1}^n b_i v_i,$$

$$x^2 = Ax^1 + b = A\left(\sum_{i=1}^d \lambda_i x_i^0 v_i + \sum_{i=1}^d b_i v_i\right) + \sum_{i=1}^n b_i v_i$$
$$= \sum_{i=1}^d \lambda_i^2 x_i^0 v_i + \sum_{i=1}^d \lambda_i b_i v_i + \sum_{i=1}^d b_i v_i,$$

$$x^3 = Ax^2 + b = A\left(\sum_{i=1}^d \lambda_i^2 x_i^0 v_i + \sum_{i=1}^d \lambda_i b_i v_i + \sum_{i=1}^n b_i v_i\right) + \sum_{i=1}^d b_i v_i$$
$$= \sum_{i=1}^d \lambda_i^3 x_i^0 v_i + \sum_{i=1}^d \lambda_i^2 b_i v_i + \sum_{i=1}^d \lambda_i b_i v_i + \sum_{i=1}^d b_i v_i.$$

The first three iterations suggest that

$$x^n = \sum_{i=1}^{d} \lambda_i^n x_i^0 v_i + \sum_{j=1}^{n} \sum_{i=1}^{d} \lambda_i^{j-1} b_i v_i . \tag{2}$$

Assume (2) as an induction hypothesis. Then we see that it also holds for $n+1$:

$$\begin{aligned} x^{n+1} = Ax^n + b &= A\left(\sum_{i=1}^{d} \lambda_i^n x_i^0 v_i + \sum_{j=1}^{n} \sum_{i=1}^{d} \lambda_i^{j-1} b_i v_i\right) + \sum_{i=1}^{d} b_i v_i \\ &= \sum_{i=1}^{d} \lambda_i^{n+1} x_i^0 v_i + \sum_{j=1}^{n} \sum_{i=1}^{d} \lambda_i^{j} b_i v_i + \sum_{i=1}^{d} b_i v_i \\ &= \sum_{i=1}^{d} \lambda_i^{n+1} x_i^0 v_i + \sum_{j=1}^{n+1} \sum_{i=1}^{d} \lambda_i^{j-1} b_i v_i . \end{aligned}$$

The last expression in (2) is not particularly revealing until we rewrite the double sum as

$$\sum_{j=1}^{n} \sum_{i=1}^{d} \lambda_i^{j-1} b_i v_i = \sum_{i=1}^{d} b_i v_i \sum_{j=1}^{n} \lambda_i^{j-1} = \sum_{i=1}^{d} b_i \frac{1-\lambda_i^n}{1-\lambda_i} v_i$$

and remember that we are looking for periodic points such that $x^0 = x^m \mod 1$. We are looking for $x = \sum_{i=1}^{n} x_i v_i \mod 1$ such that

$$\sum_{i=1}^{d} x_i v_i = \sum_{i=1}^{d} \lambda_i^n x_i v_i + \sum_{i=1}^{d} b_i \frac{1-\lambda_i^n}{1-\lambda_i} v_i \mod 1,$$

which leads to

$$\begin{aligned} 0 &= \sum_{i=1}^{d} \lambda_i^n x_i v_i - \sum_{i=1}^{d} x_i v_i + \sum_{i=1}^{d} b_i \frac{1-\lambda_i^n}{1-\lambda_i} v_i \mod 1 \\ &= \sum_{i=1}^{d} \left(\lambda_i^n x_i v_i - x_i v_i + b_i \frac{1-\lambda_i^n}{1-\lambda_i} v_i\right) \mod 1 \\ &= \sum_{i=1}^{d} \left((\lambda_i^n - 1) x_i v_i + b_i \frac{1-\lambda_i^n}{1-\lambda_i} v_i\right) \mod 1 \\ &= \sum_{i=1}^{d} (\lambda_i^n - 1)\left(x_i v_i + \frac{b_i}{\lambda_i - 1} v_i\right) \mod 1 \\ &= \sum_{i=1}^{d} (\lambda_i^n - 1)\left(x_i + \frac{b_i}{\lambda_i - 1}\right) v_i \mod 1, \end{aligned}$$

from which we can conclude that the periodic points of the affine map are precisely those of the linear map translated by $\sum_{i=1}^{d}(b_i/(\lambda_i - 1))v_i$. Since the periodic points of the linear map are dense, so too are the periodic points of the affine map. In addition, if U, V are open in $\mathbb{T}^d$, then there exists an n such that the n-th iterate of the linear map of U intersects V. Thus, the affine map is chaotic. □

There is another proof of the main result that uses far less calculation, but does not give the new locations of periodic points. We use the fact that the linear and affine hyperbolic toral automorphisms $f(x)$ and $g(x) = f(x) + b$ are topologically conjugate, so that the following diagram commutes:

$$\begin{tikzcd} \mathbb{T}^d \arrow[r, "f"] \arrow[d, "h"] & \mathbb{T}^d \arrow[d, "h"] \\ \mathbb{T}^d \arrow[r, "g"] & \mathbb{T}^d \end{tikzcd}$$

Alternate proof of Theorem 4.1. If we suppose f is defined by multiplication by a matrix A and g is multiplication by A followed by translation by b, we must define the homeomorphism h so that $f = h^{-1} \circ g \circ h$. This homeomorphism h is simply translation by some $\tilde{b}$:

$$Ax = A(x+\tilde{b}) + b - \tilde{b} = Ax + A\tilde{b} + b - \tilde{b} = Ax + (A - I_d)\tilde{b} + b$$
$$\implies \quad (A - I_d)\tilde{b} = -b \quad \implies \quad \tilde{b} = -(A - I_d)^{-1}b.$$

We know $A - I_d$ is invertible because A has no eigenvalues equal to 1.

Since f and g are topologically conjugate, they have the same properties regarding transitivity and periodic points. The transitivity and dense periodic points of linear f are known, so they hold also for the affine g. □

Corollary 4.2. *Any affine hyperbolic transformation of $\mathbb{T}^2$ is chaotic.*

Proof. Suppose the linear part of the transformation is multiplication by a matrix A. Then the roots of the characteristic polynomial are

$$\det(A - \lambda I_2) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc)$$
$$\implies \quad \lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

The discriminant $(a+d)^2 - 4(ad-bc) = (a+d)^2 - 4$ cannot be a perfect square because 2 is not part of any Pythagorean triple. Thus the eigenvalues are irrational, which in turn implies that each of the eigenvectors has irrational slope. This is sufficient since all affine hyperbolic functions on $\mathbb{T}^2$ meet the criteria of the main result, and are thus chaotic. □

Example 4.3. We now give a specific example of the previous corollary based on Example 3.3. Let $L_{A,b} : \mathbb{T}^2 \to \mathbb{T}^2$ be defined as

$$L_{A,b}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{3}/3 \end{bmatrix}. \tag{3}$$

Recall that the eigenvectors of the matrix A are $v_1 = \left[\frac{1}{2}(1+\sqrt{5}),\, 1\right]^\top$ and $v_2 = \left[\frac{1}{2}(1-\sqrt{5}),\, 1\right]^\top$, with eigenvalues $\lambda_1 = \frac{1}{2}(3+\sqrt{5})$ and $\lambda_2 = \frac{1}{2}(3-\sqrt{5})$, so that $\lambda_1 > 1 > \lambda_2 > 0$. The points

$$p_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \quad \text{and} \quad p_{10} = \begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix}$$

are fixed and periodic of periods 3 and 10 respectively under Example 3.3. To find the corresponding periodic points q_1, q_3, q_{10} under $L_{A,b}$, first calculate b_1, b_2, which are the projections of b against v_1, v_2.[1] Then add the translation prescribed by Theorem 4.1:

$$\frac{b_1}{1-\lambda_1}v_1 + \frac{b_2}{1-\lambda_2}v_2 = \frac{\langle b, v_1\rangle}{1-\lambda_1}v_1 + \frac{\langle b, v_2\rangle}{1-\lambda_2}v_2 \approx \begin{bmatrix} .4227 \\ .8702 \end{bmatrix},$$

$$q_1 \approx p_1 + \begin{bmatrix} .4227 \\ .8702 \end{bmatrix} = \begin{bmatrix} .4227 \\ .8702 \end{bmatrix},$$

$$q_3 \approx p_3 + \begin{bmatrix} .4227 \\ .8702 \end{bmatrix} = \begin{bmatrix} .9227 \\ .3702 \end{bmatrix},$$

$$q_{10} \approx p_{10} + \begin{bmatrix} .4227 \\ .8702 \end{bmatrix} = \begin{bmatrix} .6227 \\ .4702 \end{bmatrix}.$$

One can check numerically that indeed

$$L_{A,b}(q_1) = q_1, \quad L^3_{A,b}(q_3) = q_3, \quad L^{10}_{A,b}(q_{10}) = q_{10},$$

and for each this is the minimum number of iterations required.

Corollary 4.4. *Any function $f_{k,\alpha}$ on the circle given by $f_{k,\alpha} : \theta \mapsto k\theta + \alpha$ with $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and $\alpha \in S^1$ is chaotic.*

Proof. The slope of an eigenvector degenerates for $S^1 = \mathbb{T}^1$, and in any case the function $f_{k,\alpha}$ is not an automorphism. In the sense that $f_{k,\alpha}$ is a function that is known to be chaotic followed by a (possibly irrational) rotation, the main result holds.

First, an alternative explanation for the transitivity of $f_{k,\alpha}$ is in order. Any open set U in S^1 contains an open arc (θ_1, θ_2) with length $\theta_2 - \theta_1$. Define $m = 2\pi/(\theta_2 - \theta_1)$. After $n \geq k^m$ iterations of $f_{k,\alpha}$ on U, we have $f^n_{k,\alpha}(U) = S^1$. (A function with this

[1] Remember that all arithmetic is performed modulo 1.

condition is said to be *locally eventually onto*.) Since every open set is "eventually onto" the circle, any U will certainly have nontrivial intersection with any nonempty open $V \in S^1$ after a finite number of iterations.

The periodic points of $f_k = f_{k,0}$ are the rational points in [0, 1) with denominator one less than a power of k. Moreover, if this denominator is $q = k^n - 1$, then the point $p/q \in \mathbb{Q} \cap [0, 1)$ will have period n if p/q is fully reduced. To see this, note that

$$f_k^n\left(\frac{p}{q}\right) = k^n \frac{p}{k^n - 1} = (k^n - 1 + 1)\frac{p}{k^n - 1} = (k^n - 1)\frac{p}{k^n - 1} + \frac{p}{k^n - 1} = p + \frac{p}{q} = \frac{p}{q}$$

because $p \in \mathbb{N}$ and all of our arithmetic is modulo 1. Since q can be chosen arbitrarily large and $p = 0, 1, \ldots, q - 1$, with p/q evenly spaced about the circle, the periodic points of f_k are dense in S^1.

We can now use the closed form from the proof of Theorem 4.1 to find the periodic points of $f_{k,\alpha}$. We are searching for points such that $f_{k,\alpha}^n(x) = x \mod 1$, or

$$(k^n - 1)\left(x + \frac{\alpha}{k - 1}\right) = x \mod 1.$$

This shows that the periodic points of $f_{k,\alpha}$ are rational points of f_k rotated about the circle by $\alpha/(k - 1)$. Since the locally eventually onto property is preserved, and periodic points are still dense, $f_{k,\alpha}$ is chaotic. □

5. Conclusion

Our main result shows that affine hyperbolic toral automorphisms are chaotic. The added translation by a vector b preserves the transitivity of the map and translates all of the periodic points by $\sum_{i=1}^{d}(b_i/(\lambda_i - 1))v_i$, where the v_i are eigenvectors, λ_i the corresponding eigenvalues, and b_i the coordinates of the translation vector b in the basis defined by the eigenvectors. Note that in the case that $b = 0$, the periodic points are not translated at all, which coincides with a linear hyperbolic toral automorphism.

Using this translation result, one can construct an automorphism of the torus in which any specified point y has a specified period n: Find an x such that x has period n under a linear hyperbolic toral automorphism. By Lemma 3.2, x will have rational coordinates in the standard basis (but not necessarily in the basis defined by the eigenvectors of the linear automorphism). Then define b such that $b_i = (\lambda_i - 1)(y_i - x_i) \mod 1$, where b_i, x_i, y_i are the coordinates in the basis defined by the eigenvectors of the linear toral automorphism. The resulting affine hyperbolic toral automorphism will have y as a periodic point with period n.

More generally, the main result shows that the incorporation of an irrational rotation into a toral automorphism does not necessarily eliminate the possibility of periodic points.

References

[Banks et al. 1992] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, "On Devaney's definition of chaos", *Amer. Math. Monthly* **99**:4 (1992), 332–334. MR 1157223 Zbl 0758.58019

[Crannell 1995] A. Crannell, "The role of transitivity in Devaney's definition of chaos", *Amer. Math. Monthly* **102**:9 (1995), 788–797. MR 1357723 Zbl 0849.58046

[Değirmenci and Koçak 2010] N. Değirmenci and Ş. Koçak, "Chaos in product maps", *Turkish J. Math.* **34**:4 (2010), 593–600. MR 2721970 Zbl 1208.37010

[Elaydi 2008] S. N. Elaydi, *Discrete chaos: With applications in science and engineering*, 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2008. MR 2364977 Zbl 1153.39002

[Katok and Hasselblatt 1995] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications **54**, Cambridge Univ. Press, 1995. MR 1326374 Zbl 0878.58020

[Li and Zhou 2013] R. Li and X. Zhou, "A note on chaos in product maps", *Turkish J. Math.* **37**:4 (2013), 665–675. MR 3070943 Zbl 1293.37015

[Vellekoop and Berglund 1994] M. Vellekoop and R. Berglund, "On intervals, transitivity = chaos", *Amer. Math. Monthly* **101**:4 (1994), 353–355. MR 1270961 Zbl 0886.58033

Received: 2014-01-30 Accepted: 2015-08-17

cfcthomson@gmail.com University of North Carolina at Chapel Hill, Phillips Hall CB#3250, Chapel Hill, NC 27599, United States

domolinek@davidson.edu Davidson College, Box 6999, Davidson, NC 28035, United States

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2016 is US $160/year for the electronic version, and $215/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2016 vol. 9 no. 4