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Rings of invariants for the three-dimensional modular representations of elementary abelian p-groups of rank four

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We show that the rings of invariants for the three-dimensional modular representations of an elementary abelian p-group of rank four are complete intersections with embedding dimension at most five. Our results confirm the conjectures of Campbell, Shank and Wehlau (*Transform. Groups* 18 (2013), 1–22) for these representations.

Introduction

We continue the investigation of the rings of invariants of modular representations of elementary abelian p-groups initiated in [Campbell et al. 2013]. We show that the rings of invariants for three-dimensional modular representations of groups of rank four are complete intersections and we confirm the conjectures of [loc. cit., §8] for these representations.

Let V denote an n-dimensional representation of a group G over a field $\mathbb{F}$ of characteristic p for a prime number p. We will usually assume that G is finite and that p divides the order of G, in other words, that V is a *modular representation* of G. We view V as a left module over the group ring $\mathbb{F}G$ and the dual, V^*, as a right $\mathbb{F}G$-module. Let $\mathbb{F}[V]$ denote the symmetric algebra on V^*. The action of G on V^* extends to an action by degree-preserving algebra automorphisms on $\mathbb{F}[V]$. By choosing a basis $\{x_1, x_2, \ldots, x_n\}$ for V^*, we identify $\mathbb{F}[V]$ with the algebra of polynomials $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Our convention that $\mathbb{F}[V]$ is a right $\mathbb{F}G$-module is consistent with the convention used by the invariant theory package in the computer algebra software Magma [Bosma et al. 1997]. The ring of invariants, $\mathbb{F}[V]^G$, is the subring of $\mathbb{F}[V]$ consisting of those polynomials fixed by the action of G. Note that elements of $\mathbb{F}[V]$ represent polynomial functions on V and that elements of $\mathbb{F}[V]^G$ represent polynomial functions on the set of orbits V/G. For G finite and $\mathbb{F}$ algebraically

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closed, $\mathbb{F}[V]^G$ is the ring of regular functions on the categorical quotient $V/\!\!/G$. For background on the invariant theory of finite groups, see [Benson 1993; Campbell and Wehlau 2011; Derksen and Kemper 2002; Neusel and Smith 2002].

Computing the ring of invariants for a modular representation is typically a difficult problem; the rings are often not Cohen–Macaulay. It is natural to take p-groups as a starting point and recent work of David Wehlau [2013] gives us a good understanding in the case of a cyclic group of order p. The next step is to look at elementary abelian p-groups. The rings of invariants for the two-dimensional modular representations of elementary abelian p-groups were computed in Section 2 of [Campbell et al. 2013] and the three-dimensional modular representations were classified in Section 4 of that paper. The only three-dimensional representations for which computing the ring of invariants is not straightforward are those of type $(1, 1, 1)$, in other words, those representations for which $\dim(V^G) = 1$ and $\dim((V/V^G)^G) = 1$. Our goal here is to compute the rings of invariants for representations of type $(1, 1, 1)$ for groups of rank four. The methods we use are essentially the same as the methods used in [loc. cit.]. As the rank increases, the complexity of the required calculations increases; we believe that it is not feasible to use the methods here for rank greater than four.

We denote by $E = \langle e_1, e_2, e_3, e_4\rangle \cong (\mathbb{Z}/p)^4$ a rank-four elementary abelian p-group. Note that E only has representations of type $(1, 1, 1)$ if $p > 2$, so we make this assumption throughout the paper. As in Section 4 of [loc. cit.], define $\sigma : \mathbb{F}^2 \to \mathrm{GL}_3(\mathbb{F})$ by

$$\sigma(c_1, c_2) := \begin{pmatrix} 1 & 2c_1 & c_1^2+c_2 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that σ defines a representation of the group $(\mathbb{F}^2, +)$. For a matrix

$$M := \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{pmatrix}$$

with $c_{ij} \in \mathbb{F}$, the assignment $e_j \mapsto \sigma(c_{1j}, c_{2j})$ determines a three-dimensional representation of E, which we denote by V_M. The action of E on $\mathbb{F}[x, y, z]$ is given by right multiplication on $x = [0\ \ 0\ \ 1]$, $y = [0\ \ 1\ \ 0]$ and $z = [1\ \ 0\ \ 0]$. Thus $x \cdot \sigma(c_1, c_2) = x$, $y \cdot \sigma(c_1, c_2) = y + c_1 x$ and $z \cdot \sigma(c_1, c_2) = z + 2c_1 y + (c_1^2 + c_2)x$. The representation V_M is of type $(1, 1, 1)$ if at least one c_{1j} is nonzero. Furthermore, by Proposition 4.1 of [loc. cit.], for every representation of type $(1, 1, 1)$, there exists a choice of basis for which the action is given by some matrix M.

In this paper, we compute $\mathbb{F}[V_M]^E$ for all $M \in \mathbb{F}^{2\times 4}$. We give a stratification of $\mathbb{F}^{2\times 4}$ and show that within each stratum there is a uniform computation of $\mathbb{F}[V_M]^E$. Note that the automorphism group of E is isomorphic to $\mathrm{GL}_4(\mathbb{F}_p)$, where $\mathbb{F}_p$ denotes the field of p elements. Since $\mathbb{F}_p \subseteq \mathbb{F}$, there is a natural right action of $\mathrm{GL}_4(\mathbb{F}_p)$

on $\mathbb{F}^{2\times 4}$. If M and M' lie in the same $\mathrm{GL}_4(\mathbb{F}_p)$-orbit, then $\mathbb{F}[V_M]^E = \mathbb{F}[V_{M'}]^E$. Essentially, we study subrings of $\mathbb{F}[x, y, z]$ parametrised by points in $\mathbb{F}^{2\times 4}/\mathrm{GL}_4(\mathbb{F}_p)$ and use elements of $\mathbb{F}[\mathbb{F}^{2\times 4}]^{\mathrm{SL}_4(\mathbb{F}_p)}$ to describe the stratification.

In Section 2, we work over the field $\Bbbk := \mathbb{F}_p(x_{ij} \mid i \in \{1, 2\},\ j \in \{1, 2, 3, 4\})$ and compute $\Bbbk[V_{\mathcal{M}}]^E$ for the generic matrix

$$\mathcal{M} := \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}.$$

We show that $\Bbbk[V_{\mathcal{M}}]^E$ is a complete intersection of embedding dimension five with generators in degrees $1,\ p^2,\ p^2+2p,\ p^3+2$ and p^4, and relations in degrees p^3+2p^2 and p^4+2p. Consider the 10×4 matrix

$$\Gamma := \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{11}^p & x_{12}^p & x_{13}^p & x_{14}^p \\ x_{21}^p & x_{22}^p & x_{23}^p & x_{24}^p \\ \vdots & \vdots & \vdots & \vdots \\ x_{11}^{p^4} & x_{12}^{p^4} & x_{13}^{p^4} & x_{14}^{p^4} \\ x_{21}^{p^4} & x_{22}^{p^4} & x_{23}^{p^4} & x_{24}^{p^4} \end{pmatrix}$$

and for a subsequence (i, j, k, ℓ) of $(1, 2, \ldots, 10)$, let $\gamma_{ijk\ell}$ denote the associated 4×4 minor of Γ. Note that $\gamma_{ijk\ell} \in \mathbb{F}[\mathbb{F}^{2\times 4}]^{\mathrm{SL}_4(\mathbb{F}_p)}$ and, for $g \in \mathrm{GL}_4(\mathbb{F}_p)$, we have $g(\gamma_{ijk\ell}) = \det(g)\gamma_{ijk\ell}$. We use zero-sets of various $\gamma_{ijk\ell}$ to define the stratification of $\mathbb{F}^{2\times 4}/\mathrm{GL}_4(\mathbb{F}_p)$. In Section 3, we show that for $M \in \mathbb{F}^{2\times 4}$ with $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$, the generic calculation survives evaluation. In Sections 4 through 10, we compute the rings of invariants for the remaining strata.

Section 4: $\gamma_{1357}(M) \neq 0,\ \gamma_{1235}(M) \neq 0,\ \gamma_{1234}(M) = 0$. We show $\mathbb{F}[V_M]^E$ is a complete intersection with generators in degrees $1,\ 2p,\ p^3,\ p^3+2$ and p^4, and relations in degrees $2p^3$ and p^4+2p.

Section 5: $\gamma_{1357}(M) \neq 0,\ \gamma_{1235}(M) = 0,\ \gamma_{1234}(M) \neq 0$. If $\gamma_{1245}(M) \neq 0$ then $\mathbb{F}[V_M]^E$ is a complete intersection with generators in degrees $1,\ p^2,\ p^2+p,\ p^3+p+2$ and p^4, and relations in degrees p^3+p^2 and p^4+p^2+2p. Otherwise, $\mathbb{F}[V_M]^E$ is a hypersurface with generators in degrees $1,\ p^2,\ p^2+2$ and p^4, with the relation in degree p^4+2p^2.

Section 6: $\gamma_{1357}(M) = 0,\ \gamma_{1235}(M) \neq 0,\ \gamma_{1234}(M) \neq 0$. We show $\mathbb{F}[V_M]^E$ is a complete intersection with generators in degrees $1,\ p^2,\ p^2+2p,\ p^3+1$ and p^4, and relations in degrees p^3+2p^2 and p^4+p.

Section 7: $\gamma_{1357}(M) \neq 0,\ \gamma_{1235}(M) = 0,\ \gamma_{1234}(M) = 0$. We show $\mathbb{F}[V_M]^E$ is a hypersurface. If $\gamma_{1257}(M) = 0$, then the generators are in degrees $1,\ 2,\ p^4$ and p^4

and the relation is in degree $2p^4$. Otherwise, the generators are in degrees 1, p, p^3+p^2+p+2, p^4 and the relation is in degree $p^4+p^3+p^2+2p$.

Section 8: $\gamma_{1357}(M)=0$, $\gamma_{1235}(M)\neq 0$, $\gamma_{1234}(M)=0$. We show $\mathbb{F}[V_M]^E$ is a complete intersection with generators in degrees 1, $2p$, p^3, p^3+1 and p^4, with relations in degrees $2p^2$ and p^4+p.

Section 9: $\gamma_{1357}(M)=0$, $\gamma_{1235}(M)=0$, $\gamma_{1234}(M)\neq 0$. If $\gamma_{1245}(M)\neq 0$, then $\mathbb{F}[V_M]^E$ is a complete intersection with generators in degrees 1, p^2, p^2+p, p^3+1 and p^4, with relations in degrees p^3+p^2 and p^4+p. Otherwise, $\mathbb{F}[V_M]^E$ is a hypersurface with generators in degrees 1, p^2, p^2+1 and p^4, with a relation in degree p^4+p^2.

Section 10: $\gamma_{1357}(M)=0$, $\gamma_{1235}(M)=0$, $\gamma_{1234}(M)=0$. If $\gamma_{1246}(M)\neq 0$ then $\mathbb{F}[V_M]^E$ is a hypersurface with generators in degrees 1, p, p^3+1, p^4 and a relation in degree p^4+p. Otherwise, the representation is either not faithful or not of type $(1, 1, 1)$; in either case, the invariants were computed in [Campbell et al. 2013].

1. Preliminaries

We make extensive use of the theory of SAGBI bases to compute rings of invariants. A SAGBI basis is the subalgebra analogue of a Gröbner basis for ideals, and is a particularly nice generating set for the subalgebra. The concept was introduced independently by Robbiano and Sweedler [1990] and Kapur and Madlener [1989]; a useful reference is Chapter 11 of Sturmfels [1996]. We adopt the convention that a monomial is a product of variables and a term is a monomial with a coefficient. We use the graded reverse lexicographic order with $x<y<z$. For a polynomial $f\in\mathbb{F}[x,y,z]$, we denote the lead monomial of f by $\mathrm{LM}(f)$ and the lead term of f by $\mathrm{LT}(f)$. For $\mathcal{B}=\{h_1,\dots,h_\ell\}\subset\mathbb{F}[x,y,z]$ and $I=(i_1,\dots,i_\ell)$, a sequence of nonnegative integers, denote $\prod_{j=1}^{\ell} h_j^{i_j}$ by h^I. A *tête-à-tête* for $\mathcal{B}$ is a pair (h^I,h^J) with $\mathrm{LM}(h^I)=\mathrm{LM}(h^J)$; we say that a tête-à-tête is *nontrivial* if the support of I is disjoint from the support of J. The reduction of an S-polynomial is a fundamental calculation in the theory of Gröbner bases. The analogous calculation for SAGBI bases is the *subduction* of a tête-à-tête. For any $f\in\mathbb{F}[x,y,z]$, if there exists a sequence I such that $\mathrm{LM}(f)=\mathrm{LM}(h^I)$, we can choose $c\in\mathbb{F}$ so that $\mathrm{LT}(f)=\mathrm{LT}(ch^I)$. Then $\mathrm{LT}(f-ch^I)<\mathrm{LT}(f)$. If by iterating this process we can write f as a polynomial in the h_i, we say that f subducts to zero (using $\mathcal{B}$). For a tête-à-tête (h^I,h^J), choose c so that $\mathrm{LT}(h^I)=\mathrm{LT}(ch^J)$. We say that the tête-à-tête subducts to zero if h^I-ch^J subducts to zero. A subset $\mathcal{B}$ of a subalgebra $A\subset\mathbb{F}[x_1,\dots,x_n]$ is a SAGBI basis for A if the lead monomials of the elements of $\mathcal{B}$ generate the lead term algebra of A or, equivalently, every nontrivial tête-à-tête for $\mathcal{B}$ subducts to zero. For background material on term orders and Gröbner bases, we recommend [Adams and Loustaunau 1994].

The following specialisation of Theorem 1.1 of [Campbell et al. 2013] is our primary computational tool. Note that under the hypotheses of the theorem, $\{x, h_1, h_\ell\}$ is a homogeneous system of parameters and, therefore, $\mathbb{F}[V_M]^E$ is an integral extension of A.

Theorem 1.1. *For homogeneous* $h_1, \dots, h_\ell \in \mathbb{F}[V_M]^E$ *with* $\mathrm{LM}(h_1) = y^i$ *for some* $i > 0$, $\mathrm{LM}(h_\ell) = z^j$ *for some* $j > 0$ *and* $\mathrm{LM}(h_k) \in \mathbb{F}[y, z]$ *for* $k = 2, \dots, \ell - 1$, *define* $\mathcal{B} := \{x, h_1, \dots, h_\ell\}$ *and let* A *denote the algebra generated by* $\mathcal{B}$. *If* $A[x^{-1}] = \mathbb{F}[V_M]^E[x^{-1}]$ *and* $\mathcal{B}$ *is a SAGBI basis for* A, *then* $A = \mathbb{F}[V_M]^E$ *and* $\mathcal{B}$ *is a SAGBI basis for* $\mathbb{F}[V_M]^E$.

Note that, if an algebra is generated by a finite SAGBI basis, then for the corresponding presentation, the ideal of relations is generated by elements corresponding to the subductions of the nontrivial tête-à-têtes (see Corollary 11.6 of [Sturmfels 1996]). We use the term *complete intersection* to refer to an algebra with a presentation for which the ideal of relations is generated by a regular sequence. Since the Krull dimension of $\mathbb{F}[V_M]^E$ is three, the ring is a complete intersection if the number of generators minus the number of nontrivial tête-à-têtes is three.

We routinely use the *SAGBI/divide-by-x* algorithm introduced in Section 1 of [Campbell et al. 2013]. The traditional SAGBI basis algorithm proceeds by subducting tête-à-têtes and adding any nonzero subductions to the generating set. For SAGBI/divide-by-x, if a nonzero subduction is divisible by x, we divide by the highest possible power of x before adding the polynomial to the generating set. While the SAGBI algorithm extends the generating set for a given subalgebra, SAGBI/divide-by-x extends the subalgebra. If we start with a subalgebra A which contains a homogeneous system of parameters and satisfies the condition that $A[x^{-1}] = \mathbb{F}[V_M]^E[x^{-1}]$, then the SAGBI/divide-by-x algorithm will produce a generating set for $\mathbb{F}[V_M]^E$ (see Theorem 1.2 of [loc. cit.]).

For $f \in \mathbb{F}[V_M]$, we define the *norm* of f to be the orbit product

$$N_M(f) := \prod \{f \cdot g \mid g \in E\} \in \mathbb{F}[V_M]^E$$

with the action of E determined by M. When applying Theorem 1.1, we often take h_ℓ to be $N_M(z)$.

Remark 1.2. Note that the action of E restricts to an action on $\mathbb{F}[x, y]$ and that $\mathbb{F}[x, y]^E = \mathbb{F}[x, N_M(y)]$ (see Section 2 of [Campbell et al. 2013]). Therefore, if $h \in \mathbb{F}[x, y]^E$ is homogeneous with $\deg(h) = |\{y \cdot g \mid g \in E\}|$ then h is a linear combination of $N_M(y)$ and $x^{\deg(h)}$.

Define $\delta := y^2 - xz$ and observe that

$$\delta \cdot \sigma(c_1, c_2) = (y + c_1 x)^2 - x\big(z + 2c_1 y + (c_1^2 + c_2)x\big) = \delta - c_2 x^2.$$

Note that $\mathbb{F}[x, y, z][x^{-1}] = \mathbb{F}[x, y, -\delta/x][x^{-1}]$ and that the $\mathbb{F}[x, y, -\delta/x]^E$ is a polynomial algebra (see Theorem 3.9.2 of [Campbell and Wehlau 2011]). This "change of basis" can be a useful way to compute the field of fractions of $\mathbb{F}[V_M]^E$. Form the matrix $\widetilde{\Gamma}$ by augmenting Γ with the column

$$\left[\frac{y}{x} \ \left(-\frac{\delta}{x^2}\right) \ \left(\frac{y}{x}\right)^p \ \left(-\frac{\delta}{x^2}\right)^p \ \cdots \ \left(\frac{y}{x}\right)^{p^4} \ \left(-\frac{\delta}{x^2}\right)^{p^4}\right]^T.$$

For a subsequence $J = (j_1, \dots, j_5)$ of $(1, 2, \dots, 10)$, let $\tilde{f}_J \in \Bbbk[x, y, z][x^{-1}]$ denote the associated 5×5 minor of $\widetilde{\Gamma}$. Let f_J denote the element of $\Bbbk[x, y, z]$ constructed by minimally clearing the denominator of $\tilde{f}_J$. Observe that $f_J \in \Bbbk[V_{\mathcal{M}}]^E$. Furthermore, the coefficients of f_J lie in $\mathbb{F}_p[x_{ij}]^{\mathrm{SL}_4(\mathbb{F}_p)}$ and, for an arbitrary $M \in \mathbb{F}^{2\times 4}$, evaluating the coefficients of f_J at M gives an element $\bar{f}_J \in \mathbb{F}[V_M]^E$. Invariants constructed in this way are a crucial ingredient in our calculations. Define $f_1 := f_{12345}$ and observe that $\mathrm{LT}(f_1) = \gamma_{1234} y^{p^2}$. Note that $\mathrm{LT}(f_{12346}) = -\gamma_{1234} y^{2p^2}$. A straightforward calculation shows that

$$\mathrm{LT}(f_1^2 + \gamma_{1234} f_{12346}) = 2\gamma_{1234}\gamma_{1235} x^{p^2-2p} y^{p^2+2p}.$$

Therefore,

$$f_2 := \frac{f_1^2 + \gamma_{1234} f_{12346}}{2x^{p^2-2p}} \in \Bbbk[V_{\mathcal{M}}]^E$$

has lead term $\gamma_{1234}\gamma_{1235} y^{p^2+2p}$.

We make frequent use of the *Plücker relations* for the minors of Γ and $\widetilde{\Gamma}$.

Theorem 1.3. *Let N be an $n \times m$ matrix with $n > m$. Denote by $p_{i_1,\dots,i_m}$ the $m \times m$ minor of N determined by the rows $i_1, \dots, i_m$. For sequences $(i_1, \dots, i_{m-1})$ and $(j_1, \dots, j_{m+1})$, we have the following Plücker relation*

$$\sum_{a=1}^{m+1} (-1)^a p_{i_1,\dots,i_{m-1},j_a} p_{j_1,\dots,j_{a-1},j_{a+1},\dots,j_{m+1}} = 0.$$

For a proof of the above theorem, see, for example, [Lakshmibai and Raghavan 2008, §4.1.3].

Lemma 1.4. *For $2 < i < 7$,*

$$\gamma_{12i7}\gamma_{1234}^p = \gamma_{12i6}\gamma_{1235}^p - \gamma_{12i5}\gamma_{1245}^p + \gamma_{12i4}\gamma_{1345}^p - \gamma_{12i3}\gamma_{2345}^p.$$

Proof. Since taking p-th powers is $\mathbb{F}_p$-linear, $\gamma_{(i+2)(j+2)(k+2)(\ell+2)} = \gamma_{ijk\ell}^p$. For example, $\gamma_{3456} = \gamma_{1234}^p$. The desired result follows from this fact, using the $(1, 2, i)(3, 4, 5, 6, 7)$ Plücker relation for the matrix Γ. □

For $K = (k_1, k_2, \dots, k_6)$ a subsequence of $(1, 2, \dots, 10)$, let K_i denote the subsequence of K formed by omitting i and let $K_{i,j}$ denote the subsequence of K formed by omitting i and j. The following is Lemma 5.3 from [Campbell et al. 2013].

Lemma 1.5. *For any subsequence* (i_1, i_2, i_3) *of* K,

$$(-1)^{\epsilon_1}\gamma_{K_{i_1,i_2}}\tilde{f}_{K_{i_3}} + (-1)^{\epsilon_2}\gamma_{K_{i_2,i_3}}\tilde{f}_{K_{i_1}} + (-1)^{\epsilon_3}\gamma_{K_{i_1,i_3}}\tilde{f}_{K_{i_2}} = 0$$

for some choice of $\epsilon_\ell \in \{0, 1\}$.

Remark 1.6. Note that $\gamma_{1357}(M) = 0$ if and only if $\{c_{11}, c_{12}, c_{13}, c_{14}\}$ is linearly dependent over $\mathbb{F}_p$. This follows from the usual construction of the Dickson invariants; see, for example, [Wilkerson 1983]. The key observation is that $\gamma_{1357}(M)^{p-1}$ is the product of the nonzero $\mathbb{F}_p$-linear combinations of $\{c_{11}, c_{12}, c_{13}, c_{14}\}$.

2. The generic case

In this section we compute $\Bbbk[V_{\mathcal{M}}]^E$. With f_1 and f_2 defined as in Section 1, using Theorem 5.2 of [Campbell et al. 2013], we see that

$$\Bbbk[V_{\mathcal{M}}]^E[x^{-1}] = \Bbbk[x, f_1, f_2][x^{-1}].$$

Thus it is sufficient to extend $\{x, f_1, f_2, N_{\mathcal{M}}(z)\}$ to a SAGBI basis. We use the *SAGBI/divide-by-x* algorithm of [loc. cit., §1] to do this. We will show that the algorithm produces one new invariant, which we denote by f_3, and that

$$\mathrm{LT}(f_3) = \gamma_{1357} y^{p^3+2}.$$

For $p = 3$ and $p = 5$, this result follows from a Magma calculation. For the rest of this section, we assume $p > 5$.

Expanding the definitions of f_1, f_{12346} and f_2 gives

$$f_1 = \gamma_{1234} y^{p^2} + \gamma_{1235}\delta^p x^{p^2-2p} + \gamma_{1245} x^{p^2-p} y^p + \gamma_{1345}\delta x^{p^2-2} + \gamma_{2345} x^{p^2-1} y,$$

$$f_{12346} = -\gamma_{1234}\delta^{p^2} + \gamma_{1236}\delta^p x^{2p^2-2p} + \gamma_{1246} x^{2p^2-p} y^p + \gamma_{1346}\delta x^{2p^2-2} + \gamma_{2346} x^{2p^2-1} y$$

and

$$\begin{aligned}
f_2 &= \frac{f_1^2 + \gamma_{1234} f_{12346}}{2x^{p^2-2p}} \\
&= \gamma_{1234}\gamma_{1235} y^{p^2}\delta^p + \gamma_{1234}\gamma_{1245} x^p y^{p^2+p} + \gamma_{1234}\gamma_{1345}\delta x^{2p-2} y^{p^2} \\
&\quad + \gamma_{1234}\gamma_{2345} x^{2p-1} y^{p^2+1} + \tfrac{1}{2}\gamma_{1234}^2 x^{2p} z^{p^2} + \tfrac{1}{2}\gamma_{1235}^2 \delta^{2p} x^{p^2-2p} \\
&\quad + \gamma_{1235}\gamma_{1245}\delta^p x^{p^2-p} y^p + \gamma_{1235}\gamma_{1345}\delta^{p+1} x^{p^2-2} + \gamma_{1235}\gamma_{2345}\delta^p x^{p^2-1} y \\
&\quad + \tfrac{1}{2}\gamma_{1234}\gamma_{1236} x^{p^2}\delta^p + \tfrac{1}{2}\gamma_{1245}^2 x^{p^2} y^{2p} + \gamma_{1245}\gamma_{1345}\delta x^{p^2+p-2} y^p \\
&\quad + \gamma_{1245}\gamma_{2345} x^{p^2+p-1} y^{p+1} + \tfrac{1}{2}\gamma_{1234}\gamma_{1246} y^p x^{p^2+p} + \tfrac{1}{2}\gamma_{1345}^2\delta^2 x^{p^2+2p-4} \\
&\quad + \gamma_{1345}\gamma_{2345}\delta x^{p^2+2p-3} y + \tfrac{1}{2}\gamma_{2345}^2 x^{p^2+2p-2} y^2 \\
&\quad + \tfrac{1}{2}\gamma_{1234}\gamma_{1346}\delta x^{p^2+2p-2} + \tfrac{1}{2}\gamma_{1234}\gamma_{2346} x^{p^2+2p-1} y.
\end{aligned}$$

Subducting the tête-à-tête (f_1^{p+2}, f_2^p) gives

$$\tilde{f}_3 = \underbrace{\gamma_{1235}^p f_1^{p+2}}_{T_1} - \underbrace{\gamma_{1234}^2 f_2^p}_{T_2} + \underbrace{\alpha_1 x^{p^2-2p} f_1^p f_2}_{T_3} + \underbrace{\alpha_2 x^{p^2} f_1^{p+1}}_{T_4} + \underbrace{\alpha_3 x^{2p^2-2p} f_1^{p-1} f_2}_{T_5} + \underbrace{\alpha_4 x^{2p^2-p} f_1^{(p-3)/2} f_2^{(p+1)/2}}_{T_6},$$

where

$$\alpha_1 = -2\gamma_{1235}^p, \quad \alpha_2 = \gamma_{1234}\gamma_{1245}^p, \quad \alpha_3 = \frac{\gamma_{1234}^{p+1}\gamma_{1237}}{\gamma_{1235}}, \quad \alpha_4 = \frac{\gamma_{1234}^{p+3}\gamma_{1257}}{\gamma_{1235}^{(p+3)/2}}.$$

Lemma 2.1. *For $p \geq 5$, we have* $\operatorname{LT}(\tilde{f}_3) = \alpha x^{2p^2-2} y^{p^3+2}$ *with*

$$\alpha = \frac{\gamma_{1234}^{p+1}}{\gamma_{1235}}\left(\gamma_{1234}\gamma_{1345}^{p+1} + \gamma_{1235}^p\gamma_{1345}\gamma_{1236} - \gamma_{1235}^{p+1}\gamma_{1346}\right) = -\frac{\gamma_{1357}\gamma_{1234}^{2p+2}}{\gamma_{1235}}.$$

Proof. We work modulo the ideal in $\Bbbk[x, y, z]$ generated by x^{2p^2-1}. By the definition of f_2, we have

$$T_1 - T_2 + T_3 = -\gamma_{1235}^p\gamma_{1234} f_1^p f_{12346} - \gamma_{1234}^2 f_2^p.$$

As $f_1^p \equiv \gamma_{1234}^p y^{p^3}$ and

$$f_2^p \equiv \gamma_{1234}^p\gamma_{1235}^p \delta^{p^2} y^{p^3} + \gamma_{1234}^p\gamma_{1245}^p x^{p^2} y^{p^3+p^2} + \gamma_{1234}^p\gamma_{1345}^p \delta^p x^{2p^2-2p} y^{p^3} + \gamma_{1234}^p\gamma_{2345}^p x^{2p^2-p} y^{p^3+p},$$

we obtain

$$\begin{aligned} T_1-T_2+T_3 \equiv {} & -\gamma_{1234}^{p+2}\gamma_{1245}^p x^{p^2} y^{p^3+p^2} - \gamma_{1234}^{p+1}\left(\gamma_{1234}\gamma_{1345}^p + \gamma_{1235}^p\gamma_{1236}\right)\delta^p x^{2p^2-2p} y^{p^3} \\ & -\gamma_{1234}^{p+1}\left(\gamma_{1234}\gamma_{2345}^p + \gamma_{1235}^p\gamma_{1246}\right) x^{2p^2-p} y^{p^3+p} \\ & -\gamma_{1234}^{p+1}\gamma_{1235}^p\gamma_{1346}\delta x^{2p^2-2} y^{p^3}. \end{aligned}$$

Since

$$\begin{aligned} x^{p^2} f_1^{p+1} &\equiv \gamma_{1234}^p y^{p^3} x^{p^2} f_1 \\ &\equiv \gamma_{1234}^{p+1} x^{p^2} y^{p^3+p^2} + \gamma_{1234}^p\gamma_{1235}\delta^p x^{2p^2-2p} y^{p^3} \\ &\quad + \gamma_{1234}^p\gamma_{1245} x^{2p^2-p} y^{p^3+p} + \gamma_{1234}^p\gamma_{1345}\delta x^{2p^2-2} y^{p^3}, \end{aligned}$$

we see that

$$\begin{aligned} T_1 - T_2 + T_3 + T_4 \equiv {} & \gamma_{1234}^{p+1}\left(\gamma_{1235}\gamma_{1245}^p - \gamma_{1235}^p\gamma_{1236} - \gamma_{1234}\gamma_{1345}^p\right) x^{2p^2-2p} y^{p^3} \delta^p \\ & + \gamma_{1234}^{p+1}\left(\gamma_{1245}^{p+1} - \gamma_{1235}^p\gamma_{1246} - \gamma_{1234}\gamma_{2345}^p\right) x^{2p^2-p} y^{p^3+p} \\ & + \gamma_{1234}^{p+1}\left(\gamma_{1245}^p\gamma_{1345} - \gamma_{1235}^p\gamma_{1346}\right)\delta x^{2p^2-2} y^{p^3}. \end{aligned}$$

Using Lemma 1.4 for $i=3$ and $i=4$, along with the analogous result coming from the $(1,3,4)(3,4,5,6,7)$ Plücker relation for Γ, gives

$$T_1-T_2+T_3+T_4\equiv-\gamma_{1234}^{2p+1}\gamma_{1237}x^{2p^2-2p}y^{p^3}\delta^p \\ -\gamma_{1234}^{2p+1}\gamma_{1247}x^{2p^2-p}y^{p^3+p}-\gamma_{1234}^{2p+1}\gamma_{1347}\delta x^{2p^2-2}y^{p^3}.$$

Since $3p^2-4p\geq 2p^2-1$ for $p\geq 5$, we have $x^{2p^2-2p}f_1^{p-1}\equiv\gamma_{1234}^{p-1}y^{p^3-p^2}x^{2p^2-2p}$. Using the description of f_2 given above,

$$x^{2p^2-2p}f_2\equiv\gamma_{1234}x^{2p^2-2p}y^{p^2}(\gamma_{1235}\delta^p+\gamma_{1245}x^py^p+\gamma_{1345}\delta x^{2p-2}).$$

Thus

$$T_5\equiv\alpha_3\gamma_{1234}^py^{p^3}x^{2p^2-2p}(\gamma_{1235}\delta^p+\gamma_{1245}x^py^p+\gamma_{1345}\delta x^{2p-2}).$$

Using the $(1,2,4)(1,2,3,5,7)$ and $(1,3,5)(1,2,3,4,7)$ Plücker relations gives

$$T_1-T_2+T_3+T_4+T_5\equiv-\frac{\gamma_{1234}^{2p+2}\gamma_{1257}}{\gamma_{1235}}x^{2p^2-p}y^{p^3+p}-\frac{\gamma_{1234}^{2p+2}\gamma_{1357}}{\gamma_{1235}}\delta x^{2p^2-2}y^{p^3}.$$

Expanding and reducing modulo $\langle x^{2p^2-1}\rangle$, we get

$$x^{2p^2-p}f_1^{(p-3)/2}\equiv x^{2p^2-p}\gamma_{1234}^{(p-3)/2}y^{(p^3-3p^2)/2}$$

and

$$x^{2p^2-p}f_2^{(p+1)/2}\equiv\gamma_{1234}^{(p+1)/2}\gamma_{1235}^{(p+1)/2}x^{2p^2-p}y^{(p^3+3p^2)/2+p}.$$

Thus

$$\frac{T_6}{\alpha_4}\equiv\gamma_{1234}^{p-1}\gamma_{1235}^{(p+1)/2}x^{2p^2-p}y^{p^3+p}$$

and

$$\tilde{f}_3=T_1-T_2+T_3+T_4+T_5+T_6\equiv\alpha x^{2p^2-2}y^{p^3+2}.$$

Using the $(1,2,3)(1,3,4,5,6)$ and $(1,3,5)(3,4,5,6,7)$ Plücker relations, we obtain

$$\alpha=\frac{\gamma_{1234}^{p+2}}{\gamma_{1235}}(\gamma_{1345}^{p+1}-\gamma_{1356}\gamma_{1235}^p)=-\frac{\gamma_{1234}^{2p+2}\gamma_{1357}}{\gamma_{1235}}$$

and, since we are using the grevlex term order with $x<y<z$, the result follows. ☐

Define

$$f_3:=-\tilde{f}_3\frac{\gamma_{1235}}{\gamma_{1234}^{2p+2}x^{2p^2-2}}$$

so that $\mathrm{LT}(f_3)=\gamma_{1357}y^{p^3+2}$. Looking at the exponents of y modulo p, it is clear that there is only one new nontrivial tête-à-tête: $(f_3^p,\, f_2f_1^{p^2-1})$. In order to prove that $\mathcal{B}:=\{x,\, f_1,\, f_2,\, f_3,\, N_{\mathcal{M}}(z)\}$ is a SAGBI basis for $\Bbbk[V_{\mathcal{M}}]^E$, it is sufficient to show that this tête-à-tête subducts to zero. However, $N_{\mathcal{M}}(z)$ is rather complicated and it is more convenient to take an indirect approach. Subducting the tête-à-tête using only $\{x, f_1, f_2, f_3\}$ gives

$$\tilde{f}_4 := \underbrace{\beta_1 f_3^p}_{T_1'} - \underbrace{\beta_2 f_1^{p^2-1} f_2}_{T_2'} + \underbrace{\beta_3 x^p f_1^{p^2-(p+3)/2} f_2^{(p+1)/2}}_{T_3'} + \underbrace{\beta_4 x^{2p-2} f_1^{p^2-p} f_3}_{T_4'} + \underbrace{\beta_5 x^{2p-1} f_1^{(p^2-1)/2-p} f_2^{(p-1)/2} f_3^{(p+1)/2}}_{T_5'},$$

where

$$\beta_1 := \gamma_{1235}\gamma_{1234}^{p^2}, \quad \beta_2 := \gamma_{1357}^p, \quad \beta_3 := \frac{\gamma_{1234}(\gamma_{1245}\gamma_{1357}^p - \gamma_{1235}\gamma_{2357}^p)}{\gamma_{1235}^{(p+1)/2}},$$
$$\beta_4 := \gamma_{1234}^p\gamma_{1345}\gamma_{1357}^{p-1}, \quad \beta_5 := -\gamma_{1234}^{(p^2+p+2)/2}\gamma_{1235}^{(p+3)/2}\gamma_{1357}^{(p-3)/2}.$$

The lemma below proves that $\{x, f_1, f_2, f_3, \tilde{f}_4/x^{2p}\}$ is a SAGBI basis. We then use this in the proof of Theorem 2.3.

Lemma 2.2. *For $p \geq 5$, we have* $\mathrm{LT}(\tilde{f}_4) = \frac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4}$.

Proof. We work modulo the ideal in $\Bbbk[x, y, z]$ generated by x^{2p+1} and $x^{2p}y$, which we denote by $\mathfrak{n}$. Since $p \geq 5$, we have $p^2 - 2p \geq 2p + 1$. Therefore, using the expressions for f_1 and f_2 given above, we have $f_1 \equiv_{\mathfrak{n}} \gamma_{1234}y^{p^2}$ and

$$f_2 \equiv_{\mathfrak{n}} \gamma_{1234}\gamma_{1235}y^{p^2}\delta^p + \gamma_{1234}\gamma_{1245}y^{p^2+p}x^p + \gamma_{1234}\gamma_{1345}\delta x^{2p-2}y^{p^2} + \gamma_{1234}\gamma_{2345}x^{2p-1}y^{p^2+1} + \tfrac{1}{2}\gamma_{1234}^2x^{2p}z^{p^2}.$$

We will need expressions modulo $\mathfrak{n}$ for f_3^p, $x^{2p-2}f_3$ and $x^{2p-1}f_3^{(p+1)/2}$. Let $\mathfrak{m}$ denote the ideal generated by x^2y and x^3. Reworking the calculations of the proof of Lemma 2.1 to keep additional terms of f_3 gives

$$f_3 \equiv_{\mathfrak{m}} \gamma_{1357}\delta y^{p^3} + \gamma_{2357}xy^{p^3+1} + \tfrac{1}{2}\gamma_{1235}x^2z^{p^3}.$$

Thus

$$f_3^p \equiv_{\mathfrak{n}} \gamma_{1357}^p\delta^p y^{p^4} + \gamma_{2357}^p x^p y^{p^4+p} + \tfrac{1}{2}\gamma_{1235}^p x^{2p}z^{p^4},$$
$$x^{2p-2}f_3 \equiv_{\mathfrak{n}} \gamma_{1357}\delta x^{2p-2}y^{p^3} + \gamma_{2357}x^{2p-1}y^{p^3+1} + \tfrac{1}{2}\gamma_{1235}x^{2p}z^{p^3},$$
$$x^{2p-1}f_3^{(p+1)/2} \equiv_{\mathfrak{n}} \gamma_{1357}^{(p+1)/2}x^{2p-1}y^{(p^3+2)(p+1)/2}.$$

Therefore

$$T_1' - T_2' \equiv_{\mathfrak{n}} \gamma_{1234}^{p^2}\big(\gamma_{1235}\gamma_{2357}^p - \gamma_{1245}\gamma_{1357}^p\big)x^p y^{p^4+p} - \gamma_{1234}^{p^2}\gamma_{1345}\gamma_{1357}^p\delta x^{2p-2}y^{p^4} - \gamma_{1234}^{p^2}\gamma_{2345}\gamma_{1357}^p x^{2p-1}y^{p^4+1} + \tfrac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4}.$$

Since $x^p f_2^{(p+1)/2} \equiv_{\mathfrak{n}} \gamma_{1234}^{(p+1)/2}\gamma_{1235}^{(p+1)/2} x^p y^{(p^3+3p^2)/2+p}$, we have

$$T_1' - T_2' + T_3' \equiv_{\mathfrak{n}} -\gamma_{1234}^{p^2}\gamma_{1345}\gamma_{1357}^{p}\delta x^{2p-2}y^{p^4} - \gamma_{1234}^{p^2}\gamma_{2345}\gamma_{1357}^{p}x^{2p-1}y^{p^4+1} + \tfrac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4}.$$

Using the description of $x^{2p-2}f_3$ given above, we see that

$$T_1' - T_2' + T_3' + T_4' \equiv_{\mathfrak{n}} \gamma_{1234}^{p^2}\gamma_{1357}^{p-1}\big(\gamma_{1345}\gamma_{2357} - \gamma_{1357}\gamma_{2345}\big)x^{2p-1}y^{p^4+1} + \tfrac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4}.$$

The $(2,3,5)(1,3,4,5,7)$ Plücker relation gives

$$\gamma_{2345}\gamma_{1357} - \gamma_{2357}\gamma_{1345} = -\gamma_{1235}\gamma_{3457}.$$

Thus

$$T_1' - T_2' + T_3' + T_4' \equiv_{\mathfrak{n}} \gamma_{1234}^{p^2}\gamma_{1357}^{p-1}\gamma_{1235}\gamma_{3457}x^{2p-1}y^{p^4+1} + \tfrac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4}.$$

Observe that

$$x^{2p-1}f_1^{(p^2-1)/2-p} \equiv_{\mathfrak{n}} \gamma_{1234}^{(p^2-1)/2-p}x^{2p-1}y^{(p^4-p^2)/2-p^3}$$

and

$$x^{2p-1}f_2^{(p-1)/2} \equiv_{\mathfrak{n}} \gamma_{1234}^{(p-1)/2}\gamma_{1235}^{(p-1)/2}x^{2p-1}y^{(p^3+p^2)/2-p}.$$

Therefore, using the description of $x^{2p-1}f_3^{(p+1)/2}$ given above, we obtain

$$\tilde{f}_4 := T_1' - T_2' + T_3' + T_4' + T_5' \equiv_{\mathfrak{n}} \tfrac{1}{2}\gamma_{1234}^{p^2}\gamma_{1235}^{p+1}x^{2p}z^{p^4},$$

and, since we are using the grevlex term order with $x < y < z$, the result follows. □

Theorem 2.3. *The set $\mathcal{B} := \{x, f_1, f_2, f_3, N_{\mathcal{M}}(z)\}$ is a SAGBI basis, and hence a generating set, for $\Bbbk[V_{\mathcal{M}}]^E$. Furthermore, $\Bbbk[V_{\mathcal{M}}]^E$ is a complete intersection with generating relations coming from the subduction of the tête-à-têtes (f_2^p, f_1^{p+2}) and $(f_3^p, f_2 f_1^{p^2-1})$.*

Proof. Define $f_4 := \tilde{f}_4/x^{2p}$, $\mathcal{B}' := \{x, f_1, f_2, f_3, f_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-têtes for $\mathcal{B}'$ are (f_2^p, f_1^{p+2}) and $(f_3^p, f_2 f_1^{p^2-1})$. From Lemmas 2.1 and 2.2, these tête-à-têtes subduct to zero. Therefore $\mathcal{B}'$ is a SAGBI basis for A. From Theorem 5.2 of [Campbell et al. 2013], $\Bbbk[V_{\mathcal{M}}]^E[x^{-1}] = \Bbbk[x, f_1, f_2][x^{-1}]$. Thus $A[x^{-1}] = \Bbbk[V_{\mathcal{M}}]^E[x^{-1}]$. Note that $\mathrm{LM}(f_4) = z^{p^4}$. Therefore, by Theorem 1.1, $A = \Bbbk[V_{\mathcal{M}}]^E$ and $\mathcal{B}'$ is a SAGBI basis for $\Bbbk[V_{\mathcal{M}}]^E$. Hence the lead term algebra of $\Bbbk[V_{\mathcal{M}}]^E$ is generated by $\{x, y^{p^2}, y^{p^2+2p}, y^{p^3+2}, z^{p^4}\}$. Since the orbit of z has size p^4, we see that $\mathrm{LM}(N_{\mathcal{M}}(z)) = z^{p^4}$. Thus $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$ and $\mathcal{B}$ is also a SAGBI basis for $\Bbbk[V_{\mathcal{M}}]^E$. For any subalgebra with a SAGBI basis, the relations are generated by the nontrivial tête-à-tête. Hence (f_2^p, f_1^{p+2}) and $(f_3^p, f_2 f_1^{p^2-1})$ generate the ideal of relations and $\Bbbk[V_{\mathcal{M}}]^E$ is a complete intersection with embedding dimension five. □

3. The essentially generic case

In this section we consider representations V_M for $M \in \mathbb{F}^{2\times 4}$ for which $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$. With this restriction on M, we can evaluate the coefficients of the polynomials $\{f_i \mid i = 1, 2, 3, 4\}$, as defined in Section 2, at M to get $\{\bar{f}_i \mid i = 1, 2, 3, 4\} \subset \mathbb{F}[V_M]^E$. Note that $\mathrm{LT}(\bar{f}_1) = \gamma_{1234}(M) y^{p^2}$ so that $\mathrm{LM}(\bar{f}_1) = y^{p^2}$. Similarly $\mathrm{LM}(\bar{f}_2) = y^{p^2+2p}$, $\mathrm{LM}(\bar{f}_3) = y^{p^3+2}$ and $\mathrm{LM}(\bar{f}_4) = z^{p^4}$. Also, note that $\gamma_{1357}(M) = 0$ if and only if $\{c_{11}, c_{12}, c_{13}, c_{14}\}$ is linearly dependent over $\mathbb{F}_p$. Thus, if $\gamma_{1357}(M) \neq 0$, the orbit of z has size p^4 and $\mathrm{LM}(N_M(z)) = z^{p^4}$.

Theorem 3.1. *If $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$, then the set $\mathcal{B} := \{x, \bar{f}_1, \bar{f}_2, \bar{f}_3, N_M(z)\}$ is a SAGBI basis, and hence a generating set, for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a complete intersection with generating relations coming from the subduction of the tête-à-têtes $(\bar{f}_2^p, \bar{f}_1^{p+2})$ and $(\bar{f}_3^p, \bar{f}_2\bar{f}_1^{p^2-1})$.*

Proof. Define $\mathcal{B}' := \{x, \bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-têtes for $\mathcal{B}'$ are $(\bar{f}_2^p, \bar{f}_1^{p+2})$ and $(\bar{f}_3^p, \bar{f}_2\bar{f}_1^{p^2-1})$. The calculations in the proofs of Lemmas 2.1 and 2.2 survive evaluation at M, proving that these tête-à-têtes subduct to zero and $\mathcal{B}'$ is a SAGBI basis for A. Thus, to use Theorem 1.1 to prove $A = \mathbb{F}[V_M]^E$, we need only show that $A[x^{-1}] = \mathbb{F}[V_M]^E[x^{-1}]$.

Consider

$$f_{12357} = \gamma_{1235} y^{p^3} - \gamma_{1237} y^{p^2} x^{p^3-p^2} + \gamma_{1257} y^p x^{p^3-p} + \gamma_{1357}\delta x^{p^3-2} + \gamma_{2357} y x^{p^3-1}$$

and evaluate the coefficients at M to get $\bar{f}_{12357} \in \mathbb{F}[V_M]^E$ with lead monomial y^{p^3}. Since $\gamma_{1357}(M) \neq 0$, we know that $\bar{f}_{12357}$ has degree one as a polynomial in z. Furthermore, the coefficient of z is $-\gamma_{1357}(M)x^{p^3-1}$. Therefore, using Theorem 2.4 of [Campbell and Chuai 2007], $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12357}][x^{-1}]$. Thus, to prove $A = \mathbb{F}[V_M]^E$, it is sufficient to show that $\{N_M(y), \bar{f}_{12357}\} \subset A[x^{-1}]$.

Using Lemma 1.5 for the subsequence $(1, 2, 4)$ of $(1, 2, 3, 4, 5, 7)$ shows that

$$\gamma_{1235}^p \tilde{f}_{12357} = \gamma_{3457} \tilde{f}_{12357} \in \mathrm{Span}_{\mathbb{F}_p}\{\gamma_{2357}\tilde{f}_{13457}, \gamma_{1357}\tilde{f}_{23457}\}.$$

Thus $\bar{f}_{12357} \in \mathrm{Span}_{\mathbb{F}[x,x^{-1}]}\{\bar{f}_{13457}, \bar{f}_{23457}\}$. Similarly, using the $(1, 6, 7)$ subsequence of $(1, 3, 4, 5, 6, 7)$, we have that $\bar{f}_{13457} \in \mathrm{Span}_{\mathbb{F}[x,x^{-1}]}\{\bar{f}_{13456}, \bar{f}_{12345}^p\}$. Iterating this process gives

$$\bar{f}_{12357} \in \mathrm{Span}_{\mathbb{F}[x,x^{-1}]}\{\bar{f}_{12345}, \bar{f}_{12345}^p, \bar{f}_{12346}\}.$$

Since $\bar{f}_{12345} = \bar{f}_1$ and $\bar{f}_{12346} = 2\bar{f}_2 x^{p^2-2p} - \bar{f}_1^2$, we see that $\bar{f}_{13457} \in A[x^{-1}]$. A similar argument shows that

$$\bar{f}_{13579} \in \mathrm{Span}_{\mathbb{F}[x,x^{-1}]}\left\{\bar{f}_{12345}^{p^i}, \bar{f}_{12346}^{p^j} \,\middle|\, i, j \in \{0, 1, 2\}\right\},$$

giving $\bar{f}_{13579} \in A[x^{-1}]$. Since $\bar{f}_{13579} = \gamma_{1357}(M)N_M(y)$ (see Remark 1.2), we have $N_M(y) \in A[x^{-1}]$. Therefore $A = \mathbb{F}[V_M]^E$. As in the proof of Theorem 2.3, observe that $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

Remark 3.2. Lemmas 2.1 and 2.2 are only valid for $p > 5$. However, for the Magma calculations used to verify Theorem 2.3 for $p = 3$ and $p = 5$, only γ_{1234} and γ_{1235} are inverted. Thus Theorem 3.1 remains valid for $p = 3$ and $p = 5$.

4. The $\gamma_{1234} = 0$, $\gamma_{1235} \neq 0$, $\gamma_{1357} \neq 0$ stratum

In this section we consider representations V_M for $M \in \mathbb{F}^{2\times 4}$ for which $\gamma_{1234}(M) = 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$. Evaluating coefficients gives

$$\bar{f}_1 = \bar{\gamma}_{1235}\delta^p x^{p^2-2p} + \bar{\gamma}_{1245} y^p x^{p^2-p} + \bar{\gamma}_{1345}\delta x^{p^2-2} + \bar{\gamma}_{2345} y x^{p^2-1}.$$

Define

$$h_1 := \frac{\bar{f}_1}{\bar{\gamma}_{1235} x^{p^2-2p}} \quad \text{and} \quad h_2 := \frac{\bar{f}_{12357}}{\bar{\gamma}_{1235}}$$

so that $\mathrm{LT}(h_1) = y^{2p}$ and $\mathrm{LT}(h_2) = y^{p^3}$. Note that $h_1, h_2 \in \mathbb{F}[V_M]^E$. Furthermore, arguing as in the proof of Theorem 3.1, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), h_2][x^{-1}]$.

Lemma 4.1. $$N_M(y) = h_2^p + \left(\frac{\bar{\gamma}_{1237}^p}{\bar{\gamma}_{1235}^p} - \frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}}\right) h_2 x^{p^4-p^3} - \frac{\bar{\gamma}_{1357}^p}{\bar{\gamma}_{1235}^p} h_1 x^{p^4-2p}.$$

Proof. Since $\bar{f}_{13579} = \bar{\gamma}_{1357} N_M(y)$ (see Remark 1.2), we have

$$N_M(y) = y^{p^4} - \frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}} y^{p^3} x^{p^4-p^3} + \frac{\bar{\gamma}_{1379}}{\bar{\gamma}_{1357}} y^{p^2} x^{p^4-p^2} - \frac{\bar{\gamma}_{1579}}{\bar{\gamma}_{1357}} y^p x^{p^4-p} + \frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}} y x^{p^4-1}.$$

Using the definition gives

$$h_2 := y^{p^3} - \frac{\bar{\gamma}_{1237}}{\bar{\gamma}_{1235}} y^{p^2} x^{p^3-p^2} + \frac{\bar{\gamma}_{1257}}{\bar{\gamma}_{1235}} y^p x^{p^3-p} + \frac{\bar{\gamma}_{1357}}{\bar{\gamma}_{1235}} \delta x^{p^3-2} + \frac{\bar{\gamma}_{2357}}{\bar{\gamma}_{1235}} y x^{p^3-1}.$$

Thus

$$N_M(y) - h_2^p = \left(\frac{\bar{\gamma}_{1237}^p}{\bar{\gamma}_{1235}^p} - \frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}}\right) y^{p^3} x^{p^4-p^3} - \left(\frac{\bar{\gamma}_{1257}^p}{\bar{\gamma}_{1235}^p} - \frac{\bar{\gamma}_{1379}}{\bar{\gamma}_{1357}}\right) y^{p^2} x^{p^4-p^2}$$
$$- \left(\frac{\bar{\gamma}_{2357}^p}{\bar{\gamma}_{1235}^p} + \frac{\bar{\gamma}_{1579}}{\bar{\gamma}_{1357}}\right) y^p x^{p^4-p} - \frac{\bar{\gamma}_{1357}^p}{\bar{\gamma}_{1235}^p} \delta^p x^{p^4-2p} + \frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}} y x^{p^4-1}.$$

Using the $(1,3,5)(3,4,5,7,9)$, $(1,3,7)(3,4,5,7,9)$ and $(1,5,7)(3,4,5,7,9)$ Plücker relations gives

$$N_M(y) - h_2^p = \frac{\bar{\gamma}_{1357}^{p-1}}{\bar{\gamma}_{1235}^p}\big(\bar{\gamma}_{1345} y^{p^3} x^{p^4-p^3} - \bar{\gamma}_{1347} y^{p^2} x^{p^4-p^2}$$
$$- \bar{\gamma}_{1457} y^p x^{p^4-p} - \bar{\gamma}_{1357}\delta^p x^{p^4-2p}\big) + \frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}} y x^{p^4-1}.$$

Using the $(1,2,3)(1,3,4,5,7)$ and $(1,2,5)(1,3,4,5,7)$ Plücker relations,

$$\bar{\gamma}_{1347} = \frac{\bar{\gamma}_{1237}\bar{\gamma}_{1345}}{\bar{\gamma}_{1235}} \quad \text{and} \quad \bar{\gamma}_{1457} = \frac{\bar{\gamma}_{1245}\bar{\gamma}_{1357} - \bar{\gamma}_{1257}\bar{\gamma}_{1345}}{\bar{\gamma}_{1235}}.$$

Thus

$$N_M(y) = h_2^p + \frac{\bar{\gamma}_{1357}^{p-1}}{\bar{\gamma}_{1235}^p}\left(\bar{\gamma}_{1345}h_2x^{p^4-p^3} - \frac{\bar{\gamma}_{1357}\bar{\gamma}_{1245}}{\bar{\gamma}_{1235}}y^px^{p^4-p} - \frac{\bar{\gamma}_{1357}\bar{\gamma}_{1345}}{\bar{\gamma}_{1235}}\delta x^{p^4-2}\right) - \frac{\bar{\gamma}_{1357}^p}{\bar{\gamma}_{1235}^p}\delta^p x^{p^4-2p} + \left(\frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}} - \frac{\bar{\gamma}_{1357}^{p-1}\bar{\gamma}_{1345}\bar{\gamma}_{2357}}{\bar{\gamma}_{1235}^{p+1}}\right)yx^{p^4-1}.$$

Using the $(1,3,5)(2,3,4,5,7)$ Plücker relation, $\bar{\gamma}_{1345}\bar{\gamma}_{2357} = \bar{\gamma}_{1357}\bar{\gamma}_{2345} + \bar{\gamma}_{1235}^{p+1}$, giving

$$\frac{\bar{\gamma}_{1357}^{p-1}\bar{\gamma}_{1345}\bar{\gamma}_{2357}}{\bar{\gamma}_{1235}^{p+1}} = \frac{\bar{\gamma}_{2345}\bar{\gamma}_{1357}^p}{\bar{\gamma}_{1235}^p} + \bar{\gamma}_{1357}^{p-1}.$$

From the definition of h_1,

$$N_M(y) = h_2^p + \frac{\bar{\gamma}_{1357}^{p-1}\bar{\gamma}_{1345}}{\bar{\gamma}_{1235}^p}h_2x^{p^4-p^3} - \frac{\bar{\gamma}_{1357}^p}{\bar{\gamma}_{1235}^p}h_1x^{p^4-2p} + \left(\frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}} - \bar{\gamma}_{1357}^{p-1}\right)yx^{p^4-1}.$$

The result follows from the fact that $\bar{\gamma}_{3579} = \bar{\gamma}_{1357}^p$. □

As a consequence of the lemma, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, h_2][x^{-1}]$. Thus applying the SAGBI/divide-by-x algorithm to $\{x, h_1, h_2, N_M(z)\}$ produces a generating set for $\mathbb{F}[V_M]^E$. Subducting the tête-à-tête $(h_2^2, h_1^{p^2})$ gives

$$\tilde{h}_3 := h_2^2 - h_1^{p^2} + 2\frac{\bar{\gamma}_{1237}}{\bar{\gamma}_{1235}}h_1^{p(p+1)/2}x^{p^3-p^2} - 2\frac{\bar{\gamma}_{1257}}{\bar{\gamma}_{1235}}h_1^{(p^2+1)/2}x^{p^3-p}.$$

Lemma 4.2.
$$\mathrm{LT}(\tilde{h}_3) = \frac{2\bar{\gamma}_{1357}}{\bar{\gamma}_{1235}}y^{p^3+2}x^{p^3-2}.$$

Proof. We work modulo the ideal in $\mathbb{F}[x, y, z]$ generated by x^{p^3-1}. Therefore $h_1^{p^2} \equiv y^{2p^3}$, $h_1x^{p^3-p} \equiv y^{2p}x^{p^3-p}$ and

$$h_2^2 \equiv y^{2p^3} - 2\frac{\bar{\gamma}_{1237}}{\bar{\gamma}_{1235}}y^{p^3+p^2}x^{p^3-p^2} + 2\frac{\bar{\gamma}_{1257}}{\bar{\gamma}_{1235}}y^{p^3+p}x^{p^3-p} + 2\frac{\bar{\gamma}_{1357}}{\bar{\gamma}_{1235}}\delta y^{p^3}x^{p^3-2}.$$

Since $x^{p^3-p^2}h_1^p \equiv x^{p^3-p^2}y^{2p^2}$, we have $(h_1^p)^{(p+1)/2}x^{p^3-p^2} \equiv x^{p^3-p^2}y^{p^3+p^2}$. Thus

$$h_2^2 \equiv h_1^{p^2} - 2\frac{\bar{\gamma}_{1237}}{\bar{\gamma}_{1235}}h_1^{p(p+1)/2}x^{p^3-p^2} + 2\frac{\bar{\gamma}_{1257}}{\bar{\gamma}_{1235}}h_1^{(p^2+1)/2}x^{p^3-p} + 2\frac{\bar{\gamma}_{1357}}{\bar{\gamma}_{1235}}\delta y^{p^3}x^{p^3-2}.$$

Hence $\tilde{h}_3 \equiv 2(\bar{\gamma}_{1357}/\bar{\gamma}_{1235})\delta y^{p^3}x^{p^3-2}$, and the result follows. □

Define $h_3 := \bar\gamma_{1235}\tilde h_3/(2\bar\gamma_{1357}x^{p^3-2})$ so that $\mathrm{LT}(h_3) = y^{p^3+2}$. Subducting the tête-à-tête $(h_3^p, h_2^p h_1)$ gives

$$\tilde h_4 := h_3^p - h_1h_2^p - \alpha_1 x^p h_1^{(p^3+1)/2} + \alpha_2 x^{2p-2}h_3h_1^{(p^3-p^2)/2} - \alpha_3 x^{2p-1}h_1^{(p^2-1)/2}h_2^{(p-3)/2}h_3^{(p+1)/2},$$

with

$$\alpha_1 := \left(\frac{\bar\gamma_{2357}}{\bar\gamma_{1357}}\right)^p - \frac{\bar\gamma_{1245}}{\bar\gamma_{1235}}, \quad \alpha_2 := \frac{\bar\gamma_{1345}}{\bar\gamma_{1235}}, \quad \alpha_3 := \alpha_2\frac{\bar\gamma_{2357}}{\bar\gamma_{1357}} - \frac{\bar\gamma_{2345}}{\bar\gamma_{1235}}.$$

Lemma 4.3.
$$\mathrm{LT}(\tilde h_4) = \left(\frac{\bar\gamma_{1235}}{\bar\gamma_{1357}}\right)^p x^{2p}z^{p^4}.$$

Proof. We work modulo the ideal $\mathfrak{n} := \langle x^{2p+1}, x^{2p}y\rangle$. Using the definition of h_3 and methods analogous to the proof of Lemma 4.2, it is not hard to show that

$$h_3 \equiv_{\langle x^3,x^2y\rangle} \delta y^{p^3} + \frac{\bar\gamma_{2357}}{\bar\gamma_{1357}}xy^{p^3+1} + \frac{\bar\gamma_{1235}}{\bar\gamma_{1357}}x^2z^{p^3}.$$

Thus

$$h_3^p \equiv_{\mathfrak{n}} \delta^p y^{p^4} + \left(\frac{\bar\gamma_{2357}}{\bar\gamma_{1357}}\right)^p x^p y^{p^4+p} + \left(\frac{\bar\gamma_{1235}}{\bar\gamma_{1357}}\right)^p x^{2p}z^{p^4}.$$

Since $h_2 \equiv_{\mathfrak{n}} y^{p^3}$, we have

$$h_1h_2^p \equiv_{\mathfrak{n}} y^{p^4}\left(\delta^p + \frac{\bar\gamma_{1245}}{\bar\gamma_{1235}}y^px^p + \frac{\bar\gamma_{1345}}{\bar\gamma_{1235}}\delta x^{2p-2} + \frac{\bar\gamma_{2345}}{\bar\gamma_{1235}}yx^{2p-1}\right).$$

Furthermore, since $x^ph_1 \equiv_{\mathfrak{n}} x^p\delta^p$, expanding gives $x^ph_1^{(p^3+1)/2} \equiv_{\mathfrak{n}} x^py^{p^4+p}$. Thus

$$h_3^p - h_1h_2^p - \alpha_1x^ph_1^{(p^3+1)/2} \equiv_{\mathfrak{n}} -\frac{\bar\gamma_{1345}}{\bar\gamma_{1235}}\delta x^{2p-2}y^{p^4} - \frac{\bar\gamma_{2345}}{\bar\gamma_{1235}}x^{2p-1}y^{p^4+1} + \left(\frac{\bar\gamma_{1235}}{\bar\gamma_{1357}}\right)^p x^{2p}z^{p^4}.$$

Note that $x^{2p-2}h_1^{(p^3-p^2)/2} \equiv_{\mathfrak{n}} x^{2p-2}y^{p^4-p^3}$. Thus

$$x^{2p-2}h_3h_1^{(p^3-p^2)/2} \equiv_{\mathfrak{n}} x^{2p-2}\left(y^{p^4}\delta + \frac{\bar\gamma_{2357}}{\bar\gamma_{1357}}xy^{p^4+1}\right).$$

Hence

$$h_3^p - h_1h_2^p - \alpha_1x^ph_1^{(p^3+1)/2} + \alpha_2x^{2p-2}h_3h_1^{(p^3-p^2)/2} \equiv_{\mathfrak{n}} \alpha_3x^{2p-1}y^{p^4+1} + \left(\frac{\bar\gamma_{1235}}{\bar\gamma_{1357}}\right)^p x^{2p}z^{p^4}.$$

Since $x^{2p-1}h_1^{(p^2-1)/2}h_2^{(p-3)/2}h_3^{(p+1)/2} \equiv_{\mathfrak{n}} x^{2p-1}y^{p^4+1}$, the result follows. □

Define $h_4 := \bar\gamma_{1357}\tilde h_4/(\bar\gamma_{1357}x^{2p})$ so that $\mathrm{LT}(h_4) = z^{p^4}$.

Theorem 4.4. *If $\gamma_{1234}(M) = 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$, then the set $\mathcal{B} := \{x, h_1, h_2, h_3, N_M(z)\}$ is a SAGBI basis, and hence a generating set, for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a complete intersection with generating relations coming from the subduction of the tête-à-têtes $(h_2^2, h_1^{p^2})$ and $(h_3^p, h_1 h_2^p)$.*

Proof. Define $\mathcal{B}' := \{x, h_1, h_2, h_3, h_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-têtes for $\mathcal{B}'$ are $(h_2^2, h_1^{p^2})$ and $(h_3^p, h_1 h_2^p)$. Using Lemmas 4.2 and 4.3, these tête-à-têtes subduct to zero, proving that $\mathcal{B}'$ is a SAGBI basis for A. Since $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, h_2][x^{-1}]$, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Finally, observe that $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

5. The $\gamma_{1234} \neq 0$, $\gamma_{1235} = 0$, $\gamma_{1357} \neq 0$ strata

In this section we consider representations V_M for $M \in \mathbb{F}^{2\times 4}$ for which $\gamma_{1235}(M) = 0$, $\gamma_{1234}(M) \neq 0$ and $\gamma_{1357}(M) \neq 0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$.

Lemma 5.1. *If $\bar{\gamma}_{1234} \neq 0$, $\bar{\gamma}_{1235} = 0$ and $\bar{\gamma}_{1357} \neq 0$, then $\bar{\gamma}_{1345} \neq 0$.*

Proof. Let r_i denote row i of the matrix $\Gamma(M)$. Since $\bar{\gamma}_{1234} \neq 0$, the set $\{r_1, r_2, r_3, r_4\}$ is linearly independent. Using this and the hypothesis that $\bar{\gamma}_{1235} = 0$, we conclude that r_5 is a linear combination of $\{r_1, r_2, r_3\}$, say $r_5 = a_1 r_1 + a_2 r_2 + a_3 r_3$. Since r_3 is nonzero and the entries of r_5 are the p-th powers of the entries of r_3, we see that r_5 is nonzero. Suppose, by way of contradiction, that $\bar{\gamma}_{1345} = 0$. Then r_5 is a nonzero linear combination of $\{r_1, r_3, r_4\}$, say $r_5 = b_1 r_1 + b_3 r_3 + b_4 r_4$. Thus $b_1 r_1 + b_3 r_3 + b_4 r_4 = a_1 r_1 + a_2 r_2 + a_3 r_3$. Since $\{r_1, r_2, r_3, r_4\}$ is linearly independent, $b_4 = a_2 = 0$, $a_1 = b_1$, $a_3 = b_3$ and $r_5 = a_1 r_1 + a_3 r_3$, contradicting the assumption that $\bar{\gamma}_{1357} \neq 0$. □

Take f_1 as defined in Section 2, evaluate coefficients and divide by $\bar{\gamma}_{1234}$ to get

$$\hat{f}_1 := y^{p^2} + \frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1234}} y^p x^{p^2-p} + \frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1234}} \delta x^{p^2-2} + \frac{\bar{\gamma}_{2345}}{\bar{\gamma}_{1234}} y x^{p^2-1}.$$

Note that $\hat{f}_1$ is of degree one in z with coefficient $x^{p^2-2}\bar{\gamma}_{1345}/\bar{\gamma}_{1234}$ and so, using Theorem 2.4 of [Campbell and Chuai 2007], $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \hat{f}_1][x^{-1}]$. Define

$$\tilde{h}_2 := N_M(y) - \hat{f}_1^{p^2} + \alpha_1 \hat{f}_1^p x^{p^4-p^3} + \alpha_2 \hat{f}_1^2 x^{p^4-2p^2},$$

with

$$\alpha_1 := \frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}} + \frac{\bar{\gamma}_{1245}^{p^2}}{\bar{\gamma}_{1234}^{p^2}} \quad \text{and} \quad \alpha_2 := \frac{\bar{\gamma}_{1345}^{p^2}}{\bar{\gamma}_{1234}^{p^2}}.$$

We work modulo the ideal $\mathfrak{n} := \langle x^{p^4-p^2-1} \rangle$. Since $\bar{\gamma}_{1357} N_M(y) = \bar{f}_{13579}$ (see Remark 1.2), we have $N_M(y) \equiv_{\mathfrak{n}} y^{p^4} - (\bar{\gamma}_{1359}/\bar{\gamma}_{1357}) y^{p^3} x^{p^4-p^3}$. Therefore

$$N_M(y) - \hat{f}_1^{p^2} \equiv_{\mathfrak{n}} -\left(\frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}} + \frac{\bar{\gamma}_{1245}^{p^2}}{\bar{\gamma}_{1234}^{p^2}}\right) y^{p^3} x^{p^4-p^3} - \frac{\bar{\gamma}_{1345}^{p^2}}{\bar{\gamma}_{1234}^{p^2}} \delta^{p^2} x^{p^4-2p^2}.$$

Thus

$$N_M(y) - \hat{f}_1^{p^2} + \alpha_1 \hat{f}_1^p x^{p^4-p^3} \equiv_{\mathfrak{n}} -\frac{\bar{\gamma}_{1345}^{p^2}}{\bar{\gamma}_{1234}^{p^2}} \delta^{p^2} x^{p^4-2p^2} \equiv_{\mathfrak{n}} -\frac{\bar{\gamma}_{1345}^{p^2}}{\bar{\gamma}_{1234}^{p^2}} y^{2p^2} x^{p^4-2p^2}.$$

Hence

$$\begin{aligned}\tilde{h}_2 &= N_M(y) - \hat{f}_1^{p^2} + \alpha_1 \hat{f}_1^p x^{p^4-p^3} + \alpha_2 \hat{f}_1^2 x^{p^4-2p^2} \\ &\equiv_{\mathfrak{n}} \frac{2\alpha_2}{\bar{\gamma}_{1234}} (\bar{\gamma}_{1245} y^{p^2+p} x^{p^4-p^2-p} + \bar{\gamma}_{1345} y^{p^2+2} x^{p^4-p^2-2}).\end{aligned}$$

We first consider the case $\bar{\gamma}_{1245} \neq 0$. Define

$$h_2 := \bar{\gamma}_{1234}^{p^2+1} \tilde{h}_2 / (2x^{p^4-p^2-p} \bar{\gamma}_{1345}^{p^2} \bar{\gamma}_{1245})$$

so that $\mathrm{LT}(h_2) = y^{p^2+p}$. Since $N_M(y) \in \mathbb{F}[x, \hat{f}_1, h_2]$, we have

$$\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat{f}_1, h_2][x^{-1}].$$

Subducting the tête-à-tête $(h_2^p, \hat{f}_1^{p+1})$ gives

$$\tilde{h}_3 := \hat{f}_1^{p+1} - h_2^p + \left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^p \hat{f}_1^{p-2} h_2^2 x^{p^2-2p}.$$

Lemma 5.2. $$\mathrm{LT}(\tilde{h}_3) = 2\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^{p+1} y^{p^3+p+2} x^{p^2-p-2}.$$

Proof. We work modulo the ideal $\langle x^{p^2-p-1} \rangle$. Thus $\hat{f}_1 \equiv y^{p^2}$. Reviewing the definition of h_2, we see that

$$h_2^p \equiv y^{p^3+p^2} + \left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^p y^{p^3+2p} x^{p^2-2p}$$

and

$$h_2^2 x^{p^2-2p} \equiv y^{2p^2+2p} x^{p^2-2p} + 2\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right) y^{2p^2+p+2} x^{p^2-p-2}.$$

Thus

$$\hat{f}_1^{p+1} - h_2^p + \left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^p \hat{f}_1^{p-2} h_2^2 x^{p^2-2p} \equiv 2\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^{p+1} y^{p^3+p+2} x^{p^2-p-2}$$

and the result follows. □

Define $h_3 := \bar{\gamma}_{1245}^{p+1} \tilde{h}_3 / (2\bar{\gamma}_{1345}^{p+1} x^{p^2-p-2})$ so that $\mathrm{LT}(h_3) = y^{p^3+p+2}$.

Lemma 5.3. *Subducting the tête-à-tête $(h_3^p, \hat{f}_1^{p^2-1} h_2^2)$ gives an invariant with lead term $-\bar{\gamma}_{1245}^p \bar{\gamma}_{1234}^{p^2} z^{p^4} x^{p^2+2p} / (4\bar{\gamma}_{1345}^{p^2+p})$.*

Proof. Modulo the ideal $\langle x^{p^2+2p+1}, x^{p^2+2p}y\rangle$, the expression

$$\begin{aligned}h_3^p - \hat{f}_1^{p^2-1}h_2^2 + \beta_1 h_3 \hat{f}_1^{p^2-p+1}x^{p-2} + \beta_2 h_2 \hat{f}_1^{p^2}x^p + \beta_3 \hat{f}_1^{p^2+1}x^{2p}\\ + \beta_4 h_2^4 \hat{f}_1^{p^2-4}x^{p^2-2p} + \beta_5 h_3 h_2^2 \hat{f}_1^{p^2-p-2}x^{p^2-p-2}\\ + \beta_6 h_3^2 \hat{f}_1^{p^2-2p}x^{p^2-4} + \beta_7 h_2^2 \hat{f}_1^{p^2-2}x^{p^2} + \beta_8 h_3 \hat{f}_1^{p^2-p}x^{p^2+p-2}\\ + \beta_9 h_2 \hat{f}_1^{p^2-1}x^{p^2+p} + \beta_{10} h_3 h_2^{p-1} \hat{f}_1^{p^2-2p}x^{p^2+2p-2}\\ + \beta_{11} h_3^{(p+1)/2} h_2^{(p-3)/2} \hat{f}_1^{(p^2+1)/2-p}x^{p^2+2p-1},\end{aligned}$$

with

$$\beta_1 := 2\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}, \quad \beta_2 := -\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1234}}\right)^{p+1}\left(\frac{\bar{\gamma}_{1234}}{\bar{\gamma}_{1345}}\right)^{p},$$

$$\beta_3 := \frac{1}{2}\left(\frac{\bar{\gamma}_{1245,}}{\bar{\gamma}_{1345}}\right)^{p}\left(\left(\frac{\bar{\gamma}_{2345}}{\bar{\gamma}_{1345}}\right)^{p^2} - \left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1345}}\right)^{p^2}\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1234}}\right)^{p}\right),$$

$$\beta_4 := -\frac{1}{2}\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^{p}, \quad \beta_5 := 2\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^{p+1}, \quad \beta_6 := -2\left(\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1245}}\right)^{p+2},$$

$$\beta_7 := \frac{1}{2}\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1345}}\right)^{p}\left(\frac{\bar{\gamma}_{1234}}{\bar{\gamma}_{1345}}\right)^{p^2-p}\frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}}, \quad \beta_8 := -\frac{\bar{\gamma}_{1345}}{\bar{\gamma}_{1234}}\beta_7,$$

$$\beta_9 := -\frac{1}{2}\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1345}}\right)^{p}\left(\frac{\bar{\gamma}_{1234}}{\bar{\gamma}_{1345}}\right)^{p^2}\left(\frac{\bar{\gamma}_{1245}\bar{\gamma}_{1379}}{\bar{\gamma}_{1234}\bar{\gamma}_{1357}} + \frac{\bar{\gamma}_{1579}}{\bar{\gamma}_{1357}}\right),$$

$$\beta_{10} := \frac{1}{2}\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1345}}\right)^{p-1}\left(\frac{\bar{\gamma}_{1234}}{\bar{\gamma}_{1345}}\right)^{p^2}\frac{\bar{\gamma}_{1379}}{\bar{\gamma}_{1357}}, \quad \beta_{11} := \frac{1}{2}\left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1345}}\right)^{p}\left(\frac{\bar{\gamma}_{1234}}{\bar{\gamma}_{1345}}\right)^{p^2}\frac{\bar{\gamma}_{3579}}{\bar{\gamma}_{1357}},$$

is congruent to $-\bar{\gamma}_{1245}^p\bar{\gamma}_{1234}^{p^2}z^{p^4}x^{p^2+2p}/(4\bar{\gamma}_{1345}^{p^2+p})$. □

Theorem 5.4. *If* $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) = 0$, $\gamma_{1357}(M) \neq 0$ *and* $\gamma_{1245}(M) \neq 0$, *then the set* $\mathcal{B} := \{x, \hat{f}_1, h_2, h_3, N_M(z)\}$ *is a SAGBI basis for* $\mathbb{F}[V_M]^E$. *Furthermore,* $\mathbb{F}[V_M]^E$ *is a complete intersection with generating relations coming from the subduction of the tête-à-têtes* $(h_2^p, \hat{f}_1^{p+1})$ *and* $(h_3^p, \hat{f}_1^{p^2-1}h_2^2)$.

Proof. Use the subduction of $(h_3^p, \hat{f}_1^{p^2-1}h_2^2)$ given in Lemma 5.3 to construct an invariant h_4 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, \hat{f}_1, h_2, h_3, h_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-têtes for $\mathcal{B}'$ are $(h_2^p, \hat{f}_1^{p+1})$ and $(h_3^p, \hat{f}_1^{p^2-1}h_2^2)$. Using Lemmas 5.2 and 5.3, these tête-à-têtes subduct to zero, proving that $\mathcal{B}'$ is a SAGBI basis for A. Since $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat{f}_1, h_2][x^{-1}]$, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Finally, observe that $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

We now consider the case $\bar{\gamma}_{1245} = 0$. Define $\hat{h}_2 := \bar{\gamma}_{1234}^{p^2+1}\tilde{h}_2/(2x^{p^4-p^2-2}\bar{\gamma}_{1345}^{p^2+1})$ so that $\mathrm{LT}(\hat{h}_2) = y^{p^2+2}$. Since $N_M(y) \in \mathbb{F}[x, \hat{f}_1, \hat{h}_2]$, we have $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat{f}_1, \hat{h}_2][x^{-1}]$.

Lemma 5.5. *Subducting the tête-à-tête $(\hat{h}_2^{p^2}, \hat{f}_1^{p^2+2})$ gives an invariant with lead term $z^{p^4}(\bar{\gamma}_{1234}x^2/(2\bar{\gamma}_{1345}))^{p^2}$.*

Proof. Modulo the ideal $\langle x^{p^2+1}, x^{p^2}y\rangle$, the expression

$$\hat{f}_1^{p^2+2} - \hat{h}_2^{p^2} - \big(\alpha_1\hat{h}_2\hat{f}_1^{p^2}x^{p^2-2} + \alpha_2\hat{f}_1^{p^2+1}x^{p^2} + \alpha_3\hat{h}_2^{p}\hat{f}_1^{p^2-p}x^{2p^2-2p} + \alpha_4\hat{h}_2^{p(p+1)/2}\hat{f}_1^{(p^2-p-2)/2}x^{2p^2-p} + \alpha_5\hat{h}_2\hat{f}_1^{p^2-1}x^{2p^2-2} + \alpha_6\hat{h}_2^{(p^2+1)/2}\hat{f}_1^{(p^2-3)/2}x^{2p^2-1}\big),$$

with

$$\alpha_1 := \frac{2\bar{\gamma}_{1345}}{\bar{\gamma}_{1234}}, \qquad \alpha_2 := -\frac{\bar{\gamma}_{1379}\bar{\gamma}_{1234}^{p^2}}{\bar{\gamma}_{1357}\bar{\gamma}_{1345}^{p^2}}, \quad \alpha_3 := -\frac{\bar{\gamma}_{1359}\bar{\gamma}_{1234}^{p^2-p}}{\bar{\gamma}_{1357}\bar{\gamma}_{1345}^{p^2-p}},$$

$$\alpha_4 := \frac{\bar{\gamma}_{1579}\bar{\gamma}_{1234}^{p^2}}{\bar{\gamma}_{1357}\bar{\gamma}_{1345}^{p^2}}, \quad \alpha_5 := \frac{\bar{\gamma}_{1379}\bar{\gamma}_{1234}^{p^2-1}}{\bar{\gamma}_{1357}\bar{\gamma}_{1345}^{p^2-1}}, \quad \alpha_6 := -\frac{\bar{\gamma}_{3579}\bar{\gamma}_{1234}^{p^2}}{\bar{\gamma}_{1357}\bar{\gamma}_{1345}^{p^2}},$$

is congruent to $z^{p^4}(\bar{\gamma}_{1234}x^2/(2\bar{\gamma}_{1345}))^{p^2}$. □

Theorem 5.6. *If $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) = 0$, $\gamma_{1357}(M) \neq 0$ and $\gamma_{1245}(M) = 0$, then the set $\mathcal{B} := \{x, \hat{f}_1, \hat{h}_2, N_M(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a hypersurface with the relation coming from the subduction of the tête-à-tête $(\hat{h}_2^{p^2}, \hat{f}_1^{p^2+2})$.*

Proof. Use the subduction of $(\hat{h}_2^{p^2}, \hat{f}_1^{p^2+2})$ given in Lemma 5.5 to construct an invariant $\hat{h}_3$ with lead term z^{p^4}. Define $\mathcal{B}' := \{x, \hat{f}_1, \hat{h}_2, \hat{h}_3\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-tête for $\mathcal{B}'$ is $(\hat{h}_2^{p^2}, \hat{f}_1^{p^2+2})$, which subducts to zero using Lemma 5.5. Thus $\mathcal{B}'$ is a SAGBI basis for A. Since $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat{f}_1, \hat{h}_2][x^{-1}]$, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Finally, observe that $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

6. The $\gamma_{1234} \neq 0$, $\gamma_{1235} \neq 0$, $\gamma_{1357} = 0$ stratum

In this section we consider representations V_M for $M \in \mathbb{F}^{2\times 4}$ for which $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) = 0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$. Evaluating the coefficients of f_1 and dividing by $\bar{\gamma}_{1234}$ gives $\hat{f}_1$ with lead term y^{p^2}. Since $\bar{\gamma}_{1357} = 0$ and $\bar{\gamma}_{1235} \neq 0$, the orbit of y has size p^3 and $N_M(y) = \bar{f}_{12357}/\bar{\gamma}_{1235}$ (see Remark 1.2). For convenience, write

$$N_M(y) = y^{p^3} + \alpha_2 y^{p^2}x^{p^3-p^2} + \alpha_1 y^p x^{p^3-p} + \alpha_0 y x^{p^3-1}$$

and

$$\hat{f}_1 = y^{p^2} + \beta_3\delta^p x^{p^2-2p} + \beta_2 y^p x^{p^2-p} + \beta_1\delta x^{p^2-2} + \beta_0 y x^{p^2-1},$$

with

$$\alpha_2 = -\frac{\bar\gamma_{1237}}{\bar\gamma_{1235}}, \quad \alpha_1 = \frac{\bar\gamma_{1257}}{\bar\gamma_{1235}}, \quad \alpha_0 = \frac{\bar\gamma_{2357}}{\bar\gamma_{1235}},$$
$$\beta_3 = \frac{\bar\gamma_{1235}}{\bar\gamma_{1234}}, \quad \beta_2 = \frac{\bar\gamma_{1245}}{\bar\gamma_{1234}}, \quad \beta_1 = \frac{\bar\gamma_{1345}}{\bar\gamma_{1234}}, \quad \beta_0 = \frac{\bar\gamma_{2345}}{\bar\gamma_{1234}}.$$

Subducting $N_M(y)$ gives

$$\tilde h_2 := N_M(y) - \hat f_1^p + \beta_3^p x^{p^3-2p^2} \hat f_1^2.$$

Lemma 6.1.
$$\mathrm{LT}(\tilde h_2) = 2\left(\frac{\bar\gamma_{1235}}{\bar\gamma_{1234}}\right)^{p+1} y^{p^2+2p} x^{p^3-p^2-2p}.$$

Proof. We work modulo the ideal $\langle x^{p^3-p^2-p}\rangle$. Using the definitions of f_{12357} and f_{12345}, we have $N_M(y) \equiv y^{p^3}$ and $\hat f_1^p \equiv y^{p^3} + (\bar\gamma_{1235}/\bar\gamma_{1234})^p y^{2p^2} x^{p^3-2p^2}$. The result follows from the observation that

$$\hat f_1 x^{p^3-2p^2} \equiv y^{p^2} x^{p^3-2p^2} + \left(\frac{\bar\gamma_{1235}}{\bar\gamma_{1234}}\right) y^{2p} x^{p^3-p^2-2p}. \qquad \square$$

Define $h_2 := \tilde h_2 \bar\gamma_{1234}^{p+1}/(2\bar\gamma_{1235}^{p+1} x^{p^3-p^2-2p})$ so that $\mathrm{LT}(h_2) = y^{p^2+2p}$ and

$$h_2 \equiv_{\langle x^{2p}\rangle} y^{p^2}\left(\delta^p + \frac{\beta_2}{\beta_3} y^p x^p + \frac{\beta_1}{\beta_3}\delta x^{2p-2} + \frac{\beta_0}{\beta_3} y x^{2p-1}\right). \tag{1}$$

Lemma 6.2.
$$\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat f_1, h_2][x^{-1}].$$

Proof. Since $\bar\gamma_{1357} = 0$ and the first row of M is nonzero, we can use a change of coordinates, see [Campbell et al. 2013, §4], and the $\mathrm{GL}_4(\mathbb{F}_p)$-action to write

$$M = \begin{pmatrix} 1 & c_{12} & c_{13} & 0 \\ 0 & c_{22} & c_{23} & c_{24} \end{pmatrix}.$$

Since $\bar\gamma_{1235} \neq 0$, we have $c_{24} \neq 0$. With this choice of generators for E, let H denote the subgroup generated by e_1 and e_4. Using the calculation of $\mathbb{F}[x, y, z]^H$ from Theorem 6.4 of [loc. cit.], we see that $\mathbb{F}[V_M]^H[x^{-1}] = \mathbb{F}[x, N_H(y), N_H(\delta)][x^{-1}]$ with $N_H(y) := y^p - yx^{p-1}$ and $N_H(\delta) = \delta^p - \delta(c_{24}x^2)^{p-1}$. Thus, to compute $\mathbb{F}[V_M]^G[x^{-1}] = (\mathbb{F}[V_M]^H[x^{-1}])^{G/H}$, it is sufficient to compute

$$(\mathbb{F}[x, N_H(y), N_H(\delta)][x^{-1}])^{G/H} = \mathbb{F}[x, N_H(y)/x^{p-1}, N_H(\delta)/x^{2p-1}]^{G/H}[x^{-1}].$$

Note that $\deg(N_H(y)/x^{p-1}) = \deg(N_H(\delta)/x^{2p-1}) = 1$. Furthermore

$$\mathbb{F}[x, N_H(y)/x^{p-1}]^{G/H} = \mathbb{F}[x, N_{G/H}(N_H(y)/x^{p-1})]$$

and $N_{G/H}(N_H(y)/x^{p-1}) = N_M(y)/x^{p^3-p^2}$. Using the form of M given above, we see that $\bar\gamma_{1345} = -c_{24}^{p-1}\bar\gamma_{1235}$. If we evaluate $\widetilde\Gamma$ at M and set $x = 1$, $y = 1$

and $z = 1$, then first and last columns of the resulting matrix are equal. Thus $\bar{f}_{12345}(1, 1, 1) = \bar{\gamma}_{1234} + \bar{\gamma}_{1245} + \bar{\gamma}_{2345} = 0$. Using these two relations, we can write

$$\hat{f}_1 = N_H(y)^p - \frac{\bar{\gamma}_{2345}}{\bar{\gamma}_{1234}} N_H(y) x^{p^2-p} + \frac{\bar{\gamma}_{1235}}{\bar{\gamma}_{1234}} N_H(\delta) x^{p^2-2p}.$$

Thus we have $\hat{f}_1/x^{p^2-p} \in \mathbb{F}[x, N_H(y)/x^{p-1}, N_H(\delta)/x^{2p-1}]^{G/H}$ is of degree one in $N_H(\delta)/x^{2p-1}$ with coefficient $x^{p-1}\bar{\gamma}_{1235}/\bar{\gamma}_{1234}$. Thus by Theorem 2.4 of [Campbell and Chuai 2007], we have

$$\mathbb{F}[x, N_H(y)/x^{p-1}, N_H(\delta)/x^{2p-1}]^{G/H}[x^{-1}] = \mathbb{F}[x, N_M(y)/x^{p^3-p^2}, \hat{f}_1/x^{p^2-p}][x^{-1}].$$

Therefore $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \hat{f}_1][x^{-1}]$. The result then follows from the fact that $N_M(y) \in \mathbb{F}[x, \hat{f}_1, h_2]$. □

Subducting the tête-à-tête $(h_2^p, \hat{f}_1^{p+2})$ gives

$$\begin{aligned}\tilde{h}_3 := h_2^p - \hat{f}_1^{p+2} + 2\beta_3 \hat{f}_1^p h_2 x^{p^2-2p} \\ - \beta_3^{-p}(\alpha_2 \hat{f}_1^{p+1} x^{p^2} - \alpha_2\beta_3 \hat{f}_1^{p-1} h_2 x^{2p^2-2p} + \alpha_1 \hat{f}_1^{(p-3)/2} h_2^{(p+1)/2} x^{2p^2-p})\end{aligned}$$

for $p \geq 5$ and

$$\begin{aligned}\tilde{h}_3 := h_2^3 - \hat{f}_1^5 + 2\beta_3 \hat{f}_1^3 h_2 x^3 \\ - (\alpha_2\beta_3^{-3} + \beta_3^3)(\hat{f}_1^4 x^9 - \beta_3 \hat{f}_1^2 h_2 x^{12}) - (\alpha_1\beta_3^{-3} + \alpha_2\beta_3^{-1} + \beta_3^5) h_2^2 x^{15}\end{aligned}$$

for $p = 3$.

Lemma 6.3. $\mathrm{LT}(\tilde{h}_3) = \alpha_0 \beta_3^{-p} y^{p^3+1} x^{2p^2-1}$.

Proof. For $p = 3$, this is a Magma calculation. Suppose $p \geq 5$. We work modulo the ideal $\langle x^{2p^2} \rangle$. Since $p^3 - 2p^2 > 2p^2$, we have $\hat{f}_1^p \equiv y^{p^3}$. Furthermore, $3p^2 - 4p > 2p^2$, giving $\hat{f}_1 x^{2p^2-2p} \equiv y^{p^2} x^{2p^2-2}$. Using congruence (1) given above, we have

$$h_2 x^{2p^2-2p} \equiv x^{2p^2-2p} y^{p^2} \left(\delta^p + \frac{\beta_2}{\beta_3} y^p x^p + \frac{\beta_1}{\beta_3} \delta x^{2p-2} + \frac{\beta_0}{\beta_3} y x^{2p-1}\right)$$

and

$$h_2^p \equiv y^{p^3} \left(\delta^p + \frac{\beta_2}{\beta_3} y^p x^p + \frac{\beta_1}{\beta_3} \delta x^{2p-2} + \frac{\beta_0}{\beta_3} y x^{2p-1}\right)^p.$$

Using the definition of h_2, we get

$$\begin{aligned}\hat{f}_1^2 - 2\beta_3 h_2 x^{p^2-2p} &= \beta_3^{-p} x^{2p^2-p^3}(\hat{f}_1^p - N_M(y)) \\ &= \delta^{p^2} + \beta_3^{-p}\big((\beta_2^p - \alpha_2) y^{p^2} x^{p^2} + \beta_1^p \delta^p x^{2p^2-2p} \\ &\qquad + (\beta_0^p - \alpha_1) y^p x^{2p^2-p} - \alpha_0 y x^{2p^2-1}\big).\end{aligned}$$

Thus

$$h_2^p - \hat{f}_1^p(\hat{f}_1^2 - 2\beta_3 h_2 x^{p^2-2p}) \equiv \frac{y^{p^3}}{\beta_3^p}(\alpha_2 y^{p^2} x^{p^2} + \alpha_1 y^p x^{2p^2-p} + \alpha_0 y x^{2p^2-1}).$$

Furthermore, using the above expressions,

$$\hat{f}_1^{p+1}x^{p^2} - \beta_3\hat{f}_1^{p-1}h_2x^{2p^2-2p} \equiv y^{p^3-p^2}x^{p^2}(y^{p^2}\hat{f}_1 - \beta_3h_2x^{p^2-2p}) \equiv x^{p^2}y^{p^3+p^2}.$$

Therefore

$$h_2^p - \hat{f}_1^p(\hat{f}_1^2 - 2\beta_3h_2x^{p^2-2p}) - \frac{\alpha_2}{\beta_3^p}(\hat{f}_1^{p+1}x^{p^2} - \beta_3\hat{f}_1^{p-1}h_2x^{2p^2-2p}) \equiv \frac{y^{p^3}}{\beta_3^p}(\alpha_1y^px^{2p^2-p} + \alpha_0yx^{2p^2-1}).$$

Note that $h_2x^{2p^2-p} \equiv y^{p^2+2p}x^{2p^2-p}$ and $\hat{f}_1x^{2p^2-p} \equiv y^{p^2}x^{2p^2-p}$. Hence

$$\hat{f}_1^{(p-3)/2}h_2^{(p+1)/2}x^{2p^2-p} \equiv y^{p^3+p}x^{2p^2-p},$$

giving $\tilde{h}_3 \equiv \alpha_0y^{p^3+1}x^{2p^2-1}/\beta_3^p$, as required. □

Note that $\alpha_0/\beta_3^p = \bar{\gamma}_{2357}\bar{\gamma}_{1234}^p/\bar{\gamma}_{1235}^{p+1}$. Since $\bar{\gamma}_{1357} = 0$, $\bar{\gamma}_{1235} \neq 0$ and $\bar{\gamma}_{3457} = \bar{\gamma}_{1235}^p \neq 0$, arguing as in the proof of Lemma 5.1, we see that $\bar{\gamma}_{2357} \neq 0$. Define $h_3 := \bar{\gamma}_{1235}^{p+1}\tilde{h}_3/(x^{2p^2-1}\bar{\gamma}_{2357}\bar{\gamma}_{1234}^p)$ so that $\mathrm{LT}(h_3) = y^{p^3+1}$.

Lemma 6.4. $\mathrm{LM}(h_3^p - h_2^{(p^2+1)/2}\hat{f}_1^{(p^2-2p-1)/2}) = x^pz^{p^4}$.

Proof. Working modulo the ideal $\mathfrak{n} := \langle x^{p+1}, x^py\rangle$, we see that $\hat{f}_1 \equiv_{\mathfrak{n}} y^{p^2}$ and $h_2 \equiv_{\mathfrak{n}} y^{p^2+2p}$, giving $h_3^p - h_2^{(p^2+1)/2}\hat{f}_1^{(p^2-2p-1)/2} \equiv_{\mathfrak{n}} h_3^p - y^{p^4+p}$. Thus it is sufficient to identify the lead monomial of $h_3 - y^{p^3+1}$. Note that y^{p^3+1} and xz^{p^3} are consecutive monomials in the grevlex term order. Therefore, if xz^{p^3} appears with nonzero coefficient in h_3, then $\mathrm{LM}(h_3 - y^{p^3+1}) = xz^{p^3}$, and the result follows. Work modulo the ideal $\mathfrak{m} := \langle y\rangle$. Then $\hat{f}_1 \equiv_{\mathfrak{m}} -\beta_3z^px^{p^2-p} - \beta_1zx^{p^2-1}$ and $N_M(y) \equiv_{\mathfrak{m}} 0$. Therefore

$$h_2 \equiv_{\mathfrak{m}} \frac{1}{2\beta_3}\left(z^{p^2}x^{2p} + \frac{\beta_1^p}{\beta_3^p}z^px^{p^2+p} + x^{p^2}(\beta_3z^p + \beta_1zx^{p-1})^2\right).$$

Hence h_3 has degree p^3 as a polynomial in z, with leading coefficient $x/2\alpha_0$ and the result follows. □

Theorem 6.5. *If $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) \neq 0$ and $\gamma_{1357}(M) = 0$, then the set $\mathcal{B} := \{x, \hat{f}_1, h_2, h_3, N_M(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a complete intersection with generating relations coming from the subduction of the tête-à-têtes $(h_2^p, \bar{f}_1^{p+2})$ and $(h_3^p, \bar{f}_1^{(p^2-2p-1)/2}h_2^{(p^2+1)/2})$.*

Proof. Use the subduction given in Lemma 6.4 to construct an invariant h_4 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, \hat{f}_1, h_2, h_3, h_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-têtes for $\mathcal{B}'$ are

$$(h_2^p, \bar{f}_1^{p+2}) \quad \text{and} \quad (h_3^p, \bar{f}_1^{(p^2-2p-1)/2}h_2^{(p^2+1)/2}).$$

Using Lemmas 6.3 and 6.4, these tête-à-têtes subduct to zero, proving that $\mathcal{B}'$ is a SAGBI basis for A. By Lemma 6.2, we have $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, \hat{f}_1, h_2][x^{-1}]$.

Using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z)) = z^{p^k}$ for $k \le 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k = 4$, giving $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

7. The $\gamma_{1234} = 0$, $\gamma_{1235} = 0$, $\gamma_{1357} \ne 0$ strata

In this section we consider representations V_M for $M \in \mathbb{F}^{2\times 4}$ for which $\gamma_{1235}(M) = 0$, $\gamma_{1234}(M) = 0$ and $\gamma_{1357}(M) \ne 0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$.

We first consider the case $\bar{\gamma}_{1257} = 0$. Let r_i denote row i of the matrix $\Gamma(M)$. Since $\gamma_{1357}(M) \ne 0$, the set $\{r_1, r_3, r_5, r_7\}$ is linearly independent. Thus r_2 is a linear combination of r_1, r_5 and r_7. Since $\bar{\gamma}_{1235} = 0$, we know that r_2 is a linear combination of r_1, r_3 and r_5. Using the $(1,2,3)(3,4,5,7,9)$ Plücker relation, $\bar{\gamma}_{1237} = 0$. Thus r_2 is a linear combination of r_1, r_3 and r_7. Combining these observations, we see that r_2 is a scalar multiple of r_1. Using a change of coordinates (see Section 4 of [Campbell et al. 2013]), we may assume that r_2 is zero. If the second row of M is zero, then V_M is a symmetric square representation and the invariants are generated by x, δ, $N_M(y)$ and $N_M(z)$. Since $\bar{\gamma}_{1357} \ne 0$, we have that $N_M(y)$ and $N_M(z)$ are both of degree p^4 and there is a single relation in degree $2p^4$ which can be constructed by subducting the tête-à-tête $(\delta^{p^4}, N_M(y)^2)$ (see Theorem 3.3 of [loc. cit.]).

For the rest of this section, we assume $\bar{\gamma}_{1257} \ne 0$. Evaluating coefficients gives the invariant $\bar{f}_{12357}$. Using the $(1,2,3)(3,4,5,7,9)$ Plücker relation, $\bar{\gamma}_{1237}^{p+1} = 0$. Thus $\bar{\gamma}_{1237} = 0$, and we have $\bar{f}_{12357} = \bar{\gamma}_{1257} y^p x^{p^3-p} + \bar{\gamma}_{1357} \delta x^{p^3-2} + \bar{\gamma}_{2357} y x^{p^3-1}$. Divide by $\bar{\gamma}_{1257} x^{p^3-p}$ to get

$$h_1 := y^p + \frac{\bar{\gamma}_{1357}}{\bar{\gamma}_{1257}} \delta x^{p-2} + \frac{\bar{\gamma}_{2357}}{\bar{\gamma}_{1257}} y x^{p-1}.$$

Observe that $N_M(y) = \bar{f}_{13579}/\bar{\gamma}_{1357}$. Subducting $N_M(y)$ gives

$$\tilde{h}_2 = N_M(y) - h_1^{p^3} + \alpha^{p^3} h_1^{2p^2} x^{p^4-2p^3} - 2\alpha^{p^3+p^2} h_1^{p^2+2p} x^{p^4-p^3-2p^2} + 4\alpha^{p^3+p^2+p} h_1^{p^2+p+2} x^{p^4-p^3-p^2-2p},$$

with $\alpha := \bar{\gamma}_{1357}/\bar{\gamma}_{1257}$.

Lemma 7.1. $\mathrm{LT}(\tilde{h}_2) = 8\alpha^{p^3+p^2+p+1} y^{p^3+p^2+p+2} x^{p^4-p^3-p^2-p-2}$.

Proof. It will be convenient to work modulo the ideal $\langle x^{p^4-p^3}, x^{p^4-p^3-p^2-p-1} y\rangle$, so that $N_M(y) \equiv y^{p^4}$ and $h_1^{p^3} \equiv y^{p^4} + \alpha^{p^3} \delta^{p^3} x^{p^4-2p^3}$. Thus $N_M(y) - h_1^{p^3} \equiv -\alpha^{p^3} \delta^{p^3} x^{p^4-2p^3}$. Expanding gives

$$x^{p^4-2p^3} (h_1^{p^2})^2 \equiv x^{p^4-2p^3} y^{p^3} (y^{p^3} + 2\alpha^{p^2} \delta^{p^2} x^{p^3-2p^2}).$$

Thus

$$N_M(y) - h_1^{p^3} + \alpha^{p^3} h_1^{2p^2} x^{p^4-2p^3} \equiv 2\alpha^{p^3+p^2} y^{p^3} \delta^{p^2} x^{p^4-p^3-2p^2}.$$

Again expanding gives

$$h_1^{p^2+2p}x^{p^4-p^3-2p^2} \equiv x^{p^4-p^3-2p^2}y^{p^3+p^2}(y^{p^2}+2\alpha^p\delta^p x^{p^2-2p}).$$

Hence

$$N_M(y)-h_1^{p^3}+\alpha^{p^3}h_1^{2p^2}x^{p^4-2p^3}-2\alpha^{p^3+p^2}h_1^{p^2+2p}x^{p^4-p^3-2p^2} \equiv -4\alpha^{p^3+p^2+p}\delta^p y^{p^3+p^2}x^{p^4-p^3-p^2-2p}.$$

Since $h_1^{p^2+p+2}x^{p^4-p^3-p^2-2p} \equiv x^{p^4-p^3-p^2-2p}y^{p^3+p^2+p}(y^p+2\alpha\delta x^{p-2})$, we have

$$\tilde{h}_2 \equiv 8\alpha^{p^3+p^2+p+1}y^{p^3+p^2+p+2}x^{p^4-p^3-p^2-p-2}$$

and the result follows. □

Define $h_2 := \tilde{h}_2/(8\alpha^{p^3+p^2+p+1}x^{p^4-p^3-p^2-p-2})$ so that $\mathrm{LT}(h_2)=y^{p^3+p^2+p+2}$.

Lemma 7.2. *Subducting the tête-à-tête* $(h_2^p, h_1^{p^3+p^2+p+2})$ *gives an invariant with lead term*

$$\left(\frac{\bar{\gamma}_{1257}}{2\bar{\gamma}_{1357}}\right)^{p^3+p^2+p} z^{p^4}x^{p^3+p^2+2p}.$$

Proof. For $p=3$, this is a Magma calculation. For $p>3$, the subduction is given by

$$\begin{aligned} &h_2^p-h_1^{p^3+p^2+p+2}+2\alpha h_2 h_1^{p^3}x^{p-2}\\ &\quad+\frac{1}{4\alpha^{p^3+p^2+p}}\big(\beta_1 h_1^{p^3+p^2}x^{p^2+2p}-\beta_1\alpha^{p^2}h_1^{p^3+2p}x^{p^3-p^2+2p}\\ &\qquad+2\beta_1\alpha^{p^2+p}h_1^{p^3+p+2}x^{p^3}-4\beta_1\alpha^{p^2+p+1}h_2h_1^{p^3-p^2}x^{p^3+p-2}\\ &\qquad-\beta_2x^{p^3}(h_1^{p^3+p}x^{2p}-\alpha^p h_1^{p^3+2}x^{p^2}+2\alpha^{p+1}h_2h_1^{p^3-p^2-p}x^{p^2+p-2})\\ &\qquad+\beta_3x^{p^3+p^2+p}(h_1^{p^3+1}-\alpha h_2h_2^{p^3-p^2-p-1}x^{p-2})\\ &\qquad-\beta_4h_2^{(p+1)/2}h_1^{(p^2+p+1)(p-3)/2}x^{p^3+p^2+2p-1}\big),\end{aligned}$$

with

$$\alpha := \frac{\bar{\gamma}_{1357}}{\bar{\gamma}_{1257}},\quad \beta_1 := \frac{\bar{\gamma}_{1359}}{\bar{\gamma}_{1357}},\quad \beta_2 := \frac{\bar{\gamma}_{1379}}{\bar{\gamma}_{1357}},\quad \beta_3 := \frac{\bar{\gamma}_{1579}}{\bar{\gamma}_{1357}},\quad \beta_4 := \bar{\gamma}_{1357}^{p-1}.$$

To calculate the lead term, work modulo the ideal generated by $x^{p^3+p^2+2p+1}$ and $x^{p^3+p^2+2p}y$. □

Theorem 7.3. *If* $\gamma_{1234}(M)=0,\ \gamma_{1235}(M)=0,\ \gamma_{1357}(M)=0$ *and* $\gamma_{1257}(M)\neq 0$, *then the set* $\mathcal{B} := \{x, h_1, h_2, N_M(z)\}$ *is a SAGBI basis for* $\mathbb{F}[V_M]^E$. *Furthermore,* $\mathbb{F}[V_M]^E$ *is a hypersurface with the relation coming from the subduction of the tête-à-tête* $(h_2^p, h_1^{p^3+p^2+p+2})$.

Proof. Use the subduction given in Lemma 7.2 to construct an invariant h_3 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, h_1, h_2, h_3\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-tête for $\mathcal{B}'$ is $(h_2^p, h_1^{p^3+p^2+p+2})$, which subducts to zero using the definition of h_3. Thus $\mathcal{B}'$ is a SAGBI basis for A. Since h_1 is of degree one in z with coefficient $-\alpha x^{p-1}$, it follows from [Campbell and Chuai 2007] that $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, N_M(y)][x^{-1}]$. Since $N_M(y) \in \mathbb{F}[x, h_1, h_2]$, we have $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, h_2][x^{-1}]$. Using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z)) = z^{p^k}$ for $k \le 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k = 4$, giving $\mathrm{LM}(\mathcal{B}) \subset \mathrm{LM}(\mathcal{B}')$. □

8. The $\gamma_{1234} = 0$, $\gamma_{1235} \neq 0$, $\gamma_{1357} = 0$ stratum

In this section we consider representations V_M with $\gamma_{1235}(M) \neq 0$, $\gamma_{1234}(M) = 0$ and $\gamma_{1357}(M) = 0$. The results of this section are valid for $p \ge 3$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$. Observe that $N_M(y) = \bar{f}_{12357}/\bar{\gamma}_{1235}$ (see Remark 1.2). Thus $N_M(y)$ has lead term y^{p^3}. Furthermore, $\bar{f}_{12345}$ has lead term $\bar{\gamma}_{1235} y^{2p} x^{p^2-2p}$. Define $h_1 := \bar{f}_{12345}/(\bar{\gamma}_{1235} x^{p^2-2p})$ so that $\mathrm{LT}(h_1) = y^{2p}$.

Lemma 8.1. $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, N_M(y)][x^{-1}]$.

Proof. We argue as in the proof of Theorem 4.4 of [Campbell et al. 2013]. Since $N_M(y)$ and h_1/x^p are algebraically independent elements of $\mathbb{F}[x, y, \delta/x]^E$ with $\deg(N_M(y)) \deg(h_1/x^p) = p^4 = |E|$, applying Theorem 3.7.5 of [Derksen and Kemper 2002] gives $\mathbb{F}[x, y, \delta/x]^E = \mathbb{F}[x, N_M(y), h_1/x^p]$. The result then follows from the observation that

$$\mathbb{F}[x, y, z]^E[x^{-1}] = \mathbb{F}[x, y, \delta/x]^E[x^{-1}]. \qquad \square$$

Subducting the tête-à-tête $(N_M(y)^2, h_1^{p^2})$ gives

$$\tilde{h}_2 := N_M(y)^2 - h_1^{p^2} + \frac{2}{\bar{\gamma}_{1235}} (\bar{\gamma}_{1237} x^{p^3-p^2} h_1^{(p^2+p)/2} - \bar{\gamma}_{1257} x^{p^3-p} h_1^{(p^2+1)/2}).$$

Lemma 8.2. $\mathrm{LT}(\tilde{h}_2) = 2\bar{\gamma}_{2357} y^{p^3+1} x^{p^3-1}/\bar{\gamma}_{1235}$.

Proof. We work modulo the ideal $\langle x^{p^3} \rangle$. Expand $N_M(y)^2$ and observe that $h_1^{p^2} \equiv y^{2p^3}$, $h_1^p x^{p^3-p^2} \equiv y^{2p^2} x^{p^3-p^2}$ and $h_1 x^{p^3-p} \equiv y^{2p} x^{p^3-p}$. □

Using the $(1,3,5)(2,3,4,5,7)$ Plücker relation, we have $\bar{\gamma}_{1345}\bar{\gamma}_{2357} = \bar{\gamma}_{1235}^{p+1}$. Thus $\bar{\gamma}_{2357} \neq 0$. Define $h_2 := \bar{\gamma}_{1235}\tilde{h}_2/(2\bar{\gamma}_{2357} x^{p^3-1})$ so that $\mathrm{LT}(h_2) = y^{p^3+1}$.

Lemma 8.3. $\mathrm{LM}(h_2^p - h_1^{(p^3+1)/2}) = z^{p^4} x^p$.

Proof. A careful calculation shows that

$$\mathrm{LT}(h_2^p - h_1^{(p^3+1)/2}) = \frac{\bar{\gamma}_{1235}^p}{2\bar{\gamma}_{2357}^p} x^p z^{p^4}. \qquad \square$$

Theorem 8.4. *If $\gamma_{1234}(M)=0$, $\gamma_{1235}(M)\neq 0$ and $\gamma_{1357}(M)=0$, then the set $\mathcal{B}:=\{x, h_1, h_2, N_M(y), N_M(z)\}$ is a SAGBI basis for $\mathbb{F}[V_M]^E$. Furthermore, $\mathbb{F}[V_M]^E$ is a complete intersection with relations coming from the subduction of the tête-à-têtes $(N_M(y)^2, h_1^{p^2})$ and $(h_2^p, h_1^{(p^3+1)/2})$.*

Proof. Use the subduction from Lemma 8.3 to construct an invariant h_3 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, N_M(y), h_1, h_2, h_3\}$ and let A denote the algebra generated by $\mathcal{B}'$. The nontrivial tête-à-têtes for $\mathcal{B}'$ subduct to zero using Lemmas 8.2 and 8.3. Thus $\mathcal{B}'$ is a SAGBI basis for A. From Lemma 8.1, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, h_1, N_M(y)][x^{-1}]$. Thus, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z)) = z^{p^k}$ for $k \leq 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k=4$, giving $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

9. The $\gamma_{1234}\neq 0$, $\gamma_{1235}=0$, $\gamma_{1357}=0$ strata

In this section we consider representations V_M with $\gamma_{1235}(M)=0$, $\gamma_{1234}(M)\neq 0$ and $\gamma_{1357}(M)=0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$. Using the $(1,3,5)(3,4,5,6,7)$ Plücker relation, $\bar{\gamma}_{1345}=0$. Thus

$$\bar{f}_1 = \bar{\gamma}_{1234} y^{p^2} + \bar{\gamma}_{1245} y^p x^{p^2-p} + \bar{\gamma}_{2345} y x^{p^2-1} \in \mathbb{F}[x, y].$$

Since $\bar{\gamma}_{1234}\neq 0$, the orbit of y contains at least p^2 elements. Thus $N_M(y) = \bar{f}_1/\bar{\gamma}_{1234}$ (see Remark 1.2).

Lemma 9.1. $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12346}][x^{-1}]$.

Proof. We argue as in the proof of Lemma 8.1 (and Theorem 4.4 of [Campbell et al. 2013]). Since $N_M(y)$ and $\bar{f}_{12346}/x^{p^2}$ are algebraically independent elements of $\mathbb{F}[x, y, \delta/x]^E$ with $\deg(N_M(y))\deg(\bar{f}_{12346}/x^{p^2}) = p^4 = |E|$, applying Theorem 3.7.5 of [Derksen and Kemper 2002] gives

$$\mathbb{F}[x, y, \delta/x]^E = \mathbb{F}[x, N_M(y), \bar{f}_{12346}/x^{p^2}].$$

The result then follows from the observation that

$$\mathbb{F}[x, y, z]^E[x^{-1}] = \mathbb{F}[x, y, \delta/x]^E[x^{-1}]. \qquad \square$$

We first consider the case $\bar{\gamma}_{1245}\neq 0$. Define $\hat{f}_2 := \bar{f}_2/(\bar{\gamma}_{1234}\bar{\gamma}_{1245}x^p)$ so that $\mathrm{LT}(\hat{f}_2) = y^{p^2+p}$. Subduct the tête-à-tête $(\hat{f}_2^p, N_M(y)^{p+1})$ to get

$$\tilde{h}_3 := N_M(y)^{p+1} - \hat{f}_2^p - \left(\frac{\bar{\gamma}_{1245}}{\bar{\gamma}_{1234}} - \frac{\bar{\gamma}_{2345}^p}{\bar{\gamma}_{1245}^p}\right)\hat{f}_2 N_M(y)^{p-1} x^{p^2-p}.$$

Lemma 9.2. $$\mathrm{LT}(\tilde{h}_3) = \left(\frac{\bar{\gamma}_{2345}^{p+1}}{\bar{\gamma}_{1245}^{p+1}}\right) x^{p^2-1} y^{p^3+1}.$$

Proof. Expand and reduce modulo the ideal $\langle x^{p^2}\rangle$. □

Define
$$h_3 := \frac{\bar{\gamma}_{1245}^{p+1}}{x^{p^2-1}\bar{\gamma}_{2345}^{p+1}}\tilde{h}_3$$
so that $\mathrm{LT}(h_3) = y^{p^3+1}$.

Lemma 9.3. *Subducting the tête-à-tête* $(h_3^p, N_M(y)^{p^2-1}\hat{f}_2)$ *gives an invariant with lead monomial* $x^p z^{p^4}$.

Proof. Work modulo the ideal $\langle x^{p+1}, x^p y\rangle$ and expand to get
$$h_3^p - \hat{f}_2 N_M(y)^{p^2-1} + \frac{\bar{\gamma}_{2345}}{\bar{\gamma}_{1234}} x^{p-1} h_3 N_M(y)^{p^2-p} \equiv \left(\frac{\bar{\gamma}_{1234}^{p^2}\bar{\gamma}_{1245}^{p}}{\bar{\gamma}_{2345}^{p^2+p}}\right) z^{p^4} x^p. \qquad \square$$

Theorem 9.4. *If* $\gamma_{1234}(M) \neq 0$, $\gamma_{1235}(M) = \gamma_{1357}(M) = 0$ *and* $\gamma_{1245}(M) \neq 0$, *then the set* $\mathcal{B} := \{x, N_M(y), \hat{f}_2, h_3, N_M(z)\}$ *is a SAGBI basis for* $\mathbb{F}[V_M]^E$. *Furthermore,* $\mathbb{F}[V_M]^E$ *is a complete intersection with relations coming from the subduction of the tête-à-têtes* $(\hat{f}_2^p, N_M(y)^{p+1})$ *and* $(h_3^p, N_M(y)^{p^2-1}\hat{f}_2)$.

Proof. Use the subduction given in Lemma 9.3 to construct an invariant h_4 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, N_M(y), \hat{f}_2, h_3, h_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The nontrivial tête-à-têtes for $\mathcal{B}'$ subduct to zero using Lemmas 9.2 and 9.3. Thus $\mathcal{B}'$ is a SAGBI basis for A. From Lemma 9.1, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12346}][x^{-1}]$. However, since $f_2 = (f_1^2 + \gamma_{1234} f_{12346})/(2x^{p^2-2p})$, we see that
$$\mathbb{F}[x, N_M(y), \bar{f}_{12346}][x^{-1}] = \mathbb{F}[x, N_M(y), \hat{f}_2][x^{-1}].$$
Thus, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z)) = z^{p^k}$ for $k \leq 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k = 4$, giving $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. $\square$

Suppose $\bar{\gamma}_{1245} = 0$ and let r_i denote row i of the matrix $\Gamma(M)$. Since $\bar{\gamma}_{1234} \neq 0$, we see that $\{r_1, r_2, r_3, r_4\}$ is linearly independent. Using the assumptions that $\bar{\gamma}_{1235} = \bar{\gamma}_{1245} = 0$, we see that $r_5 \in \mathrm{Span}(r_1, r_2, r_3) \cap \mathrm{Span}(r_1, r_2, r_4)$. Therefore $r_5 \in \mathrm{Span}(r_1, r_2)$. However, since $\bar{\gamma}_{1357} = 0$, using a change of coordinates (see [Campbell et al. 2013, §4]) and the $\mathrm{GL}_4(\mathbb{F}_p)$-action, we may assume
$$M := \begin{pmatrix} 1 & c_{12} & c_{13} & 0 \\ 0 & c_{22} & c_{23} & c_{24} \end{pmatrix}$$
with $c_{24} \neq 0$. Since $r_5 = r_1^{p^2}$, we conclude that $r_5 = r_1$. Thus $\bar{\gamma}_{2345} = -\bar{\gamma}_{1234}$. Hence $N_M(y) = \bar{f}_1/\bar{\gamma}_{1234} = y^{p^2} - yx^{p^2-1}$. Define $\hat{h}_2 := -\bar{f}_2/(\bar{\gamma}_{1234}^2 x^{2p-1})$ so that $\mathrm{LT}(\hat{h}_2) = y^{p^2+1}$.

Theorem 9.5. *If* $\gamma_{1234}(M) \neq 0$ *and* $\gamma_{1235}(M) = \gamma_{1357}(M) = \gamma_{1245}(M) = 0$, *then the set* $\mathcal{B} := \{x, N_M(y), \hat{h}_2, N_M(z)\}$ *is a SAGBI basis for* $\mathbb{F}[V_M]^E$. *Furthermore,* $\mathbb{F}[V_M]^E$ *is a hypersurface with the relation coming from the subduction of the tête-à-tête* $(\hat{h}_2^{p^2}, N_M(y)^{p^2+1})$.

Proof. Using the definition of $\hat{h}_2$ and the description given above of $N_M(y)$, we see that

$$\mathrm{LT}\big(\hat{h}_2^{p^2} - N_M(y)^{p^2+1} - \hat{h}_2(xN_M(y))^{p^2-1}\big) = -\tfrac{1}{2} z^{p^4} x^{p^2}.$$

Thus we can use the subduction of the tête-à-tête $(\hat{h}_2^{p^2}, N_M(y)^{p^2+1})$ to construct an invariant h_4 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, N_M(y), \hat{h}_2, h_4\}$ and let A denote the algebra generated by $\mathcal{B}'$. The only nontrivial tête-à-tête subducts to zero. Therefore $\mathcal{B}'$ is a SAGBI basis for A. From Lemma 9.1, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12346}][x^{-1}]$. However, it follows from the definition of $\hat{h}_2$ that $\mathbb{F}[x, N_M(y), \bar{f}_{12346}][x^{-1}] = \mathbb{F}[x, N_M(y), \hat{h}_2][x^{-1}]$. Thus, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z)) = z^{p^k}$ for $k \le 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k = 4$, giving $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

10. The $\gamma_{1234} = 0$, $\gamma_{1235} = 0$, $\gamma_{1357} = 0$ strata

In this section we consider representations V_M with $\gamma_{1235}(M) = 0$, $\gamma_{1234}(M) = 0$ and $\gamma_{1357}(M) = 0$. For convenience, we write $\bar{\gamma}_{ijk\ell}$ for $\gamma_{ijk\ell}(M)$. We assume that the first row of M is nonzero; otherwise, the representation is of type (2, 1) and the calculation of $\mathbb{F}[V_M]^E$ can be found in Section 4 of [Campbell et al. 2013]. Using a change of coordinates, see Proposition 4.3 of [loc. cit.], the $\mathrm{GL}_4(\mathbb{F}_p)$-action, and the hypothesis that $\bar{\gamma}_{1357} = 0$, we may take

$$M = \begin{pmatrix} 1 & c_{12} & c_{13} & 0 \\ 0 & c_{22} & c_{23} & c_{24} \end{pmatrix}.$$

Since $\bar{\gamma}_{1235} = 0$, either $c_{24} = 0$ or $\{1, c_{12}, c_{13}\}$ is linearly dependent over $\mathbb{F}_p$. We assume $c_{24} \neq 0$; otherwise the representation is not faithful and we can view V_M as a representation of a group of rank three. Using the $\mathrm{GL}_4(\mathbb{F}_p)$-action, we replace the third column by a linear combination of the first two columns to get

$$\begin{pmatrix} 1 & c_{12} & 0 & 0 \\ 0 & c_{22} & c_{23} & c_{24} \end{pmatrix}.$$

Expanding gives

$$\bar{\gamma}_{1234} = (c_{12} - c_{12}^p) \det \begin{pmatrix} c_{23} & c_{24} \\ c_{23}^p & c_{24}^p \end{pmatrix}.$$

Since $\bar{\gamma}_{1234} = 0$, either $c_{12} \in \mathbb{F}_p$ or $\{c_{23}, c_{24}\}$ is linearly dependent over $\mathbb{F}_p$. However, if $\{c_{23}, c_{24}\}$ is linearly dependent over $\mathbb{F}_p$, then the representation is not faithful. So we may assume $c_{12} \in \mathbb{F}_p$. Using the $\mathrm{GL}_4(\mathbb{F}_p)$-action to replace the second column with a linear combination of the first two columns gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{22} & c_{23} & c_{24} \end{pmatrix}.$$

If $\bar{\gamma}_{1246} = 0$, then $\{c_{22}, c_{23}, c_{24}\}$ is linearly dependent over $\mathbb{F}_p$, and again the representation is not faithful. Thus we may assume that $\bar{\gamma}_{1246} \neq 0$. Using the above form for M, it is clear that $\bar{\gamma}_{1236} = 0$, $\bar{\gamma}_{1346} = 0$ and $\bar{\gamma}_{1246} = -\bar{\gamma}_{2346}$. Thus

$$\bar{f}_{12346} = \bar{\gamma}_{1246}(y^p x^{2p^2-p} - yx^{2p^2-1}) \in \mathbb{F}[x, y]^E.$$

Since $\mathbb{F}[x, y]^E = \mathbb{F}[x, N_M(y)]$, we have

$$N_M(y) = \bar{f}_{12346}/(\bar{\gamma}_{1246}x^{2p^2-p}) = y^p - yx^{p-1}.$$

Lemma 10.1. $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12468}][x^{-1}].$

Proof. The proof is similar to the proof of Theorem 4.4 of [Campbell et al. 2013] (and Lemmas 8.1 and 9.1). Since $N_M(y)$ and $\bar{f}_{12468}/x^{p^3}$ are algebraically independent elements of $\mathbb{F}[x, y, \delta/x]^E$ with $\deg(N_M(y)) \deg(\bar{f}_{12468}/x^{p^3}) = p^4 = |E|$, applying Theorem 3.7.5 of [Derksen and Kemper 2002] gives

$$\mathbb{F}[x, y, \delta/x]^E = \mathbb{F}[x, N_M(y), \bar{f}_{12468}/x^{p^3}].$$

The result then follows from the observation that

$$\mathbb{F}[x, y, z]^E[x^{-1}] = \mathbb{F}[x, y, \delta/x]^E[x^{-1}]. \qquad \square$$

Subducting $\bar{f}_{12468}$ gives

$$\tilde{h}_1 := \bar{f}_{12468} + \bar{\gamma}_{1246}\big(N_M(y)^{2p^2} + 2N_M(y)^{p^2+p}x^{p^3-p^2} + 2N_M(y)^{p^2+1}x^{p^3-p}\big).$$

Lemma 10.2. $\mathrm{LT}(\tilde{h}_1) = -2\bar{\gamma}_{1246}x^{p^3-1}y^{p^3+1}.$

Proof. We work modulo the ideal $\langle x^{p^3}\rangle$. Using the definition, $\bar{f}_{12468} \equiv -\bar{\gamma}_{1246}y^{2p^3}$. Since $N_M(y) = y^p - yx^{p-1}$, we have

$$N_M(y)^{2p^2} = y^{2p^3} - 2y^{p^3+p^2}x^{p^3-p^2} + y^{2p^2}x^{2p^3-2p^2} \equiv y^{2p^3} - 2y^{p^3+p^2}x^{p^3-p^2}.$$

Expanding and simplifying gives

$$N_M(y)^{p^2+p}x^{p^3-p^2} + N_M(y)^{p^2+1}x^{p^3-p} \equiv y^{p^3+p^2}x^{p^3-p^2} - y^{p^3+1}x^{p^3-1}.$$

Thus

$$\begin{aligned}\tilde{h}_1 &= \bar{f}_{12468} + \bar{\gamma}_{1246}\big(N_M(y)^{2p^2} + 2N_M(y)^{p^2+p}x^{p^3-p^2} + 2N_M(y)^{p^2+1}x^{p^3-p}\big)\\ &\equiv -2\bar{\gamma}_{1246}x^{p^3-1}y^{p^3+1}. \qquad \square\end{aligned}$$

Define $h_1 := -\tilde{h}_1/(2\bar{\gamma}_{1246}x^{p^3-1})$ so that $\mathrm{LT}(h_1) = y^{p^3+1}$. Note that

$$\mathbb{F}[x, N_M(y), h_1][x^{-1}] = \mathbb{F}[x, N_M(y), \bar{f}_{12468}][x^{-1}].$$

Lemma 10.3. *Subducting the tête-à-tête $(h_1^p, N_M(y)^{p^3+1})$ gives an invariant with lead monomial $x^p z^{p^4}$.*

Proof. Refining the calculation in the proof of the previous lemma gives

$$\tilde{h}_1 \equiv_{\langle x^{p^3+1},\, x^{p^3}y\rangle} \bar{\gamma}_{1246}(-2y^{p^3+1}x^{p^3-1} + x^{p^3}z^{p^3}).$$

Thus

$$h_1 \equiv_{\langle x^2,\, xy\rangle} y^{p^3+1} - \tfrac{1}{2}z^{p^3}x \quad \text{and} \quad h_1^p \equiv_{\langle x^{p+1},\, x^p y\rangle} y^{p^4+p} - \tfrac{1}{2}z^{p^4}x^p.$$

Furthermore

$$N_M(y)^{p^3+1} \equiv_{\langle x^{p+1},\, x^p y\rangle} y^{p^4+p} - y^{p^4+1}x^{p-1}$$

and

$$h_1 N_M(y)^{p^3-p^2}x^{p-1} \equiv_{\langle x^{p+1},\, x^p y\rangle} y^{p^4+1}x^{p-1}.$$

Thus $\mathrm{LT}(h_1^p - N_M^{p^3+1} - h_1 N_M(y)^{p^3-p^2}) = -\frac{1}{2}x^p z^{p^4}$. □

Theorem 10.4. *If* $\gamma_{1234}(M) = 0$, $\gamma_{1235}(M) = 0$, $\gamma_{1357}(M) = 0$ *and* $\gamma_{1246}(M) \neq 0$, *then the set* $\mathcal{B} := \{x, N_M(y), h_1, N_M(z)\}$ *is a SAGBI basis for* $\mathbb{F}[V_M]^E$. *Furthermore,* $\mathbb{F}[V_M]^E$ *is a hypersurface with the relation coming from the subduction of the tête-à-tête* $(h_1^p, N_M(y)^{p^3+1})$.

Proof. Use the subduction given in Lemma 10.3 to construct an invariant h_2 with lead term z^{p^4}. Define $\mathcal{B}' := \{x, N_M(y), h_1, h_2\}$ and let A denote the algebra generated by $\mathcal{B}'$. The single nontrivial tête-à-tête for $\mathcal{B}'$ subducts to zero using Lemma 10.3. Thus $\mathcal{B}'$ is a SAGBI basis for A. From Lemma 10.1, $\mathbb{F}[V_M]^E[x^{-1}] = \mathbb{F}[x, N_M(y), h_1][x^{-1}]$. Thus, using Theorem 1.1, $A = \mathbb{F}[V_M]^E$. Clearly $\mathrm{LT}(N_M(z)) = z^{p^k}$ for $k \leq 4$. Since $\mathcal{B}'$ is a SAGBI basis for $\mathbb{F}[V_E]^E$, this forces $k = 4$, giving $\mathrm{LM}(\mathcal{B}) = \mathrm{LM}(\mathcal{B}')$. □

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