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a journal of mathematics

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msp

2016 vol. 9, no. 5

INVOLVE 9:5 (2016)
dx.doi.org/10.2140/involve.2016.9.765

On the Chermak–Delgado lattices of split metacyclic p-groups

Erin Brush, Jill Dietz, Kendra Johnson-Tesch and Brianne Power

(Communicated by Kenneth S. Berenhaut)

The Chermak–Delgado measure of a subgroup H of a finite group G is defined as $m_G(H) = |H||C_G(H)|$. The subgroups with maximal Chermak–Delgado measure form a poset and corresponding lattice, known as the CD-lattice of G. We describe the symmetric nature of CD-lattices in general, and use information about centrally large subgroups to determine the CD-lattices of split metacyclic p-groups in particular. We also describe a rank-symmetric sublattice of the CD-lattice of split metacyclic p-groups.

1. Introduction

A. Chermak and A. Delgado [1989] developed a "measuring argument" for a finite group G acting on a finite group H to prove, as G. Glauberman [2006] put it, "remarkably beautiful results and powerful applications." Glauberman extended Chermak and Delgado's work to obtain, among other things, results about centrally large subgroups of p-groups. A subgroup H of a finite p-group P is *centrally large* if $|H||Z(H)| \geq |H^*||Z(H^*)|$ for every subgroup H^* of P.

For any positive real number α, Chermak and Delgado defined the measure

$$m_\alpha(G, H) = \mathrm{Sup}\{|A|^\alpha |C_H(A)|\}_{A \in S(G)},$$

where $S(G)$ is the set of all nontrivial subgroups of G. In his book, I. M. Isaacs [2008] focused on the case where $\alpha = 1$ and H is a subgroup of the finite group G. He defined the *Chermak–Delgado measure* of a subgroup H of G as $m_G(H) = |H||C_G(H)|$. In a definition analogous to Glauberman's centrally large subgroups, Isaacs defined a subgroup H of G to have *maximal Chermak–Delgado measure* if $|H||C_G(H)| \geq |H^*||C_G(H^*)|$ for every subgroup H^* of G. Finally, Isaacs showed that the subgroups with maximal Chermak–Delgado measure form a sublattice of the lattice of all subgroups of G, which he termed the *Chermak–Delgado lattice of* G.

MSC2010: 20D30.

Keywords: centrally large subgroups, Chermak–Delgado measure, lattices of subgroups, metacyclic p-groups.

Not only are the definitions similar, but one can see that H is centrally large in a p-group P if and only if H is in the Chermak–Delgado lattice of P and $C_P(H) \leq H$ [Glauberman 2006]. Thus, the study of Chermak–Delgado lattices may shed light on the subgroup structure of finite groups in general, and on centrally large subgroups in particular.

Not much is known about Chermak–Delgado lattices. To date, there are only four published papers on the subject. B. Brewster and E. Wilcox [2012] studied the connection between Chermak–Delgado lattices of direct and wreath products of groups and the Chermak–Delgado lattices of their components. Together with P. Hauck they constructed a class of p-groups whose Chermak–Delgado lattice is a chain [Brewster et al. 2014a]. Later the trio studied quasiantichains in Chermak–Delgado lattices, proving that if there is a quasiantichain interval between subgroups L and H of G, with $L \leq H$, then H/L is an elementary abelian p-group for some prime p [Brewster et al. 2014b]. The most recent contribution to the subject is a paper by L. An, J. Brennan, H. Qu, and Wilcox [An et al. 2015], who further studied groups for which the Chermak–Delgado lattice is a chain of diamonds.

The structure of a Chermak–Delgado lattice varies greatly among finite groups. For example, it is easy to see that the Chermak–Delgado lattice of an abelian group G consists of just G. There are groups whose Chermak–Delgado lattices are chains of any length, diamonds of any width, and chains of diamonds; see [Brewster et al. 2014a; An et al. 2015]. These two papers construct groups that have particular Chermak–Delgado lattice shapes, whereas we start with general split metacyclic p-groups, $p > 2$, and construct their associated Chermak–Delgado lattices.

The structure of this paper is as follows. In Section 2, we define some terms and describe the symmetry of Chermak–Delgado lattices in general. In Section 3, we give a full treatment of split metacyclic p-groups, $p > 2$, from describing their subgroup structure to determining their Chermak–Delgado measure and lattices. In Section 4, we define a rank-symmetric subposet of the poset of all subgroups of a split metacyclic p-group with maximal Chermak–Delgado measure; sometimes its corresponding lattice has the same structure as the full Chermak–Delgado lattice. Finally, in Section 5, we construct two complete Chermak–Delgado lattices using the theorems developed in Section 3.

2. Chermak–Delgado lattices and their symmetry

A good reference for Chermak–Delgado measures and lattices is [Isaacs 2008], where we find some of the definitions and properties below.

As above, if G is a finite group with subgroup H, then the *Chermak–Delgado measure* (or *CD-measure*) of H in G is $m_G(H) = |H|\,|C_G(H)|$, where $C_G(H)$ is the centralizer of H in G. We will denote the maximum possible CD-measure in a group G as $m^*(G)$. The set of all subgroups of G that have maximal measure is a poset called the *Chermak–Delgado set* of G (or *CD-set*) and denoted $\mathcal{CD}(G)$.

As with any poset, we are interested in smallest and largest elements. The greatest lower bound for the entirety of $\mathcal{CD}(G)$ is the intersection of all members of the set, which is called the *Chermak–Delgado subgroup* of G and denoted M_G. Isaacs proved that $Z(G) \leq M_G$. It follows that $Z(G) \leq H$ for any $H \in \mathcal{CD}(G)$. Isaacs further proved that if $H \in \mathcal{CD}(G)$, then $C_G(C_G(H)) = H$; hence $C_G(H)$ is also in $\mathcal{CD}(G)$. In particular, $M^G = C_G(M_G)$ is the least upper bound for the entirety of $\mathcal{CD}(G)$ and $m^*(G) = |M_G||M^G|$.

The *Chermak–Delgado lattice* of G (or *CD-lattice*) consisting of subgroups from $\mathcal{CD}(G)$ is a modular, self-dual lattice [Brewster et al. 2014b]. Usually this lattice is also denoted by $\mathcal{CD}(G)$, but we will use the notation $\mathcal{L}(G)$ to specifically denote the Hasse diagram drawn from the CD-set such that there is an edge between subgroups $H_1, H_2 \in \mathcal{CD}(G)$ if and only if H_1 *covers* H_2, meaning that $H_2 < H_1$ and there does not exist a subgroup $K \in \mathcal{CD}(G)$ for which $H_2 < K < H_1$. Although subgroup order does not necessarily define a rank function on $\mathcal{CD}(G)$, we will see that the lattice $\mathcal{L}(G)$ has several interesting properties.

Define the *height* of a subgroup H in the lattice $\mathcal{L}(G)$ as simply $|H|$. If H_1 and H_2 are two subgroups in $\mathcal{L}(G)$ of orders n_1 and n_2 respectively, with $n_1 \geq n_2$, we will call n_1/n_2 the *distance* between H_1 and H_2 and denote it $d(H_1, H_2)$. In our scheme, we have $d(H, K) \geq 1$ for all $H, K \in \mathcal{CD}(G)$.

The next theorem says $\mathcal{CD}(G)$ is order symmetric and $\mathcal{L}(G)$ can be displayed having a horizontal line of symmetry.

Theorem 2.1. *Let G be a finite group. Then $\mathcal{L}(G)$ is graph isomorphic to a lattice that is symmetric across a horizontal line of symmetry at height $\sqrt{m^*(G)}$.*

Proof. To prove the symmetry, we will show that the bijective correspondence $H \mapsto C_G(H)$, for $H \in \mathcal{CD}(G)$, satisfies three properties:

(1) If H lies above the line of symmetry, then the distance between the top of the lattice and H is equal to the distance between $C_G(H)$ and the bottom of the lattice (by duality, the analogous property will hold for any H lying below the line of symmetry).

(2) If H lies on the line of symmetry, then so does $C_G(H)$.

(3) Subgroup inclusion relationships above the line of symmetry are mirrored below the line.

Thus, a symmetric lattice can be drawn with $H \in \mathcal{L}(G)$ and its partner $C_G(H)$ placed on a vertical line above and below the line of symmetry, or both exactly on the line. (Figure 1 shows the same portion of the CD-lattice for the metacyclic p-group $P(4, 2, 2, 0)$ as described in (3-1) below. The lattice on the right is drawn

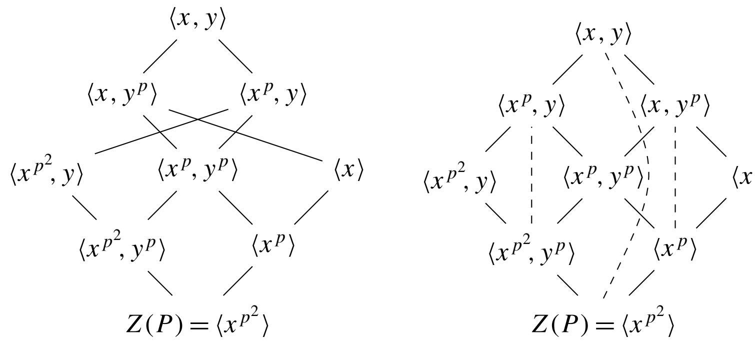

Figure 1. Isomorphic portions of the CD-lattice for $P(4, 2, 2, 0)$.

with its symmetry highlighted, and dashed lines connecting a subgroup to its centralizer.)

Proof of (1): Let $H \in \mathcal{CD}(G)$ and assume $|H| > \sqrt{m^*(G)}$. Note that

$$|H||C_G(H)| = m^*(G) = |M_G||M^G|.$$

It follows that

$$\frac{|C_G(H)|}{|M_G|} = \frac{|M^G|}{|H|}.$$

Hence, $d(C_G(H), M_G) = d(M^G, H)$.

Proof of (2): Assume $H \in \mathcal{CD}(G)$ lies on the line of symmetry, so $|H| = \sqrt{m^*(G)}$. Since $|H||C_G(H)| = m^*(G)$, we must have $|C_G(H)| = \sqrt{m^*(G)}$, so $C_G(H)$ also lies on the line of symmetry.

Proof of (3): Assume $H, K \in \mathcal{CD}(G)$ and $H \leq K$. It follows that $C_G(K) \leq C_G(H)$. Thus, subgroup inclusion relationships above the line of symmetry are mirrored below the line, and vice versa. $\square$

3. Subgroup and CD-lattices of split metacyclic p-groups

Presentations of metacyclic p-groups, p > 2. A group is metacyclic if it has a cyclic normal subgroup whose corresponding quotient is also cyclic. The group splits if it is a semidirect product of the form $\mathbb{Z}_m \rtimes \mathbb{Z}_n$. It is well known that every noncyclic metacyclic p-group, $p > 2$, has a presentation of the form

$$P = P(m, n, c, s) = \langle x, y \mid x^{p^m} = 1,\ y^{p^n} = x^{p^{m-s}},\ yxy^{-1} = x^{1+p^{m-c}}\rangle, \qquad (3\text{-}1)$$

where $1 \leq m, n,\ 0 \leq c \leq \min\{m-1, n\}$, and $0 \leq s \leq m - c$.[1] Note that the parameter s measures how far the group is from splitting (if $s = 0$, then P is split

[1] Presentations of metacyclic 2-groups are also known, but we restrict to odd primes throughout.

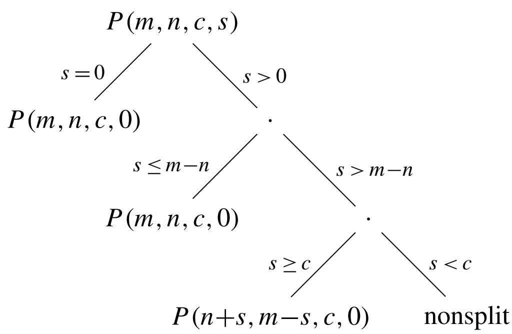

Figure 2. Conditions under which $P(m, n, c, s)$ splits.

and $P \cong \langle x \rangle \rtimes \langle y \rangle$), and the parameter c measures how far the group is from being commutative (if $c = 0$, then P is abelian). We have $|P| = p^{m+n}$ and every element of P can be written uniquely as $x^i y^j$ for some $0 \le i < p^m$ and $0 \le j < p^n$.

Note 3.1. For the rest of this paper, we will use the term "metacyclic p-group" to mean a finite, nonabelian, noncyclic metacyclic p-group, where $p > 2$.

Depending on the relative sizes of the four parameters, it is sometimes possible to find a different set of generators that yields a split presentation with $s = 0$.

Proposition 3.2 [Dietz 1993]. *Let $P(m, n, c, s)$ be as in* (3-1).

(1) *If $s = 0$, then P is split.*

(2) *If $s \ne 0$ and $m-s \ge n$, then we can find an element y^* such that $P \cong \langle x, y^* \rangle = P(m, n, c, 0)$ and thus P splits.*

(3) *If $s \ne 0$ and $m - s < \min\{n, m - c + 1\}$, then we can find elements x^* and y^* such that $P \cong \langle y^*, x^* \rangle = P(n + s, m - s, c, 0)$ and thus P splits.*

(4) *If $s \ne 0$ and $m - c < m - s < n$, then P is nonsplit.*

See Figure 2 for a decision tree illustrating this proposition.

Furthermore, we can put restrictions on the parameters in such a way that different values of s and c for a fixed pair m and n describe unique isomorphism types. We have the following isomorphism classes that we will say are in *reduced* metacyclic form.

Proposition 3.3 [King 1973]. *Every metacyclic p-group is isomorphic to exactly one of the following reduced forms*:

(1) *Split*: $P(m, n, c, s)$ *with* $0 = s \le c \le \min\{n, m - 1\}$.

(2) *Nonsplit*: $P(m, n, c, s)$ *with* $\max\{1, m - n + 1\} \le s \le \min\{c - 1, m - c\}$.

CD-measure and CD-lattice of $P(m, n, c, s)$. We first establish the CD-measures of split metacyclic p-groups, and then determine their CD-lattices.

Proposition 3.4. *Let $P = P(m, n, c, 0)$ be a split metacyclic p-group. Then $m^*(P) = p^{2(m+n-c)}$.*

Proof. By [Héthelyi and Külshammer 2011, Lemma 4.1], we have $m^*(P) = [P : P']^2$. Lemma 2.5 in [Bidwell and Curran 2010] shows that $P' = \langle x^{p^{m-c}} \rangle \cong \mathbb{Z}_{p^c}$. Hence $m^*(P) = (p^{m+n-c})^2$. □

Theorem 3.5. *Let $P = P(m, n, c, 0)$. Then $\mathcal{CD}(P)$ consists of the maximal abelian subgroups of P, all nonabelian subgroups of P, and all centralizers of the nonabelian subgroups.*

Proof. First, we show that $\mathcal{CD}(P)$ contains the subgroups mentioned in the theorem; second, we show that no other subgroups are in $\mathcal{CD}(P)$.

Proposition 2.4 in [Glauberman 2006] shows that all centrally large subgroups of a finite p-group have maximal CD-measure. Corollary 4.4 in [Héthelyi and Külshammer 2011] shows that all maximal abelian and nonabelian subgroups of P are centrally large; hence all are in $\mathcal{CD}(P)$. By duality, centralizers of these subgroups are also in $\mathcal{CD}(P)$. We will show that the maximal abelian subgroups are equal to their centralizers.

Let A be a maximal abelian subgroup of P. By Lemma 2.8 in [Héthelyi and Külshammer 2011], $|A| = [P : P'] = p^{m+n-c}$. Since A has maximal CD-measure, we know $|C_P(A)| = p^{m+n-c}$. Finally, $A \le C_P(A)$ implies $A = C_P(A)$.

Next, we must show that no other subgroups of P are contained in $\mathcal{CD}(P)$. Let H be an abelian subgroup of P that is not maximal abelian and is not the centralizer of a nonabelian subgroup of P. If $C_P(H)$ is nonabelian, then $C_P(H) \in \mathcal{CD}(P)$. By duality, $C_P(C_P(H)) = H$, but this contradicts the fact that H is not the centralizer of a nonabelian subgroup of P. If $C_P(H)$ is abelian, then neither H nor $C_P(H)$ is maximal abelian. We see that

$$m_P(H) = |H||C_P(H)| < (p^{m+n-c})^2 \le m^*(P),$$

so $H \notin \mathcal{CD}(P)$. □

A few things from the theorem and its proof are worthy of note.

- P is itself a member of $\mathcal{CD}(P)$, hence $Z(P)$ is too.
- By Theorem 2.1, the line of symmetry of $\mathcal{L}(P)$ is at height $\sqrt{m^*(P)} = p^{m+n-c}$, which is the height of the maximal abelian subgroups of P. Thus, it makes sense that the maximal abelian subgroups of P are equal to their own centralizers.
- All nonabelian subgroups of P lie above the line of symmetry in $\mathcal{L}(P)$ and have order greater than all abelian subgroups of P (this is in Proposition 3.3 of [Berkovich 2013] too).

• All nonabelian subgroups of P have abelian centralizers (that are not maximal). In fact, the proposition below shows that centralizers of nonabelian subgroups equal their centers.

• All nonabelian subgroups of P contain the center of P.

We have two further results about abelian and nonabelian subgroups of $P = P(m, n, c, 0)$.

Proposition 3.6. *Let H be a nonabelian subgroup of P. Then $C_P(H) = Z(H)$.*

Proof. As we saw in the proof above, H is centrally large, so $|H||Z(H)| \geq |H^*||Z(H^*)|$ for all $H^* \leq P$. Since a maximal abelian subgroup A of P is equal to its own centralizer, we see that $|A||C_P(A)| = |A||Z(A)|$ is maximal too. Then

$$|H||Z(H)| \geq |A||Z(A)| = |A||C_P(A)| = |H||C_P(H)|$$

implies $C_P(H) = Z(H)$. □

Proposition 3.7. *Not every nonmaximal abelian subgroup A of P with $|A| \geq |Z(P)|$ is in $\mathcal{CD}(P)$.*

Proof. First some notation: let s_j denote the total number of subgroups of P of order p^j, and let s_j^{cd} denote the total number of subgroups of order p^j that are in $\mathcal{CD}(P)$.

From Theorem 3.5 we know that if a nonmaximal abelian subgroup of P is in $\mathcal{CD}(P)$, then it has order p^k, where $m+n-2c \leq k < m+n-c$. Thus, if we can show $s_k > s_k^{\mathrm{cd}}$ for all such k, then we will have proved the proposition.

By duality, $s_k^{\mathrm{cd}} = s_i^{\mathrm{cd}}$, where $i = 2(m+n-c)-k$. Since every nonabelian subgroup of P is in $\mathcal{CD}(P)$, we know $s_i^{\mathrm{cd}} = s_i$. By a result of A. Mann [2010], the number of subgroups of P of order p^i, where $m+n-c < i \leq m+n$, is

$$s_i = \frac{p^{m+n-i+1}-1}{p-1}.$$

The same theorem in [Mann 2010] shows that the size of s_k depends on the size of k relative to m and n.

Assume $m \geq n$.

(1) If $k \geq m$, then $s_k = (p^{m+n-k+1}-1)/(p-1)$. Since $k < i$, we have $s_k > s_i$.

(2) If $n \leq k \leq m$, then $s_k = (p^{n+1}-1)/(p-1)$. Since $m+n-c < i \leq m+n$ and $c \leq n$, we have $i > m$. Thus $n+1 > m+n-i+1$ and $s_k > s_i$.

(3) If $k \leq n$, then $s_k = (p^{k+1}-1)/(p-1)$. Since $c \leq \min\{m-1, n\}$, we have $m+n-2c > 0$. Thus

$$k+1 = 2(m+n-c)-i+1 > m+n-i+1,$$

and $s_k > s_i$.

In each case, the total number of abelian subgroups of order p^k is greater than the number of abelian subgroups of order p^k that appear in $\mathcal{CD}(P)$.

A similar result holds if $m < n$. □

By duality, we need only determine the upper half of the CD-lattice in order to at least know its complete structure (if not the precise subgroups). The next subsection gives us tools for determining the nonabelian subgroups of $P(m, n, c, 0)$ in a regressive manner.

Subgroups of metacyclic p-groups. It is well known that subgroups of metacyclic groups are either cyclic or metacyclic. In this section we determine the precise metacyclic structure of the maximal subgroups of $P = P(m, n, c, s)$. First we need some computational lemmas from [Schulte 2001] and [Bidwell and Curran 2010].[2]

Lemma 3.8 [Schulte 2001]. *Let $P = P(m, n, c, s)$ be as in* (3-1).

(1) *We have $y^j x^i = x^{i(1+p^{m-c})^j} y^j$, where $i, j \geq 0$.*

(2) *Let $\alpha = 1 + p^{m-c}$ and $k \in \mathbb{N}$. Then $(x^i y^j)^k = x^{i\Lambda(j,k)} y^{jk}$, where*

$$\Lambda(j, k) = 1 + \alpha^j + \alpha^{2j} + \cdots + \alpha^{(k-1)j} \quad \textit{for } i, j, k \geq 0.$$

Lemma 3.9 [Bidwell and Curran 2010]. *For any positive integers a and b and an odd prime p, we have $(1 + p^a)^{p^b} \equiv 1 \bmod p^{a+b}$.*

From the proof of Proposition 3.7, the number of proper subgroups of P of maximal order is $s_i = p + 1$, where $i = m + n - 1$.

Theorem 3.10. *The $p+1$ maximal proper subgroups of $P = P(m, n, c, s) = \langle x, y\rangle$ are*

$$L = \langle x^p, y\rangle, \qquad M_i = \langle x^p, x^i y\rangle \quad \textit{for } i = 1, \ldots, p-1, \qquad R = \langle x, y^p\rangle.$$

Proof. It is clear that L and R each have order p^{m+n-1} and are distinct from one another. L and M_i are distinct because $y \in M_i$ only if $i \equiv 0 \bmod p$. Similarly, R and M_i are distinct because $x \notin M_i$. To show the M_i are distinct from one another, suppose that $x^i y \in M_j$ for some $i \neq j$. Then for some $u, v \in \mathbb{Z}$, we have

$$x^i y = (x^p)^u (x^j y)^v = x^{pu + j\Lambda(1,v)} y^v$$

by Lemma 3.8. We must have $v \equiv 1 \bmod p^n$ so there exists $w \in \mathbb{Z}$ such that $v = 1 + wp^n$. Now

$$x^i y = x^{pu+j\Lambda(1,v)} y^{1+wp^n} = x^{pu+j\Lambda(1,v)+wp^{m-s}} y,$$

hence $i \equiv pu + j\Lambda(1, v) + wp^{m-s} \bmod p^m$. By Lemma 3.9 we can see that $\Lambda(1, v) \equiv 1 \bmod p$. Thus $i \equiv j \bmod p$, which is a contradiction.

[2]Schulte [2001] focuses on a particular family of split metacyclic p-groups, but the result stated here clearly holds more generally.

It remains to show that $|M_i| = p^{m+n-1}$. Because the M_i are metacyclic, we know that

$$|M_i| = \frac{|\langle x^p\rangle||\langle x^i y\rangle|}{|\langle x^p\rangle \cap \langle x^i y\rangle|}.$$

Using Lemmas 3.8 and 3.9, we can show that $|x^i y| \geq |y| = p^{n+s}$. Using similar computations, we can show that $\langle x^p\rangle \cap \langle x^i y\rangle \leq \langle x^{p^{m-s}}\rangle$. Thus

$$\frac{|\langle x^p\rangle||\langle x^i y\rangle|}{|\langle x^p\rangle \cap \langle x^i y\rangle|} \geq \frac{p^{m-1}p^{n+s}}{p^s} = p^{m+n-1}.$$

Certainly M_i is a proper subgroup of P, so we must have $|M_i| = p^{m+n-1}$. □

Next we identify the metacyclic structure of the $p+1$ maximal subgroups of a metacyclic p-group, but separate the split and nonsplit cases.

Theorem 3.11. *The metacyclic forms of the maximal subgroups of the split metacyclic p-group $P = P(m, n, c, 0)$ are*

(1) $L = \langle x^p, y\rangle \cong P(m-1, n, c-1, 0)$,

(2) $M_i = \langle x^p, x^i y\rangle \cong \begin{cases} P(m-1, n, c-1, 0) \cong L & \text{if } m \leq n, \\ P(m, n-1, c-1, 0) \cong R & \text{if } m > n \text{ and } n \leq m-c+1, \\ P(m-1, n, c-1, m-n) & \text{if } m > n > m-c+1, \end{cases}$

(3) $R = \langle x, y^p\rangle \cong P(m, n-1, c-1, 0)$.

Proof. The order relations in L are clear, so we need only check the degree of commutativity. We have

$$yx^p y^{-1} = (yxy^{-1})^p = (x^{1+p^{m-c}})^p = (x^p)^{1+p^{m-c}}.$$

Since $m-c = (m-1)-(c-1)$, we have our result.

In R we see that $y^p x y^{-p} = x^{(1+p^{m-c})^p}$ by Lemma 3.8. By Lemma 3.9 we know that $(1+p^{m-c})^p \equiv 1 \bmod p^{m-c+1}$. By [King 1973, Proposition 2.3], we can replace x and y^p with x^* and y^* respectively so that $\langle x^*\rangle = \langle x\rangle$, $\langle y^*\rangle = \langle y^p\rangle$ and $y^* x^* (y^*)^{-1} = (x^*)^{1+p^{m-c+1}}$. Since $m-c+1 = m-(c-1)$, we have our result.

In M_i we have

$$(x^i y)x^p(x^i y)^{-1} = x^i(yx^p y^{-1})x^{-i} = x^i(x^{1+p^{m-c}})^p x^{-i} = (x^p)^{1+p^{m-c}}.$$

Next, we compute the splitting degree. Since P is regular (see [Davitt 1970, Corollary 1]), we know

$$(x^i y)^{p^n} = x^{ip^n} y^{p^n} z^{p^n}$$

for some $z \in [P, P] = \langle x^{p^{m-c}}\rangle$. Suppose $z = x^{ap^{m-c}}$ for some $a \in \mathbb{Z}$. Then

$$(x^i y)^{p^n} = x^{ip^n + ap^{m-c+n}}.$$

If $m \le n$, we see that $x^i y$ has order p^n and $M_i = P(m-1, n, c-1, 0)$. If $m > n$, then

$$(x^i y)^{p^n} = (x^p)^{ip^{n-1}+ap^{m-c+n-1}} = (x^p)^{p^{n-1}(i+ai^{-1}p^{m-c})}.$$

Since $i + ai^{-1}p^{m-c}$ is a unit modulo p^m, [King 1973] again tells us that we can choose new generators x^* and y^* so that $\langle x^*\rangle = \langle x^p\rangle$, $\langle y^*\rangle = \langle x^i y\rangle$, $y^*x^*(y^*)^{-1} = (x^*)^{1+p^{m-c}}$, and $(y^*)^{p^n} = (x^*)^{p^{n-1}}$. Hence we get $M_i = P(m-1, n, c-1, m-n)$. Finally, by Proposition 3.2 we see that the reduced metacyclic form of M_i depends on whether $m-n < c-1$. □

Theorem 3.12. *The metacyclic forms of the maximal subgroups of the nonsplit metacyclic p-group $P = P(m,n,c,s)$, $\max\{1, m-n+1\} \le s \le \min\{c-1, m-c\}$, are*

(1) $L = \langle x^p, y\rangle \cong \begin{cases} P(n+s,\, m-s-1,\, c-1,\, 0) & \textit{if } s = c-1, \\ P(m-1,\, n,\, c-1,\, s) & \textit{if } s < c-1, \end{cases}$

(2) $M_i = \langle x^p, x^i y\rangle \cong L$ *for* $i = 1, \dots, p-1$,

(3) $R = \langle x, y^p\rangle \cong \begin{cases} P(m,\, n-1,\, c-1,\, 0) & \textit{if } s = m-n+1, \\ P(n+s-1,\, m-s,\, c-1,\, 0) & \textit{if } m-n+1 < s = c-1, \\ P(m,\, n-1,\, c-1,\, s) & \textit{if } m-n+1 < s < c-1. \end{cases}$

Proof. Consider the subgroup L first. As in the proof of Theorem 3.11, we have the commutativity degree is $c-1$. Since $y^{p^n} = (x^p)^{p^{m-1-s}}$, we see that the splitting degree is s. Thus $L = P(m-1, n, c-1, s)$. By Proposition 3.2 and the subsequent decision tree, we know $s > 0$ and $s \ge m-n+1 > (m-1)-n$, so that the structure of L depends on whether $s \ge c-1$. We already have $s \le c-1$, so $s \ge c-1$ if and only if $s = c-1$.

Next we consider R. As in the proof of Theorem 3.11, we can replace x and y^p with x^* and y^* respectively so that $y^*x^*(y^*)^{-1} = (x^*)^{1+p^{m-c+1}}$. Furthermore, the splitting degree remains unchanged, so $(y^p)^{p^{n-1}} = x^{p^{m-s}}$ implies $(y^*)^{p^{n-1}} = (x^*)^{p^{m-s}}$. Hence $R = P(m, n-1, c-1, s)$. From the decision tree, we see that $R = P(m, n-1, c-1, 0)$ if $s \le m-(n-1)$. We already have $m-n+1 \le s$, so this form occurs exactly when $s = m-n+1$. Finally, the structure of R depends on whether $s \ge c-1$ (which happens if and only if $s = c-1$) or $s < c-1$.

Lastly, we consider the subgroups M_i. We compute $x^i y x^p y^{-1} x^{-p} = (x^p)^{1+p^{m-c}}$ so the commutativity degree is $c-1$. The splitting degree is determined by computing $(x^i y)^{p^n} = x^{i\Lambda(1,p^n)+p^{m-s}}$, where $\Lambda(1, p^n)$ is as in Lemma 3.8. Now

$$\Lambda(1, p^n) = 1 + \alpha + \alpha^2 + \cdots + \alpha^{p^n-1} = \frac{\alpha^{p^n}-1}{\alpha-1}.$$

Since $\alpha^{p^n} \equiv 1 \bmod p^{m-c+n}$ by Lemma 3.9, there exists $a \in \mathbb{Z}$ such that $\alpha^{p^n} = 1 + ap^{m-c+n}$. Thus $\Lambda(1, p^n) = ap^n$ and we have

$$(x^i y)^{p^n} = x^{iap^n+p^{m-s}} = x^{p^{m-s}(iap^{n-m+s}+1)} = (x^p)^{p^{m-s-1}(iap^{n-m+s}+1)}.$$

Once again by [King 1973] we can replace x^p and $x^i y$ with generators x^* and y^* satisfying the same order and commutator relations, and further satisfying $(y^*)^{p^n} = (x^*)^{p^{m-1-s}}$. Hence $M_i = P(m-1, n, c-1, s)$ and has the same structure as L. □

With Theorems 3.5, 3.11, and 3.12 in hand, we can construct the subgroup lattice of a split metacyclic p-group, and hence construct its CD-lattice. In practice, even small-order metacyclic p-groups will have complicated lattices with split subgroups spawning nonsplit subgroups, and vice versa (see one example in Section 5). However, there are certain conditions under which we get particularly nice CD-lattices, including some quasiantichains as discussed in [Brewster et al. 2014b]. This is what we will describe in the next section.

4. Diamonds in the rough: the BEK-lattice

We begin with some notation. Let $P = P(m, n, c, 0) = \langle x, y\rangle$, and set $H_{ab} = \langle x^{p^a}, y^{p^b}\rangle$, where $0 \le a \le m$ and $0 \le b \le n$. These subgroups are the L- and R-types from Theorems 3.11 and 3.12, and sometimes the M_i subgroups are isomorphic to these. The set of the H_{ab} will form a sublattice of $\mathcal{L}(P)$, which we will denote $\mathcal{BEK}(P)$, and under the right circumstances $\mathcal{L}(P)$ will "collapse" to $\mathcal{BEK}(P)$. Before getting to these results, we delve into the structure of the H_{ab}.

Lemma 4.1 [Bidwell and Curran 2010]. *Let $P = P(m, n, c, s)$ be as in* (3-1). *Then*

(1) $C_{\langle x\rangle}(\langle y\rangle) = \langle x^{p^c}\rangle$,

(2) $C_{\langle y\rangle}(\langle x\rangle) = \langle y^{p^c}\rangle$,

(3) $Z(P) = \langle x^{p^c}, y^{p^c}\rangle$ *and* $|Z(P)| = p^{m+n-2c}$.

By the third property above, we have $Z(P) \le H_{ab}$ for all $0 \le a, b \le c$.

We know the metacyclic forms of the subgroups H_{ab}.

Proposition 4.2. *Let $P = P(m, n, c, 0)$ and consider $H_{ab} \le P$, where $0 \le a \le m$ and $0 \le b \le n$. Then*

$$H_{ab} \cong \begin{cases} P(m-a,\, n-b,\, c-(a+b),\, 0) & \text{if } a+b < c,\\ \mathbb{Z}_{p^{m-a}} \times \mathbb{Z}_{p^{n-b}} & \text{if } a+b \ge c.\end{cases}$$

In particular, $|H_{ab}| = p^{m+n-a-b}$.

Proof. By repeated applications of Theorem 3.11, we see that

$$\langle x^{p^a}, y^{p^b}\rangle = P(m-a,\, n-b,\, c-(a+b),\, 0)$$

as long as $c-(a+b) > 0$. On the other hand, by Lemma 3.8

$$y^{p^b} x^{p^a} = x^{p^a(1+p^{m-c})^{p^b}} y^{p^b}.$$

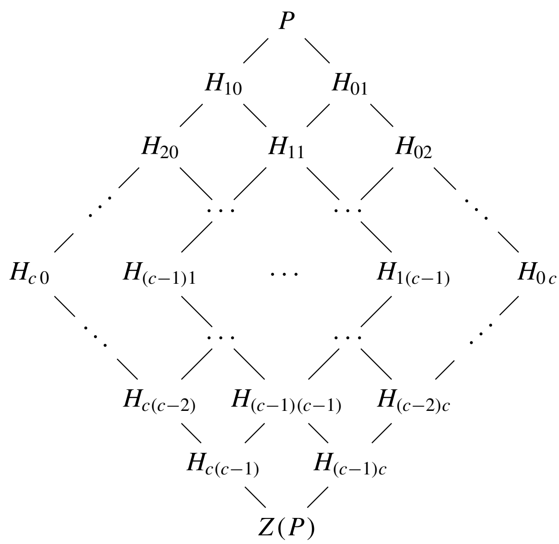

Figure 3. The BEK-lattice of a split metacyclic p-group.

By Lemma 3.9, $(1+p^{m-c})^{p^b} \equiv 1 \bmod p^{m-c+b}$. Thus

$$x^{p^a(1+p^{m-c})^{p^b}} = x^{p^a(1+dp^{m-c+b})}$$

for some $d \in \mathbb{Z}$. If $a+b \geq c$, we can see that H_{ab} is abelian and isomorphic to $\mathbb{Z}_{p^{m-a}} \times \mathbb{Z}_{p^{n-b}}$. □

Theorem 4.3. *Let $\mathcal{CD}_{\text{bek}}(P) = \{H_{ab} \mid 0 \leq a, b \leq c\}$. Then $\mathcal{CD}_{\text{bek}}(P)$ is a subposet of $\mathcal{CD}(P)$ that is rank-symmetric, and its corresponding lattice, $\mathcal{BEK}(P)$, is shown in Figure 3.*

To show that each H_{ab} is in $\mathcal{CD}(P)$, we will show that the centralizers of the nonabelian H_{ab} are the abelian ones.

Proposition 4.4. $$C_P(H_{ab}) = H_{c-b,c-a}.$$

Proof. When $a+b < c$, we know H_{ab} is nonabelian and thus is in $\mathcal{CD}(P)$ by Theorem 3.5. Therefore, $m_P(H_{ab}) = p^{2(m+n-c)}$. Since $|H_{ab}| = p^{m+n-a-b}$, it follows that $|C_P(H_{ab})| = p^{m+n-2c+a+b}$. By Lemma 4.1,

$$Z(H_{ab}) = Z(P(m-a,\, n-b,\, c-a-b,\, 0)) = \langle x^{p^{c-b}}, y^{p^{c-a}} \rangle = H_{c-b,c-a}$$

and therefore $|Z(H_{ab})| = p^{m+n-2c+a+b}$. We have $|C_P(H_{ab})| = |Z(H_{ab})|$ and so $C_P(H_{ab}) = Z(H_{ab}) = H_{c-b,c-a}$.

When $a+b = c$, we know $|H_{ab}| = p^{m+n-c}$, so it is maximal abelian. From Theorem 3.5 we know $C_P(H_{ab}) = H_{ab} = H_{c-b,c-a}$.

Let $a+b>c$. Then by Lemma 3.8,

$$y^{p^b}x^{p^{c-b}} = x^{p^{c-b}(1+p^{m-c})^{p^b}}y^{p^b} = x^{p^{c-b}}y^{p^b},$$

so $x^{p^{c-b}} \in C_P(H_{ab})$. Similarly,

$$y^{p^{c-a}}x^{p^a} = x^{p^a(1+p^{m-c})^{p^{c-a}}}y^{p^{c-a}} = x^{p^a}y^{p^{c-a}},$$

so $y^{p^{c-a}} \in C_P(H_{ab})$. Thus $H_{c-b,c-a} \le C_P(H_{ab})$. We have

$$|H_{ab}||C_P(H_{ab})| \ge |H_{ab}||H_{c-b,c-a}| = p^{2m+2n-2c}.$$

Since this is the maximal possible measure, we have $H_{c-b,c-a} = C_P(H_{ab})$. □

The next corollary follows immediately.

Corollary 4.5. $H_{ab} \in \mathcal{CD}(P)$ *for all* $0 \le a, b \le c$.

Proposition 4.6. *H_{ab} covers $H_{a'b'}$ if and only if $a+b+1 = a'+b'$ and either $a' = a+1$ or $b' = b+1$.*

Proof. First suppose $H_{a'b'} < H_{ab}$. The generators of $H_{a'b'}$ must be in H_{ab}, so there exist $i, j, u, v \in \mathbb{Z}$ such that

$$x^{p^{a'}} = (x^{p^a})^i (y^{p^b})^j, \tag{4-1}$$

$$y^{p^{b'}} = (x^{p^a})^u (y^{p^b})^v. \tag{4-2}$$

From (4-1) we see that $p^{a'} = ip^a$, so $a' \ge a$. From (4-2) we have $b' \ge b$. If $a+b = a'+b'-1$, then there are exactly two groups that cover $H_{a'b'}$ as stated in the proposition. If $a+b < a'+b'-1$, then one of three cases can occur: (i) $a \le a'-2$, (ii) $b \le b'-2$, or (iii) $a \le a'-1$ and $b \le b'-1$. In turn, we will have the three cases

(i) $H_{a'b'} < H_{a'-1,b'} < H_{a'-2,b'} \le H_{ab}$,

(ii) $H_{a'b'} < H_{a',b'-1} < H_{a',b'-2} \le H_{ab}$,

(iii) $H_{a'b'} < H_{a'-1,b'} < H_{a'-1,b'-1} \le H_{ab}$,

contradicting the fact that H_{ab} covers $H_{a'b'}$.

Conversely, if $a+b+1 = a'+b'$ and $a' = a+1$, then it is clear that $H_{a'b'} < H_{ab}$ and there does not exist H_{cd} such that $H_{a'b'} < H_{cd} < H_{ab}$. We have a similar result if $b' = b+1$. □

The proposition above proves that $\mathcal{BEK}(P)$ has the structure illustrated in Figure 3.

Finally, we define a rank function $\rho : \mathcal{CD}_{\text{bek}}(P) \to \mathbb{N}$ by $\rho(H_{ab}) = m+n-a-b$. Then $\rho(H_{ab}) = \rho(H_{a'b'})+1$ when H_{ab} covers $H_{a'b'}$ and we see that $\mathcal{CD}_{\text{bek}}(P)$ is a ranked poset that is clearly rank-symmetric from its definition, proving Theorem 4.3.

$$
\begin{tikzcd}[column sep=small]
 & & & P \arrow[dl, dash, "p"'] \arrow[dr, dash, "1"] & & & \\
 & & H_{10}(p) \arrow[dl, dash, "p"'] \arrow[dr, dash, "1"] & & H_{01}(1) \arrow[dl, dash, "p"'] \arrow[dr, dash, "1"] & & \\
 & H_{20}(p^2) \arrow[dl, dash, "p"'] \arrow[dr, dash, "1"] & & H_{11}(p) \arrow[dl, dash, "p"'] \arrow[dr, dash, "1"] & & H_{02}(1) \arrow[dl, dash, "p"'] \arrow[dr, dash, "1"] & \\
H_{30}(p^3) \arrow[dr, dash, "1"'] & & H_{21}(p^2) \arrow[dl, dash, "p"] \arrow[dr, dash, "1"'] & & H_{12}(p) \arrow[dl, dash, "p"] \arrow[dr, dash, "1"'] & & H_{03}(1) \arrow[dl, dash, "1"] \\
 & H_{31}(p^2) \arrow[dr, dash, "1"'] & & H_{22}(p) \arrow[dl, dash, "p"] \arrow[dr, dash, "1"'] & & H_{13}(1) \arrow[dl, dash, "p"] & \\
 & & H_{32}(p) \arrow[dr, dash, "1"'] & & H_{23}(1) \arrow[dl, dash, "p"] & & \\
 & & & Z(P) & & &
\end{tikzcd}
$$

Figure 4. The CD-lattice of $P(4, 6, 3, 0)$.

As we will prove below, there are conditions on the parameters of $P(m, n, c, 0)$ that guarantee its CD-lattice "collapses" to the BEK-lattice. That is, all subgroups in $\mathcal{CD}(P)$ will be isomorphic to H_{ab} for some $0 \le a, b \le c$. There will be multiple copies of some H_{ab} in $\mathcal{CD}(P)$ — for example, in $P(3, 4, 1, 0)$ there will be p copies of H_{10} by Theorem 3.11 — so that the edges in $\mathcal{L}(P)$ will be weighted versions of those in $\mathcal{BEK}(P)$.

Theorem 4.7. *Let $P = P(m, n, c, 0)$, with $m \le n$ and $c \le n - m + 1$. Then every subgroup in $\mathcal{CD}(P)$ is isomorphic to H_{ab} for some $0 \le a, b \le c$.*[3]

Proof. We begin by showing that all the nonabelian subgroups in the upper half of $\mathcal{L}(P)$ are isomorphic to some H_{ab}.

By Theorem 3.11, the maximal subgroups of P are isomorphic to H_{10} and H_{01}. Thereafter, as the order of the subgroups decreases, they are split and of the form $H_{ab} = P(m-a, n-b, c-a-b, 0)$ as long as $m-a \le n-b$. Suppose $m+k = n$ for some $k \ge 0$. The first possibility for a nonsplit subgroup to appear is when $m-a > n-b$, or $b-a > k$. Thus $H_{0,k+1}$ is the first subgroup of P that might have a nonsplit subgroup (none of the other H_{ab} of the same order satisfy $b-a = k+1$). However, $k+1 \ge c$ implies $H_{0,k+1}$ is abelian so we have reached the lower half of the lattice and all subgroups in the upper half are split and isomorphic to some H_{ab}.

Next we must show that centralizers of all nonabelian subgroups of P are isomorphic to some $H_{a'b'}$ with $a'+b' > c$. Proposition 4.4 shows that the centralizers of those nonabelian subgroups of P exactly equal to H_{ab}, where $a+b < c$, are equal to $H_{c-b,c-a}$.

[3] These are not the only conditions on the metacyclic parameters under which the result of the theorem holds, but they are the most succinct.

Now suppose that $M \le P$ is isomorphic to H_{ab}, where $a + b < c$, and let $M = P(m-a, n-b, c-a-b, 0) = \langle s, t\rangle$ as in (3-1). By Lemma 4.1, $Z(M) = \langle s^{p^{c-a-b}}, t^{p^{c-a-b}}\rangle$, which has order $p^{m+n-2c+a+b}$, and, therefore, must be equal to $C_P(M)$. Thus $C_P(M) = P(c-b, c-a, 0, 0) \cong H_{c-b,c-a}$. $\square$

Example 4.8. In Figure 4 we illustrate the CD-lattice for $P(4, 6, 3, 0)$, where $m < n$ and $c \le n - m + 1$. The weights on the edges indicate the number of isomorphic copies of a particular subgroup coming from a parent H_{ab} (for example, each copy of H_{10} has p subgroups isomorphic to H_{20}), and the numbers in parentheses indicate the number of distinct subgroups of a particular form (for example, even though there are $2p^2$ paths from P to H_{21}, there are only p^2 distinct copies of H_{21} in P). Except for the weights, one can see that $\mathcal{L}(P)$ looks like $\mathcal{BEK}(P)$.

5. Two complete CD-lattices

We begin by applying the theory of Section 3 to a particular family of split metacyclic p-groups whose Chermak–Delgado lattices we can determine completely.

Theorem 5.1. *The Chermak–Delgado lattice of $P(m, n, 1, 0)$ is as illustrated in Figure 5.*

Proof. Since $|Z(P)| = p^{m+n-2}$, we know the only subgroups in $\mathcal{L}(P)$ other than $Z(P)$ and P are the maximal proper subgroups, which coincide with the maximal abelian subgroups on the line of symmetry at height p^{m+n-1}. From Theorem 3.11 we have

(1) $L = \langle x^p, y\rangle = P(m-1, n, 0, 0) \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_p$,

(2) $M_i = \langle x^p, x^i y\rangle = \begin{cases} P(m-1, n, 0, 0) \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n} & \text{if } m \le n, \\ P(m, n-1, 0, 0) \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} & \text{if } m > n, \end{cases}$

(3) $R = \langle x, y^p\rangle = P(m, n-1, 0, 0) \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}}$.

Note that if $n = 1$, then $M_i = \langle x^i y\rangle \cong \mathbb{Z}_{p^m}$ and $R = \langle x\rangle \cong \mathbb{Z}_{p^m}$. $\square$

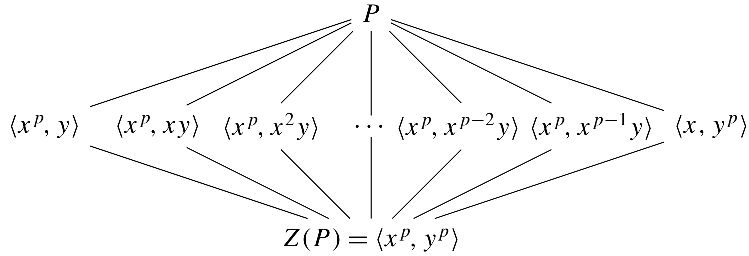

Figure 5. The Chermak–Delgado lattice of $P(m, n, 1, 0)$.

The theorem above shows that the CD-lattice of $P(m, n, 1, 0)$ is a *quasiantichain* in the sense of [Brewster et al. 2014b]. Indeed, while $\mathcal{L}(P)$ contains many intervals that are quasiantichains, the work in Section 3 shows that the whole CD-lattice of $P(m, n, c, 0)$ is a quasiantichain (of width $p+1$) if and only if $c = 1$.

The next example shows how some nonsplit subgroups can appear in $\mathcal{CD}(P)$, but modulo these irregular groups the CD-lattice looks like the BEK-lattice.

We build the CD-lattice of $P = P(6, 5, 4, 0)$ one level at a time, using Theorems 3.11 and 3.12. Lowercase letters such as m, n, c, and s will always refer to the original group P, while uppercase letters such as M, N, C, and S will refer to the parameters of the particular subgroup in question:

• *Order* p^{10}. Since $m > n > m - c + 1$, there are $p - 1$ subgroups of type $J_1 = P(5, 5, 3, 1)$ together with H_{10} and H_{01} at the maximal level.

• *Order* p^9. Since $H_{10} = P(5, 5, 3, 0)$ has $M \le N$, it has p subgroups isomorphic to type L and one of type R. That is, we have p copies of H_{20} and one of H_{11}.

Since $H_{01} = P(6, 4, 3, 0)$ has $M > N$ and $N \le M - C + 1$, it has p subgroups isomorphic to type R and one of type L. That is, we have p copies of H_{02} and one of H_{11}.

Since $J_1 = P(5, 5, 3, 1)$ has $S < C - 1$, it has p subgroups isomorphic to type $L = P(4, 5, 2, 1) = J_2$. Since $S = M - N + 1$, we know J_1 has one subgroup of type $R = P(5, 4, 2, 0) = H_{11}$.

There are only $p^2 + p + 1$ subgroups of P of order p^9, so there must be intersections among the $p^2 + 2p + 1$ subgroups listed above. The intersections can be hard to track because we often use [King 1973] to show the existence of alternative generators having nice properties. Although King shows how to construct the alternative generators, tracking all of them is a mind-numbing task that we will not illustrate.

• *Order* p^8. Since $H_{20} = P(4, 5, 2, 0)$ has $M \le N$, it has p subgroups isomorphic to type $L = P(3, 5, 1, 0) = H_{30}$ and one of type $R = P(4, 4, 1, 0) = H_{21}$.

Since $H_{11} = P(5, 4, 2, 0)$ has $M > N$ and $N \le M - C + 1$, it has p subgroups isomorphic to type $R = P(5, 3, 1, 0) = H_{12}$ and one of type $L = P(4, 4, 1, 0) = H_{21}$.

Since $H_{02} = P(6, 3, 2, 0)$ has $M > N$ and $N \le M - C + 1$, it has p subgroups isomorphic to type $R = P(6, 2, 1, 0) = H_{03}$ and one of type $L = P(5, 3, 1, 0) = H_{12}$.

Since $J_2 = P(4, 5, 2, 1)$ has $S = C - 1$, it has p subgroups isomorphic to type $L = P(6, 2, 1, 0) = H_{03}$. Since $M - N + 1 < S = C - 1$, we know J_2 has one subgroup of type $R = P(5, 3, 1, 0) = H_{12}$.

• *Order* p^7. Since $H_{30} = P(3, 5, 1, 0)$ has $M \le N$, it has p subgroups isomorphic to type $L = P(2, 5, 0, 0) = H_{40}$ and one of type $R = P(3, 4, 0, 0) = H_{31}$.

Since $H_{21} = P(4, 4, 1, 0)$ has $M \le N$, it has p subgroups isomorphic to type $L = P(3, 4, 0, 0) = H_{31}$ and one of type $R = P(4, 3, 0, 0) = H_{22}$.

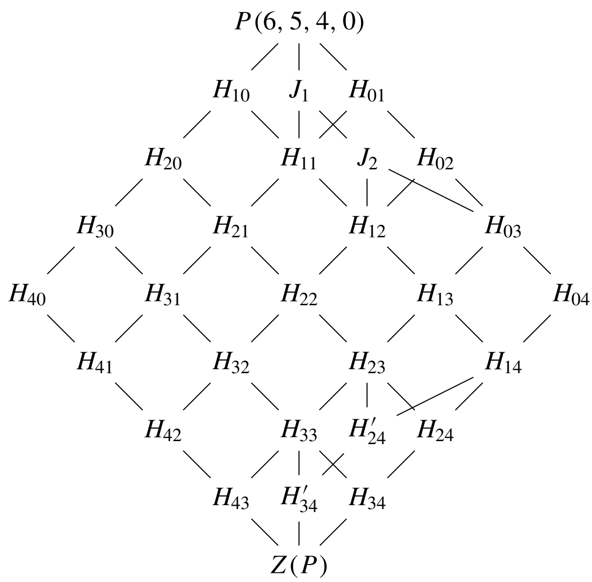

Figure 6. The collapsed CD-lattice of $P(6, 5, 4, 0)$.

Since $H_{12} = P(5, 3, 1, 0)$ has $M > N$ and $N \leq M - C + 1$, it has p subgroups isomorphic to type $R = P(5, 2, 0, 0) = H_{13}$ and one of type $L = P(4, 3, 0, 0) = H_{22}$.

Since $H_{03} = P(6, 2, 1, 0)$ has $M > N$ and $N \leq M - C + 1$, it has p subgroups isomorphic to type $R = P(6, 1, 0, 0) = H_{04}$ and one of type $L = P(5, 2, 0, 0) = H_{13}$.

• *Order* $\leq p^6$. Notice that all of the subgroups of order p^7 are abelian, so that is the row of maximal abelian subgroups of P. The rest of the CD-lattice consists of centralizers of the groups above. From Proposition 4.4 we know that $C_P(H_{ab}) = H_{c-b,c-a}$. Suppose $J_1 = P(5, 5, 3, 1) = \langle s, t\rangle$. Then Lemma 4.1 says that $Z(J_1) = \langle s^{p^3}, t^{p^3}\rangle = P(2, 2, 0, 1)$. This has order p^4, which is the order of $C_P(J_1)$, so $Z(J_1) = C_P(J_1)$. By the decision tree in Figure 2, we see that $Z(J_1)$ has an alternative presentation of the form $P(3, 1, 0, 0)$, making it isomorphic to H_{34}.

Similarly, the center of J_2 coincides with its centralizer in P and is of the form $P(2, 3, 0, 1)$. This subgroup has an alternative presentation as $P(4, 1, 0, 0)$, which is isomorphic to H_{24}.

A collapsed version of $\mathcal{L}(P)$ is shown in Figure 6, where multiple copies of subgroups are not indicated.

Acknowledgements

The authors would like to thank one of the reviewers for helpful comments that inspired us to delve deeper and write a better, more complete paper.

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Received: 2015-03-16 Revised: 2015-10-01 Accepted: 2015-10-27

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

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involve

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